6. Bott & Tu exercise II.13.6, p. 152. (First read Example 13.5 on the previous page. Note that to do the problem, you don’t need to know Proposition 12.1, though you’re welcome to read it—we’ll be proving a more general version of this in the context of spectral sequences.)


8. Similarly to the two previous problems, the fiber bundle $S^2 \to \mathbb{RP}^2$ (with the fiber being two points) yields a presheaf $\mathcal{H}^0$ on $\mathbb{RP}^2$; over a good cover $\mathcal{U}$ of $\mathbb{RP}^2$, this is locally constant with $\mathcal{H}^0(U) \cong \mathbb{R}^2$ for each nonempty finite intersection $U$ of open sets in $\mathcal{U}$, but $\mathcal{H}^0$ isn’t constant.

With that as motivation, give an explicit construction of a locally constant presheaf on $\mathbb{RP}^2$ that isn’t constant.\(^1\) More precisely, find a good cover $\mathcal{U}$ of $\mathbb{RP}^2$ and a presheaf $\mathcal{F}$ over $\mathcal{U}$ with $\mathcal{F}(U) = \mathbb{R}^2$ for all nonempty finite intersections $U$, and give all restriction maps $r_{V,U} : \mathbb{R}^2 \to \mathbb{R}^2$ for nonempty finite intersections $U \subset V$. Then prove that your presheaf is not isomorphic to the constant presheaf $\mathbb{R}^2$ over $\mathcal{U}$.

(Hint: if you start with a triangulation of $\mathbb{RP}^2$, then this becomes an exercise in graph theory.)

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\(^1\)This doesn’t have to be $\mathcal{H}^0$. In fact, over any good cover of $\mathbb{RP}^2$, there are exactly three locally constant presheaves for the group $\mathbb{R}^2$, up to isomorphism; one of these is constant and two are nonconstant. You’re welcome to verify this yourself if you’re interested.