2. $\Delta^3$ deformation retracts to $[v_0, v_1, v_3] \cup [v_0, v_2, v_3] =: Y$.

Since the homotopy fixes $Y$ pointwise, this descends to a deformation retract $\Delta^3 \to Y/\sim$.

But $Y/\sim = \text{Klein bottle}$.

Similarly, for $T^2$, we have $\Delta^3 / [v_0, v_1] \approx [v_0, v_3], [v_0, v_2] \approx [v_1, v_3]$.

For $S^2$, we have $\Delta^3 / [v_0, v_1] \approx [v_0, v_3], [v_1, v_2] \approx [v_2, v_3]$.

For $\mathbb{R}P^2$, we have $\Delta^2 / [v_0, v_1] \approx [v_0, v_2], [v_1, v_2] \approx [v_1, v_3]$.

4. \[
\begin{align*}
\sigma &\sim (v_0 = v_1 = v_2 = 0) \\
C_2 &\approx \{ \sigma \} \quad \sigma_0 = a + b + c \\
C_1 &\approx \{ a, b, c \} \quad \sigma_0 = a + b + c = 0 \\
C_0 &\approx \{ \sigma \} \quad \sigma_0 = 0 \\
H^0 &\approx \mathbb{Z} \quad H^1 \approx \mathbb{Z}^2 \quad H^2 \approx 0
\end{align*}
\]
C₂ : <a, L> \quad \forall u = a + 5 - c, \quad \forall l = c + a - 1
C₁ : <a, d, c> \quad \forall a = \forall b = \forall c = 0
C₀ : <v> \quad \forall v = 0

H₂ = 0, \quad H₁ = \mathbb{Z} \oplus \mathbb{Z}/2, \quad H₀ = \mathbb{Z}

S³ = \triangle³ / ([v₀, v₁, v₂] \sim [v₀, v₁, v₃] \text{ and } [v₀, v₂, v₃] \sim [v₁, v₂, v₃]). (**) 

Proving (**) is the heart of this problem. Given that:

\begin{align*}
C₃ : [0123] & \quad \forall [0123] = 0 \\
C₂ : [012] = [013], [023] = [123] & \quad \forall [012] = [01], \forall [023] = [23] \\
C₁ : [01], [02] = [03] = [12], [13] = [23] & \quad \forall [01] = 0, \forall [02] = [23], \forall [03] = 0 \\
C₀ : [03] = [13], [23] = [3] & \quad \forall [03] = 0
\end{align*}

⇒ H₃ \cong \mathbb{Z}, \quad H₂ = H₁ = 0, \quad H₀ = \mathbb{Z}

To see (**), first glue [v₀, v₁, v₂] to [v₀, v₁, v₃]:

\[ \triangle³ / ([v₀, v₁, v₂] \sim [v₀, v₁, v₃]) = \]

Now gluing [v₀, v₂, v₃] \sim [v₁, v₂, v₃]

is the same as gluing the top and bottom cases:

each point on the surface is identified with its mirror reflection in the plane of the center disk.
That is: \( \Delta^3/\sim = D^3/\sim \)

where \( x \sim L(x) \) if \( x \in D^3 = S^2 \)

and \( L = \) reflection in equatorial plane.

Why is this \( S^3 \)? View \( S^3 \) as the unit sphere in \( R^4 \):

northern and southern hemispheres are each \( D^3 \), and they overlap along their boundary.

(N hemisphere: \( \{(x,y,z,0) \mid x^2 + y^2 + z^2 + w^2 = 1, w \geq 0\} \to D^3 = \{x^2 + y^2 + z^2 \leq 1\} \))

by \( (x,y,z,0) \to (x,y,z) \)

and similarly S hemisphere \( \to D^3 \) by the same map.)

\[ S^3 = D^3 \cup D^3 / \sim = \frac{D^3 \cup D^3}{\sim} \]

\( (x \in D^3, \sim x \in D^3) \)

this comes from keeping \( D^3 \) fixed and reflecting \( D^3 \)

\[ = S^3 = \frac{D^3}{\sim} \]

\( x \sim L(x) \) when \( x \in \sim = D^3 \)

This completes the proof that the desired quotient space of \( \Delta^3 \)

is in fact \( S^3 \).
The tetrahedra are

\[ T_i = \{V_i, v_{i1}, w_i, w_{i2}\} \quad i \in \mathbb{N} \]

with

\[ [V_i, v_{i1}, w_i] \sim [v_{i1}, v_{i2}, w_{i2}] \]

\[ C_3: \quad T_1, \ldots, T_n \]

\[ C_2: \quad [V_i, v_{i1}, w_i] = [v_{i1}, v_{i2}, w_{i2}] \]

\[ [V_i, w_i, w_{i2}] \quad (i \in \mathbb{N}) \]

\[ C_1: \quad [V_i, v_{i2}] = [V_i, v_{i1}] \quad [V_i, w_i] = [v_{i1}, w_{i2}] \quad \text{for} \quad i \in \mathbb{N} \]

\[ C_0: \quad [w_i, w_{i2}] \]

\[ \forall T_i = [V_i, v_{i1}, w_i, w_{i2}] - [V_i, w_i, w_{i2}] + [V_i, v_{i2}, w_{i2}] - [V_i, v_{i1}, w_{i1}] \]

\[ \forall [V_i, v_{i1}, w_i] = [v_{i1}, w_{i2}] + [V_i, w_i] + [v_{i2}, w_{i2}] \]

\[ \forall [v_{i2}, w_i] = 0 \]

\[ \forall [v_i, w_i] = [w_i] - [v_i] \]

\[ H_3 \cong \mathbb{Z} \quad \text{(generated by } T_1 + \ldots + T_n) \]

\[ H_2 \cong 0 \]

\[ H_1 \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{(generated by } [v_i, v_{i2}] \text{)} \]

\[ H_0 \cong \mathbb{Z} \]