This has a cell complex structure as follows:

- an 0-cell, three 1-cells, one 2-cell.

\[ \pi_1(Y) \cong \langle a, b, c \mid aba^{-1}b^{-1}c^{-1} \rangle. \]

To get \( X \), we attach one new 2-cell along
\[ \pi_1(X) \cong \langle a, b, c \mid aba^{-1}b^{-1}c^{-1}, b \rangle \cong \langle 0, c \rangle \cong \mathbb{Z} \times \mathbb{Z}. \]

To show \( \pi_1(Y) \cong \pi_1(P^3 - 2) \), we claim \( S^3 - 2 \) does retract to \( Y \). Then
\[ \pi_1(S^3 - 2) \cong \pi_1(P^3 - 2) \cong \pi_1(Y). \]

Easy to check.
To picture $S^3 - Z$ deformation retraction to $Y$, cut $S^3 - Z$ into two pieces by dividing along $Y$ (disk on bottom of Klein bottle).

Inside:

Outside:

So both inside and outside $\partial Z$ retract to the boundary:

$\Rightarrow S^3 - Z$ deformation retracts to $Y$. 
14. Attaching 2-cells along $a^{-1}c^{-1}, ac^{-1}d^{-1}, ad^{-1}bc^{-1}$

$$xy^2z^2 xy^2x = z^{-1}y$$

Given $\pi_1 \equiv \langle x, y, z | 2x = y, xy^2z = x \rangle$.

This is the quaternion group: identity $1$, $i$, $j$, $k$, $-1$, $-i$, $-j$, $-k$.

(you should check that this is an isom.)

$$i \times j = k \times x = j \times k \times i = i \times j \times k$$

22. (a) Fix an orientation on $K$, and fix a base point $x_0$ on $T$.

Choose a loop $\alpha_i$ in $\text{Tur}_i$ by joining $x_0$

to the two sides of $R_i$ by paths in $T$,

joining the endpoints of these paths by

an arc along $R_i$, and orienting by

right hand rule (so the loop follows RH rule once we push $K$

upwards). (Note this is well-defined up to homotopy.)

Then $\text{Tur}_i \cong S^1$ and $\pi_1(\text{Tur}_i, x_0) = \mathbb{Z}$ is gen'd by $[R_i]$. 

Next let $n = \# \text{ crossings of the knot diagram} = \# \text{ of arcs}$.

Then $\text{Tur}_1 \cup \cdots \cup \text{Tur}_n \cong S^1 \cup \cdots \cup S^1$ and

Van Kampen $\Rightarrow \pi_1(\text{Tur}_1 \cup \cdots \cup \text{Tur}_n, x_0) \cong \mathbb{Z}^n$ is gen'd by $[R_1], \ldots, [R_n]$.

Call these $x_1, \ldots, x_n$. 

$1 - \text{skeleton} = \{s, c, a \}$.

$\pi_1(1 \text{-skeleton}, *) \cong \langle x, y, z \rangle$

where $x = a^{-1}, y = ac^{-1}, z = ad^{-1}$. 

22 (a) Now we just have to attach the 2-cells $S^2$ and use Van Kampen (= Prop 1.26). There are two possible pictures at crossing $i$:

(a) $\alpha_i, \alpha_j, \alpha_i', \alpha_j'$

(b) $\gamma_i, \gamma_j, \gamma_i', \gamma_j'$

(or rotate 180°).

From above, the boundary of $S^2$ looks like:

(a) $\gamma_i, \gamma_j, \gamma_i', \gamma_j'$

(b) $\gamma_i, \gamma_j, \gamma_i', \gamma_j'$

homotopic to $\gamma_i, \gamma_j, \gamma_i', \gamma_j'$

So $S^2$ produces the relation $x_i x_j x_i' = x_k$ or $x_i x_j x_i' = x_j$.

Finally: $\pi_1(X, x_0) = \langle x_1, \ldots, x_m \mid \text{relations of the form } x_i x_j x_i' = x_k \rangle$.

(b) The abelianization is $\langle x_1, \ldots, x_m \mid x_j = x_k \text{ for each crossing } \rangle$.

Since the knot is connected, the relations mean all $x_i$ are equal

$\Rightarrow$ abelianization $\cong \langle x \rangle = \mathbb{Z}$.
Extra problem

Wirtinger presentation:

\[ \pi_1(\mathbb{R}^2\setminus \{0\}) \cong \langle x_1, x_2, x_3 \mid x_1 x_2 x_1^{-1} = x_2, \ x_2 x_1 x_2^{-1} = x_3, \ x_3 x_2 x_3^{-1} = y_1 \rangle \]

\[ \cong \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2 \rangle. \]

Note in this group \((x_1 x_2)^3 = x_1 x_2 x_1 x_2 x_1 = (x_1 x_2 x_1 )^2 . \)

Here's the isomorphism between the and \(\langle x_1, y \mid x^2 = y^3 \rangle\)

and its inverse:

\[
\begin{align*}
& \langle x_1, y \mid x^2 = y^3 \rangle & \longrightarrow & \langle x_1, x_2 \mid x_2 x_1 x_2^{-1} = x_2 x_1 x_2 \rangle \\
& x_1 \rightarrow x_1 x_2 \quad y \rightarrow y x_1 x_2 \quad x_1 \rightarrow x_1 y \quad x_2 \rightarrow y^{-1} x_1 \end{align*}
\]