**Vector bundles**

A fiber bundle with fiber $\mathbb{R}^n$ is a vector bundle if transition functions are in $\text{GL}(n, \mathbb{R}) \subseteq \text{Diff}(\mathbb{R}^n)$:

\[
\begin{align*}
\pi : E & \to \mathbb{M} \\
\left( U_x, \phi_x \right) & \text{ open cover of } \mathbb{M} \\
\phi_x : E|_{U_x} & \cong \pi^{-1}(U_x) \cong U_x \times \mathbb{R}^n
\end{align*}
\]

\[
x \in U_x \cap U_{x'} \implies g_{x,x'}(x) = \phi_{x'}^{-1} \circ \phi_x
\]

\[
\phi_{x,x'} \in \text{GL}(n, \mathbb{R}) \subseteq \text{Diff}(\mathbb{R}^n)
\]

\[
\left\{ \left( U_x, \phi_x \right) \right\} \text{ is a trivialization of } E.
\]

**Def.** $E \xrightarrow{\psi} E'$ are isomorphic as vector bundles if $\exists \psi : E \to E'$ with $\psi|_{E_x} \xrightarrow{\psi|_{E_x}} E'$ and $\psi|_{E_x} : E_x \to E_x'$ is in $\text{GL}(n, \mathbb{R})$.

\[\psi|_{E_x} \quad \text{(or bundle map if } \psi|_{E_x} \text{ is linear).}\]

One way to describe a trivialization: frames.
Def A section of $E$ over $U \subset M$ is $s: U \rightarrow E|_U$, s.t. $\pi s = id$.

A frame for $E$ over $U$ is $(s_1, \ldots, s_n)$ s.t. $s_i(x)$, $\ldots$, $s_n(x)$ form a basis for $E|_x$ at $x \in U$.

Note: Trivialization $\varphi: E|_U \rightarrow U \times \mathbb{R}^n$ $\leftrightarrow$ frame for $E$ over $U$:

$\Rightarrow$: $s_i(x) = \varphi_\alpha^{-1}(x)(\overline{e_i})$.

$\Leftarrow$: $\varphi_\alpha(f_1(x) s_1(x) + \cdots + f_n(x) s_n(x)) = (f_1(x), \ldots, f_n(x))$.

Given two frames near a point, one is a linear comb of the other $\leftrightarrow$ transition furn.

Def $H \subset GL(n, \mathbb{R})$ subjgp. The structure group of $E$ can be reduced to $H$ if $\exists$ trivialization st. $\varphi_\alpha(x) \in H$.

Prop The structure group of any vector bundle can be reduced to $O(n)$.

If Place a Riemannian metric on the fibers of $E$:
Smoothly varying metric (pos def bilinear form) on $E_x$.

Any frame on $E|_U$ can be made orthonormal (and still smooth) by Gram-Schmidt. Then on overlap,

$(s_1(x), \ldots, s_n(x))$, $(s'_1(x), \ldots, s'_n(x))$ are related by an orthogonal matrix $\mathbf{A}$ with respect to the trivialization:

$\varphi_\alpha(x) \in O(n)$. $\square$
Def: $E$ is orientable as a vector bundle if the structure group can be reduced to $GL^+(n,\mathbb{R})$ or equivalently $SO(n)$.

(Note: Gram-Schmidt preserve the sign of the determinant)

Ex: $E = TM$ tangent bundle. Transition functions for TM are given by the Jacobian of transition functions for $M$. $\pi_1 M$ is orientable as TM is orientable as a vector bundle.

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CV Cohomology

$\mathbb{R}^n \to E$ vector bundle. On $E$, we can consider

$$H^*(E) \cong H^*(M)$$  Poincaré (homotopy invariance) or

$$H_c^*(E) \cong H_c^*(M)$$  if $E, M$ are orientable.

$$H_c^k(E) \cong (H^{m+n-k}(M))^* \cong H_c^{k-n}(M).$$

Or we can consider a third cohomology,

$$\Omega_c^*(E) = \{ \omega \in \Omega^*(E) \mid \forall K \subset M \text{ cpt}, \supp(\omega) \cap \pi^{-1}(K) \text{ is compact} \}.$$  forms with compact vertical support.

Note $\Omega_c^*(E) \subset \Omega_c^*(E) \subset \Omega^*(E)$.

And $d: \Omega_c^*(E) \to \Omega_{c+1}^*(E)$. Def $H_{c+1}^*(E) = \ker d / \text{im} d$. 
Then Isomorphism Theorem \[ E \mapsto \mathbb{E} \] rank n vector bundle, \( U \) a good cover of \( M \), \[ H^*_C (E) \cong H^{* - n} (U, P_C^*). \]

If \( E \) is orientable as a vector bundle, then \[ H^*_C (E) \cong H^{* - n} (M). \]

Ex. If \( M = \mathbb{R} \) then \( H^*_C (E) \cong H^*_C (\mathbb{R}^n) \cong H^{* - n} (\mathbb{R}). \)

How does one prove this?

\( U \) = good cover of \( M \), \( \pi^{-1} (U) \) = cover of \( E \). Define \[ \Omega^*_C = \text{presheaf over } M \]
\[ \Omega^*_C (M) = \Omega^*_C (\pi^{-1} (U)). \]

Then \[ 0 \to \Omega^*_C (E) \to C^0 (U, \Omega^*_C) \to C^1 (U, \Omega^*_C) \to \cdots \]
is exact (same proof as MV).

\[ \begin{array}{c}
C^* (U, \Omega^*_C) \\
\downarrow \quad E_1 \\
C^* (U, H^*_C) \\
\end{array} \quad \begin{array}{c}
E_1' \\
\downarrow \\
\end{array} \quad \begin{array}{cc}
\Omega^*_C (E) & 0 \\
\Omega^*_C (E) & 0 \\
\Omega^*_C (E) & 0 \\
\end{array} \quad \begin{array}{c}
H^*_C (E) & 0 \\
H^*_C (E) & 0 \\
\end{array} \quad \begin{array}{c}
H^*_C (E) & 0 \\
H^*_C (E) & 0 \\
\end{array} \quad \begin{array}{c}
0 \\
0 \\
0 \\
\end{array} \]

where \( H^*_C = \text{presheaf over } M \), \( H^*_C (U) = H^*_C (E|_U). \)

Poincaré Lemma: \( H^*_C (E|_U) \cong H^*_C (U \times \mathbb{R}^n) \) integrate along fibers
\[ \cong H^{* - n} (U) \quad \text{(since } C^*_C \text{ is} \mathbb{R}) \]
\[ \cong \bigoplus_{k=0}^{\infty} \mathbb{R}^{k-n} \quad \text{otherwise}. \]
\[ \varepsilon_1 = \left( C^*(U, \mathbb{H}^n) \to C^*(U, \mathbb{H}^n) \to \cdots \right) \xrightarrow{\varepsilon_2} \tilde{H}^*(U, \mathbb{H}^n) \to \tilde{H}^*(U, \mathbb{H}^n) \to \cdots \]

So:

\[ H_c^*(E) \cong \tilde{H}^{*-n}(U, \mathbb{H}^n). \] (does not need orientation)

In the oriented case:

\[ \{ \text{Choose \ } \varphi \in \Omega^k(E), \text{ \ where } \varphi = 0, \text{ \ such that } \varphi \text{ \ generates } \tilde{H}^k(\mathbb{R}^n) = \mathbb{R}. \} \]

\[ \begin{pmatrix} E_x \to E \cr H^*_c(E_x) \to H^*_c(E) \end{pmatrix} \]

Then:

\[ H^*_c(U) \cong \mathbb{R} \text{ is generated by } [\varphi|_{\pi^{-1}(U)}] \]

and for \( V \subset U \), the restriction map \( \tilde{H}^*_c(U) \to \tilde{H}^*_c(V) \) sends \( \varphi|_{\pi^{-1}(U)} \mapsto \varphi|_{\pi^{-1}(V)} \).

So:

\[ \tilde{H}^*_c(U) \to \mathbb{R} \]

and thus:

\[ \tilde{H}^*_c(E) \cong \tilde{H}^{*-n}(U, \mathbb{H}^n) \]

\[ \cong \tilde{H}^{*-n}(U, \mathbb{R}) \]

\[ = \tilde{H}^{*-n}(M). \]

\[ [\varphi] \in H^{*-n}_c(E) \text{ is called the Thom class of } E. \] (think: volume bump form on each \( E_x \))
Stranger form of Thom isomorphism \[ E \xrightarrow{\Phi} M \text{ orientable rank } n. \]
\[ \exists \phi \in H_c^0(E) \text{ with } \left[ \phi \right] \left[ \phi \right] \mid_{\text{Ex}} \in H_0^c(\mathbb{R}^n) \text{ having } f=1 \text{ s.t.} \]
\[ H^*(M) \xrightarrow{\phi} H_*^{c,n}(E) \text{ is an isom.} \]

Algebraic form of Thom (see Milnor-Stasheff)

\[ \mathbb{R}^n \to E \xrightarrow{\phi} M \text{ vector bundle with a metric } \langle , \rangle \text{ on } \mathbb{R}^n \times X. \]
Define the disk bundle \[ D(E) = \{ v \in E \mid \|v\| \leq 1 \} \]
and sphere bundle \[ S(E) = \{ v \in E \mid \|v\| = 1 \} \]

\[ H_c^*(E) \cong H^*(D(E), S(E); \mathbb{R}) \text{ relative cohomology.} \]
\[ \omega \longrightarrow (\sigma \mapsto \int_{\sigma} \omega) \]
\[ \text{supp} \omega \subset D(E) \]

Thus if \( E \) is orientable, then \( \exists \Phi \in H^m(D(E), S(E); \mathbb{R}) \) s.t.

the map

\[ H^*(M, \mathbb{Z}) \longrightarrow H^{*+m}(D(E), S(E); \mathbb{Z}) \]
\[ \lambda \mapsto \pi^* \lambda \cdot \Phi \]

is an isomorphism.

(If non-orientable, still true, with \( \mathbb{Z}/2 \) coefficients.)
Poincaré Duality

$M^n = \text{oriented manifold, not necessarily connected}$

$N^{n-k} = \text{(topologically) closed, oriented submanifold, codim } k, \ N \subset M$.

$N$ induces a map

$\varphi_N : \Omega^{n-k}(M) \rightarrow \mathbb{R}$

$\omega \mapsto \int_N \omega$

$\varphi_N \in (H^{n-k}(M))^{*}$.

By Poincaré duality, $(H^{n-k}(M))^{*} \cong H^k(M)$

So there's a class

$PD(N) \in H^k(M)$

(i.e. $\int_N \omega = \int_M \omega \cap PD(N)$)

called the Poincaré dual of $N$.

We can visualize this using Thom.

$(M, g)$ Riem manifold, $N^{n-k} \subset M$. The normal bundle to $N$ is the rank $k$ vector bundle $VN \subset TM$, $\mathbb{R}^k \rightarrow \mathbb{R}^n$, defined by

$VN_x = \{ v \in T_x M \mid v \perp T_x N \} = (TN)^{\perp}$.

For any vector bundle $E \rightarrow M$, the zero section is $s : M \rightarrow E$, $s(x) = 0 \in E_x$.

Tubular Neighborhood Theorem: $N$ has a neighborhood $\text{neb}(N) \subset M$ s.t.

$VN \cong \text{neb}(N)$

$0 \times \mathbb{R}^k \rightarrow \mathbb{R}^n$
Now \( \text{CV form on VN} \) is a form on \( M \). Use Tub. Nbd. Thm. + extend by \( 0 \) to \( M - \text{nbd} (N) \).

Thus we get a map \( \Omega^k_{\text{CV}}(VN) \xrightarrow{i_*} \Omega^k(M) \xrightarrow{H^*_{\text{CV}}(VN)} H^k(M) \).

Now VN has Thom class \( [\Theta] \in H^k_{\text{CV}}(VN) \).

Then \( (p.67) \quad i_*[\Theta] = PD(N) \in H^k(M) \).

We can interpret \( \cup \) (cup product) on \( H^k(M) \) in terms of Poncaré duals.

\[ N_{n-k_1}, N_{n-k_2} \subset M \implies PD(N_1) \in H^k(M), \ PD(N_2) \in H^k(M). \]

What's \( PD(N_1) \cup PD(N_2) \in H^{k_1+k_2}(M) \)?

Suppose \( N_1, N_2 \) intersect transversely: \( \forall x \in N_1 \cap N_2, \)

\[ \text{codim} (T_x N_1 \cap T_x N_2) = \text{codim} T_x N_1 + \text{codim} T_x N_2. \]

Then \( N_1 \cap N_2 \) is a smooth \((n-k_1-k_2)\)-dim submanifold of \( M \).

\[ \text{(Property) (p.65)} \Rightarrow \]

\[ v(N_1 \cap N_2) = v(N_1) \oplus v(N_2) \]

\[ \Phi(v(N_1 \cap N_2)) = \Phi(v(N_1)) \oplus v(N_2) = \Phi(v(N_1)) \cup \Phi(v(N_2)) \]

\[ \Rightarrow \quad PD(N_1 \cap N_2) = PD(N_1) \cup PD(N_2) \quad \text{(p.69).} \]

Cup product on cohomology \( \xrightarrow{PD} \) intersection of submanifolds.
Special case: if $k_1 + k_2 = n$:

$$PD(N) \cap [N_1] = \sum_{M} PD(N) = \sum_{M} PD(N_1) \cup PD(N_2) = \sum_{M} PD(N_1 \cap N_2)$$

$$= \sum_{N_1 \cap N_2} 1 = \# (N_1 \cap N_2).$$

**Euler class**

$$\mathbb{R}^k \to E$$

oriented v.s., $s : M \to E$ zero section, $s_0^* : H^*_c(E) \to H^*(M).$

**Def** The Euler class of $E$ is

$$e(E) = s_0^* [0] \in H^k(M).$$

This is an example of a characteristic class: inst of vector bundle up to.

(oriented $\to e(E)$; complex $\to c_k(E)$; real $\to Pontryagin$.

**Facts:**

- If $E$ trivial $= M \times \mathbb{R}^n$ then $e(E) = 0$

- $e(E)$ is PD to the zero locus of any section $s : M \to E$ (p. 134)

- If $s(E)$ sphere bundle + E then $e(E)$ is what shows up in the Gysin sequence

- If $E = TM$ and $M^n$ is compact then $e(TM) e^{+h^*(M)}$:

$$\sum_{M} e(TM) = \chi(M) \quad \text{Euler characteristic}!$$
Example: \( M = S^2 \), \( e(TS^2) = 2 \in H^2(S^2; \mathbb{Z}) \)

Sphere bundle \( S^1 \rightarrow STS^2 \)

Gysin sequence

\[
\begin{array}{ccc}
& & 0 \\
& & \downarrow \\
& 0 & \rightarrow & H^3(STS^2) & \rightarrow & H^2(S^2) \\
& \rightarrow & H^2(S^2) & \rightarrow & H^1(S^2) & \rightarrow & H^0(S^2) \\
& \times 2 & H^1(S^2) & \rightarrow & H^0(STS^2) & \rightarrow & 0 \\
\end{array}
\]

\( \Rightarrow H^0(STS^2) = \{ \mathbb{R} \} \), \( H^1(STS^2) = 0 \), \( H^2(STS^2) = \{ \mathbb{R} \} \), \( H^3(STS^2) = \{ \mathbb{R} \}. \)

In fact, point in \( STS^2 \) corresponds to a unit vector \( v, u \) which also represents an element of \( SO(3) \), hence \( STS^2 \equiv SO(3) \cong \mathbb{R}P^3 \).
Let $V$ = vector field on $M$: $V \in \mathcal{P}(TM)$. Each zero of $V$ has an index: let $x \in M$, $V(x) = 0$, assume $x$ is isolated zero (true for generic). $TM$ is locally trivial:

If we choose a metric on $M$, then

$$TM \bigg| D_\varepsilon(x) \cong D_\varepsilon(x) \times \mathbb{R}^n.$$ 

Restrict $V$ to $S^{n-1}_\varepsilon(x)$:

$$S^{n-1} = S^{n-1}_\varepsilon(x) \mathrel{\overset{V}{\longrightarrow}} \mathbb{R}^n \cong 0 \rightarrow S^{n-1}$$

and define

$$\text{index}(x) := \text{degree} \left(S^{n-1} \rightarrow S^{n-1}\right).$$

**Poincaré-Hopf Theorem** (p. 129)

$$\chi(M) = \sum_{x \in \text{zero of } V} \text{index}(x).$$