Math 612 Homework 12—due Tuesday April 23
Spring 2019

This is the final problem set of the semester. I think each problem is pretty short and is really more like half of a problem. Please note that I will be out of town on Monday April 22 and won’t have an office hour then. However, I’ll be available this Friday April 19 from 1:30 to 3:00, as a sort of substitute office hour, in addition to the usual Thursday office hour after class.

My plan is to hand out the take-home final exam on the last day of our class, April 23. It will be open book (you can use the textbooks, your notes, my notes, and the homeworks). The final will be due at the end of the day on Wednesday May 1. If this is a hardship for you then let me know and I can probably extend the due date until Friday May 3 (but not later, because of when grades are due).

1. Let \((K^{*,*}, \delta, d)\) be the double complex of \(\mathbb{R}\)-vector spaces defined as follows. A basis for \(K^{*,*}\) is given by 13 generators \(v^{0,1}, v^{0,2}, v^{1,0}, v^{1,1}, v^{1,2}, v^{2,0}, v^{2,1}, v^{2,2}, v^{3,1}, v^{3,2}, v^{2,3}, v^{2,2}\), where the superscript denotes the bigrading; e.g., \(K^{1,0}\) is the 2-dimensional vector space over \(\mathbb{R}\) generated by \(v^{1,0}\) and \(v^{1,0}\). The differentials are given by
   \[
   \delta(v^{0,1}) = v^{1,1}, \quad \delta(v^{0,2}) = v^{2,0}, \quad \delta(v^{1,1}) = v^{2,1}, \quad \delta(v^{1,2}) = v^{2,1} + v^{3,1}, \quad \delta(v^{1,3}) = v^{2,2}
   \]
   and \(\delta = 0\) on all other generators;
   \[
   d(v^{0,1}) = v^{0,2}, \quad d(v^{0,2}) = v^{1,1}, \quad d(v^{1,0}) = v^{1,1}, \quad d(v^{1,1}) = v^{2,1}, \quad d(v^{2,0}) = v^{2,1}, \quad d(v^{2,1}) = v^{2,2}
   \]
   and \(d = 0\) on all other generators.
   Calculate the (singly) graded vector space \(H^*(K, D)\) by calculating the spectral sequence \(\{E_r^{*,*}, d_r\}\) for \(K\) with the usual filtration.

2. With the same setup as problem 1, recalculate \(H^*(K, D)\) by calculating the alternate spectral sequence \(\{E'_r^{*,*}, d'_r\}\) obtained by switching the role of rows and columns \((E'_1 = H^{*,*}(K, \delta)\) and \(d'_r\) has bidegree \((1-r, r))\).

3. For arbitrary \(r \geq 1\), give an example of a finite-dimensional double complex of \(\mathbb{R}\)-vector spaces for which \(d_r\) is nonzero.

4. The Lie group \(SO(3)\) is diffeomorphic to \(\mathbb{R}P^3\). Use the fiber bundle \(SO(3) \to SO(4) \to S^3\) to calculate \(H^*_{DR}(SO(4))\) as a graded vector space. (You can assume that \(SO(4)\) has this fiber bundle structure, though you’re welcome to think about why.)

5. Suppose that \(E\) is a smooth orientable manifold that is also a fiber bundle with base \(S^2 \times S^2\) and fiber \(S^1\): that is, we have a fiber bundle \(S^1 \to E \to S^2 \times S^2\). (You can assume that \(E\) is compact; it’s a standard exercise to show that a fiber bundle with compact base and compact fiber is compact.) Find all possible choices for the graded vector space \(H^*_{DR}(E)\). Give an example to show that each possibility can be achieved.