1. \( E_1 \cong \mathbb{R} \langle v_{2,1}^{10}, v_{3,1}^{20}, v_{1,2}^{12} \rangle \), \( d_1(v_{2,1}^{10}) = v_{3,1}^{20}, \ d_1(v_{3,1}^{20}) = d_1(v_{1,2}^{12}) = 0 \)

\[ E_2 = E_0 \cong \mathbb{R} \langle v_1^{10} \rangle, \ d_2 = \cdots = 0 \Rightarrow H^*(K, \mathbb{Z}) \cong \bigoplus_{i=0}^{\infty} \mathbb{Z} \]

2. \( E_1' = H(K, \mathbb{Z}) = \mathbb{Z}^n \), \( d_1' = 0 \), \( E_2' = H(E_1', d) = E_1' \)

\( d_1 \) has bidegree \((-1, 2)\) and \( d_2' [v_1^{10}]_2 = [v_0^{01}]_2 \) since we have

\[ (dv_1^{10} + 3v_0^{01} = v_3^{11} - v_3^{11} = 0) \]

\( \Rightarrow E_3' = H(E_2', d_3') = \mathbb{Z}^n \)

\( E_1' = E_2' \cong \mathbb{R} \langle v_{1,2}^{10}, v_{1,0}^{10}, v_{2,1}^{10} \rangle, \ d_1' = 0, \ d_1'(v_{1,0}^{10}) = v_{2,1}^{10} \)

\[ (E_1')^{10}, (E_1')^{10}, (E_1')^{21} \]

\( E_3' = E_0' \cong \mathbb{R} \langle v_{2,1}^{10} \rangle, \ d_3' = \cdots = 0 \Rightarrow H^*(K, \mathbb{Z}) \cong \bigoplus_{i=0}^{\infty} \mathbb{Z} \]
3. $K$ generated by $v_i^{0,-1}, v_i^{1,-1}, v_i^{1,1}, v_i^{2,1}, \ldots, v_i^{r-1,0}, v_i^{0,0}$ with $v_i^{ij} \in K_{ij}$ and

$$d(v_i^{i,-1}) = v_i^{i+1,0}, \quad 1 \leq i \leq r-1$$

$$d(\text{other}) = 0$$

$$\delta(v_i^{i+1,0}) = v_i^{i+2,1}, \quad 0 \leq i \leq r-1$$

$$\delta(\text{other}) = 0.$$

The only differential whose homology could give something smaller is $d_{r+1}$, so $d_{r+1} \neq 0$ in this case.

4. $H^q(\mathbb{RP}^3) \cong H^i(S^3) \cong \begin{cases} \mathbb{R} & i=0 \text{ or } i=3 \\ 0 & \text{otherwise} \end{cases}$

Since $S^3$ is simply connected, the Leray spectral sequence has

$$E_2^{i,j} = H^i(S^3) \otimes H^j(\mathbb{RP}^3)$$

$$E_2 = \begin{array}{cccc}
\mathbb{R} & 0 & 0 & \mathbb{R} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\mathbb{R} & 0 & 0 & \mathbb{R}
\end{array}$$

All differentials $d_2, d_3, \ldots$ are 0 for grading reasons, so $E_\infty = E_2$.

$$H^*(SO(4)) \cong \begin{cases} \mathbb{R} & i=0 \text{ or } i=6 \\ \mathbb{R}^2 & i=3 \\ 0 & \text{otherwise} \end{cases}$$
5. Note $S^2 \times S^2$ is simply connected so the $E_2$ page in the Leray spectral sequence is $H^\ast(S^2 \times S^2) \otimes H^\ast(S^1)$.

\[
\begin{array}{c|cccc}
R & 0 & R^2 & 0 & 0 \\
\hline
R & 0 & R^2 & 0 & R \\
\end{array}
\]

The only possible nontrivial higher differential is $d_2$, which we split into $d_2^{01}$ and $d_2^{31}$.

If $d_2^{01} = d_2^{31} = 0$ then $H^\ast(E) \cong E^\ast$ is

\[
\begin{array}{c|cccc}
\ast & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
H^\ast(E) & R & R & R & R & R & R \\
\end{array}
\]

and this is achieved when $E = S^2 \times S^2 \times S^1$.

If $d_2^{01} \neq 0$ then $E_0 \cong E_2 = \begin{array}{c} R \\
0 \\
0 \\
R \\
0 \\
R \end{array}$

and Poincaré duality fills in the rest.

Choice 1

Choice 2

This is achieved when $E = S^3 \times S^2$ (this is the Hopf fibration $S^2$).

Similarly $d_2^{31} \neq 0$ yields the same thing.