In the first two problems, you will prove a version of the Küneth Theorem for \textit{compactly supported cohomology} under certain assumptions.

1. (a) Let $M$ and $N$ be smooth manifolds, and let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the projection maps. Show that the cross product map $\psi$ defined by

$$\psi(\omega \otimes \eta) = \pi_1^* \omega \wedge \pi_2^* \eta$$

induces a well-defined map $\chi : H^*_c(M) \otimes H^*_c(N) \to H^*_c(M \times N)$.

(b) If $M$ and $N$ are orientable and both have finite good covers, use the usual Küneth Theorem and Poincaré duality to prove that $\chi$ is an isomorphism.

2. With the same notation as the previous problem, but under the weaker assumption that $M$ has a finite good cover (and no assumption on orientability of $M$ and $N$), use Mayer–Vietoris and induction to prove that $\chi$ is an isomorphism.

3. (a) Let $(K^{*,*}, \delta, d)$ be a double complex. We can construct another double complex $(\tilde{K}^{*,*}, \tilde{\delta}, \tilde{d})$ by setting $\tilde{K}^{i,j} = K^{j,i}, \tilde{\delta} = d, \tilde{d} = \delta$. Prove that the single complexes $(K^*, D)$ and $(\tilde{K}^*, \tilde{D})$ are chain isomorphic, where as usual $D = \delta + (-1)^i d$ on $K^{i,j}$ and $\tilde{D} = \tilde{\delta} + (-1)^j \tilde{d}$ on $\tilde{K}^{i,j}$.

(b) Bott & Tu Exercise II.8.4, p. 93.