Math 612 HW 4 Solutions

1. \( M \text{ connected} \Rightarrow H_0^c(M) = \{0\}, \quad H_0^c(M) = \begin{cases} \mathbb{R}, & M \text{ compact} \\ \{0\}, & \text{otherwise} \end{cases} \)

\[ H_1^{Dr}: \text{if } \omega = f \, dx + g \, dy \text{ is closed} \quad \Leftrightarrow \quad \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \text{then by} \]

Green's Thm., if \( \gamma, \delta \) are homotopic paths in \( \mathbb{R}^2 \) with fixed endpoints, then \( \int_\gamma \omega = \int_\delta \omega \). Then we can define \( h: \mathbb{R}^2 \to \mathbb{R} \) by \( h(x,y) = \int_\gamma \omega \) where \( \gamma \) is any path from \((0,0)\) to \((x,y)\).

By choosing \( \gamma \) of the form \( (x,y) \) for all \((x,y)\), we see that \( \frac{\partial h}{\partial x} = f \),

by choosing \( (x,y) \), we get \( \frac{\partial h}{\partial y} = g \). Thus \( \omega = dh \) and \( H_1^{Dr}(\mathbb{R}^2) = 0 \).

\[ H_2^{Dr}: \text{if } \omega = k(x,y) \, dx \wedge dy \text{, then } \omega = d\eta \text{ where} \]

\[ \eta(x,y) = \left( \int_0^x k(t,y) \, dt \right) \, dy. \quad \text{Thus } \quad H_2^{Dr}(\mathbb{R}^2) = 0. \]

\[ H_1^c: \text{Proceed as for } H_1^{Dr}, \text{ but fix } (x_0,y_0) \text{ outside } B_2(0) \supset \text{ supp } \omega, \]

and define \( h(x,y) = \int_\gamma \omega \) where \( \gamma \) is any path from \((x_0,y_0)\) to \((x,y)\).

Then \( \omega = dh \) as before, and \( h = 0 \) outside \( B_2(0) \) since any \((x,y) \notin B_2(0) \) can be connected to \((x_0,y_0)\) by a path outside \( B_2(0) \).

\[ H_2^c: \text{Define } f: \Omega^2_c(\mathbb{R}^2) \to \mathbb{R} \text{ by } f(\omega) = \int_{\mathbb{R}^2} \omega. \]

\[ \begin{bmatrix} \Omega^1_c(\mathbb{R}^2) \xrightarrow{d} \Omega^2_c(\mathbb{R}^2) \xrightarrow{f} \mathbb{R} \to 0 \end{bmatrix} \]

is exact. Then \( H_2^c(\mathbb{R}^2) = \Omega^2_c(\mathbb{R}^2)/\text{im } d \cong \Omega^2_c(\mathbb{R}^2)/\ker f \cong \mathbb{R} \).

It’s clear that \( f \) is surjective. Also, if \( \omega = d\eta \) for \( \eta \in \Omega^1_c(\mathbb{R}^2) \), then \( \eta = 0 \) outside \( B_2(0) \), and write \( \gamma = \partial B_2(0) \) by Green’s Thm.,

\[ \int_{\mathbb{R}^2} \omega = \int_{\partial B_2(0)} \gamma \eta = 0. \]
\textbf{Claim: } \ker f \leq \text{im}(d: \mathcal{D}_c^\infty \to \mathcal{D}_c^\infty).

\textbf{Proof:} Suppose \( w = k(x,y) \, dx \, dy \) is compactly supported with
\[ \iint_{\mathbb{R}^2} k(x,y) \, dx \, dy = 0. \]
Define
\[ g_0(x,y) = \int_{-\infty}^{\infty} k(t,y) \, dt \]
\[ \eta(x,y) = g_0(x,y) \, dy. \]
Then \( w = dx \, \eta \). Now \( g_0 \) isn't necessarily compactly supported:
for fixed \( y \), if \( x \gg 0 \), then \( g_0(x,y) = h(y) \) where \( h(y) = \int_{-\infty}^{\infty} k(t,y) \, dt \).
Note \( \int_{-\infty}^{\infty} h(y) \, dy = \iint_{\mathbb{R}^2} k(x,y) \, dx \, dy = 0 \), and \( h \) is compactly supported.
Since \( k \) is. Also define \( \sigma : \mathbb{R} \to \mathbb{R} \) to be any smooth function such that
\[ \sigma(x) = \begin{cases} 0 & \text{if } x \ll 0, \\ 1 & \text{if } x \gg 0. \end{cases} \]
Then
\[ g(x,y) := g_0(x,y) - \sigma(x) \, h(y) \]
is compactly supported: if \( |y| \gg 0 \) then \( g_0(x,y) = \sigma(x) \, h(y) = 0 \),
and for fixed \( y \), if \( |x| \gg 0 \) then \( g_0(x,y) = \sigma(x) \, h(y) \).
Furthermore,
\[ d(g(x,y) \, dy) = d\eta - d(\sigma(x) \, h(y)) \, dy = \omega - \sigma'(x) \, h(y) \, dx \, dy. \]
Now define \( f : \mathbb{R}^2 \to \mathbb{R} \) by
\[ f(x,y) = \sigma'(x) \left( \int_{-\infty}^{\infty} h(t) \, dt \right). \]
Since \( \sigma(x) \) is constant for \( |x| \gg 0 \) and \( \int_{-\infty}^{\infty} h(t) \, dt = 0 \), \( f \) is compactly supported.
Finally,
\[ d \left( f(x,y) \, dx \right) = \sigma'(x) \, h(y) \, dy \, dx \]
so
\[ \omega = d \left( g(x,y) \, dy - f(x,y) \, dx \right) \]
and we im \( d \), as desired.
2. (a) $H^1_{de} (\mathbb{R}^2 - \{0\})$ is clear except $k=1,2$.

$H^1_{de}$: Define $Z, B \subseteq \Omega^1 (\mathbb{R}^2 - \{0\})$ by $Z = \{dw - \partial f\}$ and $B = \{df | f: \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R}\}$.

Define $\Phi: Z \rightarrow B$ by $\Phi(w) = \int_{\gamma_0} w$ where $\gamma_0 = \text{unit circle in } \mathbb{R}^2$.

Then $\Phi$ is surjective: if $w = "d\theta" = \frac{x dy - y dx}{x^2 + y^2}$, then $\Phi(w) = 2\pi$ and $dw = 0$.

Claim: $B = \ker \Phi$; then $\mathbb{R}^2 \cong \mathbb{R}/\mathbb{Z}$ as desired.

Note $B \subseteq \ker \Phi$ since $\int_{\gamma_0} df = 0$ by the fundamental theorem of line integrals. So it remains to prove $\ker \Phi \subseteq B$.

Suppose $w = f dx + g dy$ is closed. As in #1, we want to define $h: \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R}$ by $h(x,y) = \int_{\gamma} w$ where $\gamma$ is any path in $\mathbb{R}^2 - \{0\}$ from $(1,0)$ to $(x,y)$. If this is well-defined, then $w = dh$.

But if $\Phi(w) = 0$, then this is well-defined: if $\gamma, \gamma'$ are two paths from $(1,0)$ to $(x,y)$, then $\exists k \in \mathbb{Z}$ such that $\gamma'$ is homotopic to $k \gamma_0 + \gamma$ (i.e. trace to $k$ times, then follow with $\gamma$).

By Green's Theorem,

$$\int_{\gamma} w = \int_{\gamma_0} w = k \int_{\gamma_0} w + \int_{\gamma'} w = \int_{\gamma'} w,$$

and so $h$ is well-defined.

Note: $[c_0]$ generates $H^1_{de} (\mathbb{R}^2 - \{0\})$.

$H^2_{de}$: Use polar coordinates $(r, \theta)$ with $x = r \cos \theta, y = r \sin \theta$. Then

$$dr = \cos \theta \, dx + \sin \theta \, dy = \frac{xdx + ydy}{\sqrt{x^2 + y^2}}$$

$$d\theta = -\sin \theta \, dx + \cos \theta \, dy = \frac{-ydx + xdy}{\sqrt{x^2 + y^2}}$$

$$r \, dr \wedge d\theta = dx \wedge dy.$$
2. \textit{(a)} \quad \text{If } \omega = k(x,y) \, dx \wedge dy \text{ on } \mathbb{R}^2 \setminus 0, \text{ define } l : \mathbb{R}^2 \setminus 0 \to \mathbb{R} \text{ by}
\begin{align*}
l(r \cos \theta, r \sin \theta) &= \int_{r}^{\infty} t k(t \cos \theta, t \sin \theta) \, dt \\
\Rightarrow \frac{\partial l}{\partial r} &= r k(r \cos \theta, r \sin \theta).
\end{align*}
Thus if we define \( \eta \in \Omega^1(\mathbb{R}^2 \setminus 0) \) by \( \eta = l \, d\theta \), then
\[ d\eta = \frac{\partial l}{\partial r} \, dr \wedge d\theta = k \, dr \wedge d\theta = k \, dx \wedge dy = \omega. \]

\text{If you don’t like polar coords, this can be shown in Cartesian:}
\begin{align*}
l(x,y) &= \int_{0}^{\infty} \frac{t}{x^2 + y^2} \, dt \\
\eta &= l(x,y) \left(-\frac{x \, dx + y \, dy}{x^2 + y^2}\right)
\end{align*}
and it's an involved but straightforward computation that \( \omega = dy \).

\textbf{(b)} \quad \text{Let } k : \mathbb{R}^2 \setminus 0 \to \mathbb{R} \text{ be any function with compact support} \\
in \mathbb{R}^2 \setminus 0 \text{ and such that } \iiint_{\mathbb{R}^2 \setminus 0} k \, dx \, dy \, dz \neq 0.

\textbf{We claim}
\[ k \, dx \wedge dy \wedge dz = \text{im}(d : \Omega^2(\mathbb{R}^2 \setminus 0) \to \Omega^3(\mathbb{R}^2 \setminus 0)). \]

\textbf{Indeed, if } \omega \in \Omega^1_c(\mathbb{R}^2 \setminus 0) \text{ is given by}
\[ \omega = f_1 \, dy \wedge dx + f_2 \, dx \wedge dz + f_3 \, dz \wedge dy, \]
\text{then there are } r, R > 0 \text{ such that}
\[ (\text{supp } f_1) \cup (\text{supp } f_2) \cup (\text{supp } f_3) \subset \left\{ r < \| (x,y,z) \| < R \right\} = K. \]

\textbf{Write } \vec{F} \text{ for the vector field on } \mathbb{R}^2 \setminus 0 \text{ given by } \vec{F} = (f_1, f_2, f_3).

\textbf{Then } \omega = \div \vec{F} \, dx \wedge dy \wedge dz, \text{ and by the Divergence Theorem,}
\begin{align*}
\iiint_{\mathbb{R}^2 \setminus 0} (\div \vec{F}) \, dx \, dy \, dz &= \iiint_{K} (\div \vec{F}) \, dx \, dy \, dz = \int_{S^2_2(0)} \vec{F} \cdot d\vec{n} - \int_{S^2_2(0)} \vec{F} \cdot d\vec{n} = 0.
\end{align*}

\textbf{Thus } \omega = k \, dx \wedge dy \wedge dz.