1. (a) The isomorphism is of graded rings, where the $k$-th graded piece of \( \Lambda^k M_n \) is \( \Lambda^k M_n \).

Proof by induction: for \( n = 1 \), \( H^* (S^1 ; \mathbb{Z}) \cong \mathbb{Z}[x]/(x) \cong \Lambda^1 M_1 \).

\( \mathbb{Z}[x]/(x) \cong \mathbb{Z} \langle x \rangle \oplus \mathbb{Z} \langle 1 \rangle \), where \( x \) has degree \( 1 \) and \( 1 \) has degree \( 0 \).

Induction step: suffices to find an isomorphism

\[ \varphi : (\Lambda^2 M_{n-1}) \otimes (\mathbb{Z}[x]/(x)) \rightarrow \Lambda^2 M_n. \]

This is given by \( \varphi (w \otimes 1) = \omega, \varphi (w \otimes x) = w \wedge \nu_n \) for \( w \in \Lambda^2 M_{n-1} \).

It’s easy to check that this is an isomorphism of graded \( \mathbb{Z} \)-modules, so we just have to check that it preserve multiplication:

\[ \varphi ((w_1 \otimes 1)(w_2 \otimes 1)) = \varphi (w_1 \wedge w_2) = \varphi (w_1 \otimes 1) \varphi (w_2 \otimes 1) \]

\[ \varphi ((w_1 \otimes 1)(w_2 \otimes x)) = \varphi ((w_1 \wedge w_2) \otimes x) = w_1 \wedge \nu_n \otimes x = \varphi (w_1 \otimes 1) \varphi (w_2 \otimes x) \]

\[ \varphi ((w_1 \otimes x)(w_2 \otimes 1)) = (-1)^{w_1} \varphi ((w_1 \wedge w_2) \otimes x) = (-1)^{w_1} w_1 \wedge \nu_n \otimes x = \varphi (w_1 \otimes x) \varphi (w_2 \otimes 1) \]

\[ \varphi ((w_1 \otimes x)(w_2 \otimes x)) = 0 = \varphi (w_1 \otimes x) \varphi (w_2 \otimes x). \]

(b) If there were a homeomorphism \( f : \mathbb{C}P^3 \rightarrow S^2 \times S^4 \), we’d have a ring isomorphism

\[ f^* : H^* (S^2 \times S^4 ; \mathbb{Z}) \rightarrow H^* (\mathbb{C}P^3 ; \mathbb{Z}). \]

Say \( y \) generates \( H^2 (S^2 ; \mathbb{Z}) \cong \mathbb{Z} \) and \( 1 \) generates \( H^0 (S^4 ; \mathbb{Z}) \cong \mathbb{Z} \).

By Künneth, \( y \otimes 1 \) generates the dimension 2 part of \( H^* (S^2 \otimes S^4) \) (note \( H^*(S^2) \), \( H^*(S^4) \) are free). Thus \( f^*(y \otimes 1) = x \) where \( H^* (\mathbb{C}P^3 ; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^4) \). But then
1. (b) \[ x^2 = (\pm x)^2 = f^*(y_01) \cup f^*(y_01) = f^*(y_01)(y_01) = f^*(\zeta(y_01)) = 0, \]
   which isn't true.

2. (a) \[ H_0(X) = \mathbb{Z}, \text{ since } X \text{ connected } \Rightarrow H^0(X; \mathbb{Z}) = \text{ free part of } \mathbb{Z} = \mathbb{Z} \text{ by UCT.} \]
   \[ H_1(X) = 0, \text{ since } X \text{ simply connected } \Rightarrow H^1(X; \mathbb{Z}) = 0 \text{ by UCT.} \]
   Poincaré duality \[ H^*(X; \mathbb{Z}) = H_*(X; \mathbb{Z}) \]
   \[ H_3(X; \mathbb{Z}) = H_3(X; \mathbb{Z}) = 0, \]
   \[ H^1(X; \mathbb{Z}) = H_1(X; \mathbb{Z}) = 0. \]
   
   UCT: \[ H_2^\text{free}(X; \mathbb{Z}) = (\text{free part of } H_2(X)) \oplus (\text{torsion part of } H_2(X)) = \text{free}. \]
   PD: \[ H_2(X) \cong H^2(X; \mathbb{Z}) = \text{free}. \]

3. (c) Suppose \[ H_2^{m+1}(X) = \mathbb{Z}, \text{ then UCT } \Rightarrow \text{ free part of } H_2^{m+1}(X; \mathbb{Z}) = \mathbb{Z}. \]
   Then by PD, \[ \omega : (\text{free part of } H_2^{m+1}(X; \mathbb{Z}) \oplus \text{(some)} \to \mathbb{Z}^m \]
   is nonsingular. But if \( \alpha \) generates \( H_2^{m+1}(X; \mathbb{Z}) \), then \[ \omega(x) = \epsilon_1(x)^{m+1}. \]
   \[ \alpha \omega = \epsilon_1(x)^m \omega \alpha = \alpha \omega \alpha = 0, \]
   so \[ \omega : \mathbb{Z}^m \to \mathbb{Z} \] is the zero map, contradiction.
   
   Note: one can show more generally that \( H_2^{m+1}(X) \) must have even free rank.

3. (a) Let \( \alpha \) generate \( H^1(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2 \) and \( \beta \) generate \( H^1(\mathbb{R}P^m; \mathbb{Z}/2) = \mathbb{Z}/2 \).
   By naturality in UCT for fields, \( f^* : H^1 \to H^1 \) is an isomorphism.
   
   (since it's the dual map to \( f^* \)). Thus \( f^* \beta = \alpha. \)
   
   But \( H^*(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2 \langle x^m \rangle \) and \( H^*(\mathbb{R}P^m; \mathbb{Z}/2) = \mathbb{Z}/2 \langle \beta \rangle \).
   So \( \beta^{m+1} = 0 \Rightarrow 0 = f^*(\beta^{m+1}) = (f^* \beta)^{m+1} = \alpha^{m+1} \Rightarrow n \leq m. \)
3. (b) Such a map would induce \( \tilde{f} : \mathbb{R}P^n \to \mathbb{R}P^{n-1} \). Let \( x_0 \in S^n \) and let \( \gamma \) be any continuous path in \( S^n \) from \( x_0 \) to \(-x_0\). Then 

\[ p \circ \gamma \text{ generates } \pi_1(\mathbb{R}P^n, p(x_0)) \cong \mathbb{Z} \] 

by covering maps, and 

\[ p \circ \gamma \text{ connects } f(x_0) \text{ to } f(-x_0) = -f(x_0) \] 

so \( p \circ f \circ \gamma \) generates 

\[ \pi_1(\mathbb{R}P^n, p(f(x_0))) \cong \mathbb{Z} \] . Since \( p \circ f = \tilde{f} \circ p \), \( f \) maps the generator of \( H_1(\mathbb{R}P^n) \) to the generator of \( H_1(\mathbb{R}P^{n-1}) \), contradicting (c).

4. \( M \) connected \( \Rightarrow H^0_{de}(M) = \mathbb{R}, \quad H^0_c(M) = \{ 0 \}, \quad M \) compact \( \Rightarrow H^1_{de}(M) = 0 \). 

**\( H^1_{de} \)**: If \( \omega = f \, dx + g \, dy \) is closed \((\Leftrightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x})\), then by Green's Theorem, if \( \gamma, \tilde{\gamma} \) are homotopic paths in \( \mathbb{R}^2 \) with fixed endpoints, then 

\[ \int_{\gamma} \omega = \int_{\tilde{\gamma}} \omega \] . Then we can define \( h : \mathbb{R}^2 \to \mathbb{R} \) by 

\[ h(x, y) = \int_{\gamma} \omega \] where \( \gamma \) is any path from \((0,0)\) to \((x,y)\). 

By choosing \( \gamma \) of the form \( \gamma(t) = (xt, yt) \) for all \((x,y)\), we see that 

\[ \frac{\partial h}{\partial x} = f \] 

by choosing \( \gamma(t) = (xt, yt) \) we get \( \frac{\partial h}{\partial x} = g \). Thus \( \omega = dh \) and 

\[ H^1_{de}(\mathbb{R}^2) = 0 \] .

**\( H^2_{de} \)**: If \( \omega = k(x,y) \, dx \wedge dy \), then \( \omega = dy \wedge \eta \) when 

\[ \eta(x,y) = \left( \int_0^x k(t,y) \, dt \right) \, dy \] . Thus 

\[ H^2_{de}(\mathbb{R}^2) = 0 \] .
4. $H_c^1$: Proceed as for $H_{de}^1$, but fix $(x_0, y_0)$ outside $B_r(0) = \text{support } \omega$, and define $W(x, y) = \int_{\gamma} \omega$ where $\gamma$ is any path from $(x_0, y_0)$ to $(x, y)$. 