Math 612 HW 11 Solutions

1. \( H(K) \cong \mathbb{Z} y_0 \) \[ H(K/4K) \cong \mathbb{Z}/2\langle y_0 \rangle \]
   \( H(3K) \cong \mathbb{Z}/2 \langle y_1 \rangle \) \[ H(3K/3K) \cong \mathbb{Z}/2 \langle y_1 \rangle \]

   The map \( \phi : H(3K) \to H(K) \) sends \( y_1 \) to \( y_1 = -2y_0 \), so it's the map \( \mathbb{Z}/2 \to \mathbb{Z}/4 \).

   Thus \( \text{Gr } H(K) = (H(K)/H(3K)) \oplus (H(3K)/H(3K)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).

   Now \( E_1 = H(\text{Gr } K) = H(K/4K) \oplus H(3K/3K) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).

   Since \( E_1 \cong \text{Gr } H(K) \) and the spectral sequence converges to \( \text{Gr } H(K) \),
   it must be the case that \( d_1 = d_2 = \ldots = 0 \) since otherwise \( E_\infty \) would have fewer elements than \( E_1 \).

   Thus \[ E_1 = E_2 = \ldots = E_\infty \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \] and \[ E_\infty \neq H(K) \].

2. \[ x_0 \longrightarrow y_0 \]
   \[ x_1 \longrightarrow y_1 \]
   \[ x_2 \longrightarrow y_2 \]
   \[ y_0 \longrightarrow \mathbb{Z} \]
   \[ y_1 \longrightarrow \mathbb{Z} \]

   \( H(K) = 0 \) : all cycles are boundaries (linear combination of \( y_0, y_1, y_2, \ldots \)).

   Suppose \( E_1 \ldots \) converged to \( H(K) \). Then for some \( r \), \( E_r = 0 \),
   so from the exact couple \[ \cdots \longrightarrow A_r \longrightarrow A_r \longrightarrow \cdots \]

   be an isomorphism. Now \[ A_r = \cdots \cong H(K) \cong \mathbb{Z}/2 \langle y_0, y_1, \ldots \rangle \]
   \( i'' H(3K) = 0 \) since it's in \( H(K) = 0 \). But

   \( H(3K) \cong \mathbb{Z}/2 \langle y_0, y_1, y_2, \ldots \rangle \)
   \( H(3K) \cong \mathbb{Z}/2 \langle y_0, y_1, y_2, \ldots \rangle \)
2. \[ i^*: H(c_k) \cong \mathbb{R}<y_1, y_2, y_3, \ldots \rightarrow \mathbb{C}H(c_k) \]

and \[ i^*: H(c_k) \rightarrow i^*: H(c_k) = 0 \text{ isn't injective. Thus } i: \mathbb{R} \rightarrow \mathbb{C} \]

isn't injective, so \( E_i, \ldots \) can't converge to \( H(c_k) \).

Note: For the record, \( E_i \cong \mathbb{R}<x_i, y_{i-1}, x_{i-1}, y_{i-2}, \ldots > \) and \( d_r: E_r \rightarrow E_r \) is \( d_r(x_{i-1}) = y_{i-1} \), \( d_r = 0 \) for other generators.

3.

\[ x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \cdots \rightarrow x_{n-1} \rightarrow x_n \rightarrow \cdots \rightarrow x_{m-1} \rightarrow x_m \rightarrow \cdots \rightarrow x_n \rightarrow \cdots \]

\[ H(k) = 0 \]
\[ H(c_k) \cong \mathbb{R}<y_n \text{ for } 1 \leq m \leq n \]
\[ H(c_k) = 0 \text{ for } m > n \]

\[ H(C^m_k) = 0 \text{ for } m > n \]

\[ H(C^m_k \big/ f^{n+1}) = \mathbb{R}<y_n > \]

**The first exact couple is:**

\[ \begin{array}{cccccc}
\text{A}_i: & \cdots & \langle y_n \rangle & \langle y_{n-1} \rangle & \cdots & \langle y_1 \rangle & \langle y_0 \rangle & \rightarrow 0 \\
\text{E}_i: & \langle x_0 \rangle & 0 & 0 & \cdots & 0 & \langle y_n \rangle \\
\end{array} \]

Here \( j(y_n) = y_n \) and \( k(x_0) = y_1 = y_0 \in H(c_k) \) since \( Dx_0 = y_1 \), so \( d_0 = 0 \).

**The second exact couple is:**

\[ \begin{array}{cccccc}
\text{A}_i: & \cdots & \langle y_n \rangle & \langle y_{n-1} \rangle & \cdots & \langle y_1 \rangle & \langle y_0 \rangle & \rightarrow 0 \\
\text{E}_i: & \langle x_0 \rangle & 0 & 0 & \cdots & 0 & \langle y_n \rangle \\
\end{array} \]
3. When $k_1(x_0) = y_n$ and $j_n(y_n) = y_n$, and $d_2 = 0$. Each exact couple $C$.
Shrink the sequence of maps until we get

\[
\begin{align*}
A_n: & \quad \cdots \rightarrow 0 \xleftarrow{\alpha} \langle y_n \rangle \xrightarrow{\beta} 0 \\
E_n: & \quad \langle x_0 \rangle \quad 0 \quad 0 \rightarrow 0 \xleftarrow{\gamma} \langle y_n \rangle
\end{align*}
\]

and now $k_n(x_0) = y_n$, $j_n(y_n) = y_n \Rightarrow d_n(x_0) = y_n$, $d_n(y_n) = 0$
\[
\Rightarrow E_{n+1} = H(E_n, d_n) = 0.
\]

So
\[
\begin{align*}
E_r &= \mathbb{R} \langle x_i, y_n \rangle \cong \mathbb{R}^2 \\ r &\leq n \Rightarrow E_r = E_0 = H(k) = 0 \\
E_r &= E_0 = H(k) = 0 \quad \text{for} \ r > n.
\end{align*}
\]

4. (a) $H(k) = 0$: cycles and boundaries are both linear combinations of $x, w + x, y$.

(b) \[\begin{align*}
H(k) &= 0 \\
H(x) &= \mathbb{R} \\
H(x') &= \mathbb{R} \\
H(y) &= \mathbb{R} \\
H(z) &= \mathbb{R}
\end{align*}\]

\[\begin{align*}
A_i: & \quad \cdots \rightarrow 0 \xleftarrow{i} \langle u \rangle \xrightarrow{j} 0 \\
E_i: & \quad \langle u \rangle \quad \langle z \rangle \rightarrow \langle x, y \rangle
\end{align*}\]

\[\begin{align*}
k_1(u) &= w + z = z \in H(x') \\
j_1(z) &= z
\end{align*}\]

\[\begin{align*}
d_1(u) &= 0, \quad d_1(z) = 0
\end{align*}\]

\[\Rightarrow \begin{align*}
E_2 &= \mathbb{R} \langle \alpha_1, \beta_1 \rangle = 0 = H(k)
\end{align*}\]
4. (c) \[ \begin{align*}
H(\emptyset) &= 0 \\
H(\{x\}) &= \langle u, v \rangle \\
H(\{x, y\}) &= \langle w, x, z \rangle \\
H(\{x, y, z\}) &= \langle u, v \rangle
\end{align*} \]

\[ E_1 \cong \langle u, v, w, x, y, z \rangle \]

\[ \begin{align*}
A_1 &= 0 \\
\xi_1 \quad \xi_2 \quad \xi_3 \\
E_1 &= \langle u, v \rangle \\
\langle w, x \rangle \\
\langle y, z \rangle
\end{align*} \]

- \( i_1(y) = 0, \quad i_1(z) = 2 \)
- In \( H(\emptyset) \), \( j_1(w) = w, j_1(z) = 0 \); in \( H(\{x\}) \), \( j_1(y) = y, j_1(z) = 2 \)
- In \( H(\{x, y\}) \), \( k_1(w) = w + x, k_1(v) = w + x, k_1(z) = y, k_1(x) = -y \)
- \( d_1(w) = w + x, d_1(v) = w + x, d_1(w) = y, d_1(x) = -y \)

\[ E_2 = H(E_1, a_1) \cong \langle u - v, z \rangle \]

\[ \begin{align*}
A_2 &= 0 \\
\xi_2 \\
E_2 &= \langle u - v \rangle \\
\langle z \rangle
\end{align*} \]

- \( j_2(z) = z \); \( k_2(u - v) = \tau \); \( k_2(z) = 2 \)
- \( d_2(u - v) = 2, \quad d_2(z) = 0 \)

\[ E_3 = \ldots = E_\infty = 0 = H(\emptyset) \]