Math 411 — part 2

Note Title 11/13/2017

Homotopy — important technique for studying topological spaces.
(cf. path independence of line integrals: $\vec{\nabla}\times\vec{F}=\vec{0} \Rightarrow \int_{\gamma_1}\vec{F}\cdot d\vec{s} = \int_{\gamma_2}\vec{F}\cdot d\vec{s}$)

Def $f,g: X\to Y$ continuous. Then f is <u>homotopic</u> to g, write $f\simeq g$, if there is continuous $F: X\times I\to Y$ such that

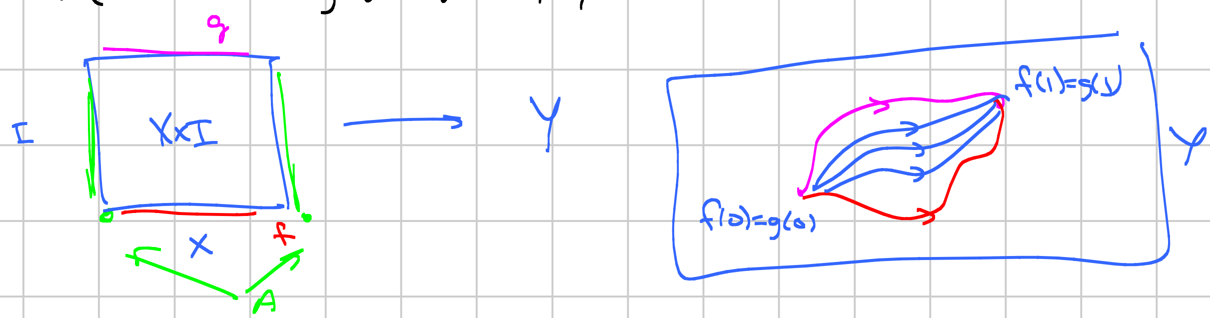
$$F(x,0)=f(x) \quad\text{and}\quad F(x,1)=g(x) \quad \forall\, x\in X.$$

F = "homotopy from f to g"

note: for $t\in[0,1]$ get $f_t: X\to Y$, $f_t(x)=F(x,t)$, with $f_0=f$ and $f_1=g$, continuously varying.

For paths, need to also fix endpoints.

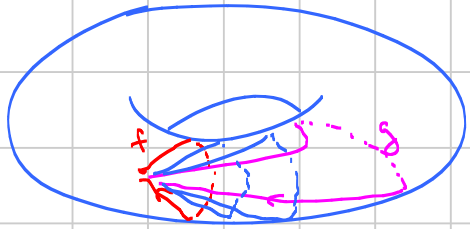
Def $f,g: X\to Y$ continuous, $A\subset X$, $f|_A=g|_A$. Then f is <u>homotopic to g rel A</u> if $\exists$ $F: X\times I\to Y$ such that $F(x,0)=f(x)$, $F(x,1)=g(x)$, <u>and</u> $F(a,t)=f(a)=g(a)$ $\forall$ $a\in A$, $t\in I$.



Def $f,g:[0,1]\to Y$ are <u>path homotopic</u> if they're homotopic rel $\{0,1\}$ (in particular, f,g have same endpoints): write $f\simeq_p g$.

Special case: if $f(0)=f(1)=g(0)=g(1)=p$ then f,g are loops.

In particular, suppose $g(t)=p$ $\forall t$ (constant path); if $f\simeq_p g$ then f is <u>null-homotopic</u>.

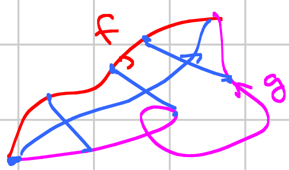


Ex $Y=\mathbb{R}^n$. Any maps $f,g: X\to\mathbb{R}^n$ are homotopic via straight-line homotopy $F(x,t)=(1-t)f(x)+tg(x)$

(also works for convex subsets of $\mathbb{R}^n$: if $p,q\in Y$ then the line segment between p,q is in Y).

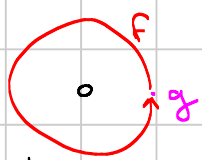
If $A\subset X$ and $f=g$ on A, then f,g are homotopic rel A.

ex: $f,g:[0,1]\to\mathbb{R}^n$: any paths in $\mathbb{R}^n$ with same endpoints are path htpic.



Straight-line htpy doesn't work for arbitrary subsets of $\mathbb{R}^n$:

eg. $Y=\mathbb{R}^2\setminus\{0\}$.



$f(s)=(\cos 2\pi s, \sin 2\pi s)$ $\quad g(s)=(1,0)$

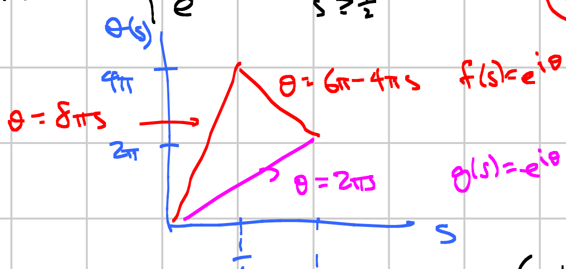
Straight line htpy has $F(\frac{1}{2},\frac{1}{2})=(0,0)\notin Y$.

More involved ex of path homotopy: loops in S^1.

View $S^1=\{z\in\mathbb{C}\mid |z|=1\}$. $\quad f,g:[0,1]\to S^1$

$$f(s)=\begin{cases} e^{8\pi i s} & s\le \frac{1}{2} \\ e^{-4\pi i s} & s\ge\frac{1}{2}\end{cases}$$

$$g(s)=e^{2\pi i s}$$



$$F(s,t)=\begin{cases}\exp\left(i\left((1-t)8\pi s+t\cdot 2\pi s\right)\right) & t\le\frac{1}{2}\\ \exp\left(i\left((1-t)(6\pi-4\pi s)+t\cdot 2\pi s\right)\right) & t\ge\frac{1}{2}\end{cases}$$

(note continuous by pasting lemma).

11/14

Lemma Homotopy ($\simeq$) is an equivalence relation, as is homotopy rel A if $A \subset X$.

Pf Reflexive ✓ Symmetric: given $F: X \times I \to Y$ htpy from f to g, $G(x,t) = F(x, 1-t)$ is a htpy from g to f.

Transitive: Suppose $f \simeq g \simeq h$, $F: X \times I \to Y$ htpy from f to g, $G: X \times I \to Y$ g to h.

$$H(x,t) = \begin{cases} F(x, 2t) & t \leq \frac{1}{2} \\ G(x, 2t-1) & t \geq \frac{1}{2} \end{cases}$$

$\Rightarrow f \simeq h$.

If rel A: then for $a \in A$, $F(a,t) = f(a) = g(a)$, $G(a,t) = g(a) = h(a) \Rightarrow H(a,t) = f(a) = h(a)$. ☐

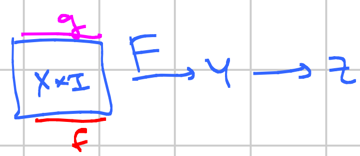
Lemma
$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[r, "h"] & Z \end{tikzcd}$$
$A \subset X$

if $f \simeq g$ rel A then $h \circ f \simeq h \circ g$ rel A

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & Z \end{tikzcd}$$
$A \subset Y$

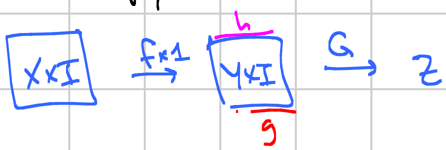
if $g \simeq h$ rel A then $g \circ f \simeq h \circ f$ rel $f^{-1}(A)$.

Pf $f \simeq g$ with homotopy $F: X \times I \to Y \Rightarrow h \circ f \simeq h \circ g$ with htpy $h \circ F: X \times I \to Z$.



and $(h \circ F)(a,t) = h(f(a)) = h(g(a))$.

$g \simeq h$ with homotopy $G: Y \times I \to Z \Rightarrow g \circ f \simeq h \circ f$ with htpy $H: X \times I \to Z$



$H(x,t) = G(f(x), t)$. ☐

ϕ

Fundamental Group

X = top. space, $p \in X$ "base point." A **loop** at p is a path whose endpts are p.

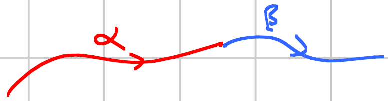
Lemma $\Rightarrow$ path homotopy = equiv. relation.

α = loop $\rightsquigarrow$ $[\alpha]$ = equivalence class of α under path homotopy = homotopy class of α.

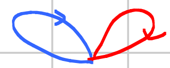
Let $\pi_1(X, p) = \{$equivalence classes of loops at $p\}$.

Multiplication of paths: $\alpha, \beta : [0,1] \to X$ with $\alpha(1) = \beta(0)$

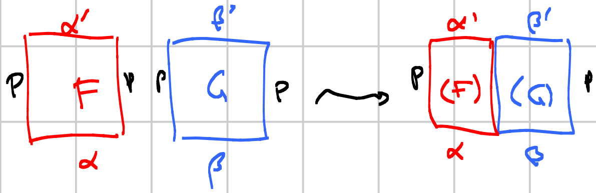
$\rightsquigarrow \alpha \cdot \beta : [0,1] \to X$ defined by $(\alpha\cdot\beta)(t) = \begin{cases} \alpha(2t) & t \le \frac{1}{2} \\ \beta(2t-1) & t \ge \frac{1}{2} \end{cases}$.



In particular if α, β are loops at p, So is $\alpha\cdot\beta$.



Claim 1 This descends to a map on homotopy classes $[\alpha]\cdot[\beta] = [\alpha\cdot\beta]$: need to check well-defined. Say α, α' homotopic, β, β' homotopic.

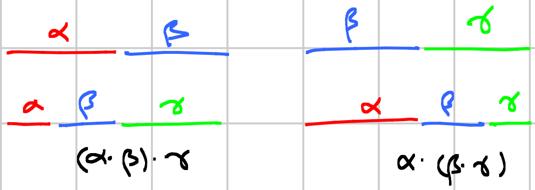


$$H(s,t) = \begin{cases} F(2s,t), & s \le \frac{1}{2} \\ G(2s-1,t), & s \ge \frac{1}{2} \end{cases}$$

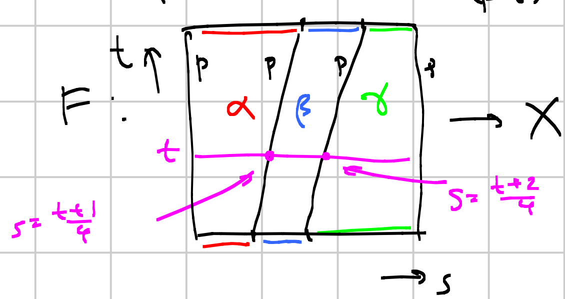
$\Rightarrow \alpha\cdot\beta,\ \alpha'\cdot\beta'$ are homotopic.

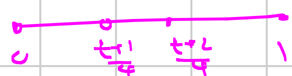
Claim 2 Multiplication is associative (important: on htpy classes, not loops!)

α, β, γ loops at $p \Rightarrow [(\alpha\cdot\beta)\cdot\gamma] = [\alpha\cdot(\beta\cdot\gamma)]$



$: [0,1] \to X$ $\qquad (\alpha\cdot\beta)\cdot\gamma\,(s) = \begin{cases} \alpha(4s) & s \le \frac{1}{4} \\ \beta(4s-1) & \frac{1}{4} \le s \le \frac{1}{2} \\ \gamma(2s-1) & s \ge \frac{1}{2} \end{cases}$





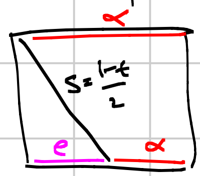
$$F(s,t) = \begin{cases} \alpha\left(\frac{s}{(t+1)/4}\right) & s \le \frac{t+1}{4} \\ \beta\left(\frac{s - \frac{t+1}{4}}{1/4}\right) & \frac{t+1}{4} \le s \le \frac{t+2}{4} \\ \gamma\left(\frac{s - \frac{t+2}{4}}{1 - \frac{t+2}{4}}\right) & \frac{t+2}{4} \le s \le 1 \end{cases}$$

Def A group is a set G with a binary operation $\cdot : G\times G \to G$ satisfying

1. associative: $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ $\quad \forall a,b,c \in G$
2. identity: $\exists$ element $e \in G$ such that $a\cdot e = e\cdot a = a$ $\quad \forall a \in G$
3. inverse: $\forall a \in G\ \exists b \in G$ such that $a\cdot b = b\cdot a = e$ (write $b = a^{-1}$).

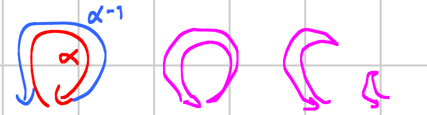
Thm X top space, $p \in X$. The set of htpy classes of loops at p, with the above multiplication, forms a group: $\pi_1(X,p)$.

Pf Identity: define $e:[0,1]\to X$ by $e(s)=p$. Then $[e\cdot\alpha]=[\alpha\cdot e]=[\alpha]$:

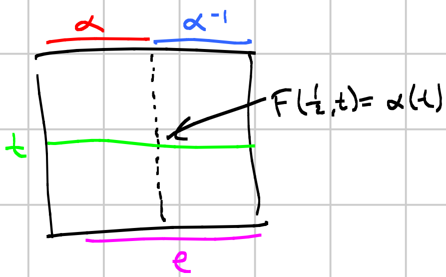


$$F(s,t)=\begin{cases} p & s\le \frac{1-t}{2} \\ \alpha\left(\dfrac{s-\frac{1-t}{2}}{1-\frac{1-t}{2}}\right) & s\ge\frac{1-t}{2}\end{cases}.$$

Inverse: define α^{-1} by $\alpha^{-1}(s)=\alpha(1-s)$.



Then $[\alpha\cdot\alpha^{-1}]=[\alpha^{-1}\cdot\alpha]=[e]$:



$$F(s,t)=\begin{cases}\alpha(2st) & s\le\frac12\\ \alpha((2-2s)t) & s\ge\frac12\end{cases}. \qquad \square$$

11/16 ②

Def A homomorphism between groups G,H is a map $\varphi: G\to H$ such that $\varphi(g_1g_2)=\varphi(g_1)\cdot\varphi(g_2)$ $\forall g_1,g_2\in G$ (follows that $\varphi(e)=e$ and $\varphi(g^{-1})=\varphi(g)^{-1}$).

An isomorphism is a homomorphism $\varphi: G\to H$ that is bijective. (follows that φ^{-1} is a homomorphism). write $G\cong H$.

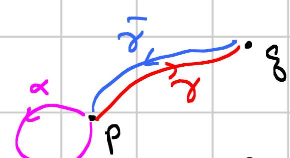
Thm If p,q are in the same path component of X, then $\pi_1(X,p)\cong\pi_1(X,q)$.

Pf Let γ = path from p to q, $\bar\gamma$ = opposite path from q to p: $\bar\gamma(t)=\gamma(1-t)$.

Define $\varphi: \pi_1(X,p)\longrightarrow\pi_1(X,q)$

Munkres calls this "$\hat\gamma$"

$$[\alpha]\longmapsto[\bar\gamma\cdot\alpha\cdot\gamma]$$



If α,α' are homotopic then so are $\bar\gamma\cdot\alpha\cdot\gamma$, $\bar\gamma\cdot\alpha'\cdot\gamma$ so this is well-defined.

φ is a homomorphism: $\varphi([\alpha]\cdot[\beta])=\varphi([\alpha\cdot\beta])=[\bar\gamma\cdot\alpha\cdot\beta\cdot\gamma]$

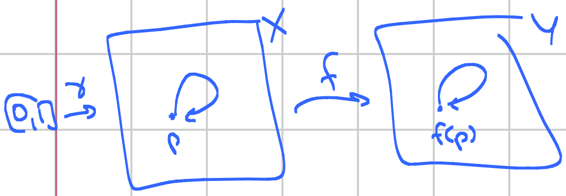
$\varphi([\alpha])\cdot\varphi([\beta])=[\bar\gamma\cdot\alpha\cdot\gamma]\cdot[\bar\gamma\cdot\beta\cdot\gamma]=[\bar\gamma\cdot\alpha\cdot\gamma\cdot\bar\gamma\cdot\beta\cdot\gamma]$

and $\gamma\cdot\bar\gamma\simeq$ constant path at p so $\bar\gamma\cdot\alpha\cdot(\gamma\cdot\bar\gamma)\cdot\beta\cdot\gamma\simeq\bar\gamma\cdot\alpha\cdot e\cdot\beta\cdot\gamma\simeq\bar\gamma\alpha\beta\gamma$.

φ is bijective: the inverse map $\varphi^{-1}: \pi_1(X,q)\to\pi_1(X,p)$ is

$$[\alpha]\longmapsto[\gamma\cdot\alpha\cdot\bar\gamma]. \qquad \square$$

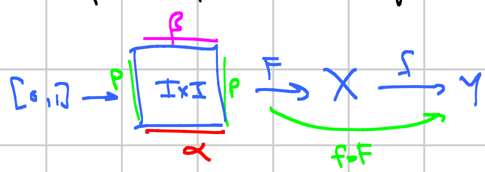
Next: suppose we have a continuous map $f: X \to Y$. This gives a map



$(\text{Loops in } X \text{ at } p) \longrightarrow (\text{Loops in } Y \text{ at } q = f(p))$:

$\gamma \longmapsto f \circ \gamma$.

If α, β are path homotopic in X then $f\circ\alpha, f\circ\beta$ are path htpic in Y:



are path homotopic in Y with homotopy $f \circ F$.

Thus we get a map $f_*: \pi_1(X,p) \longrightarrow \pi_1(Y,q)$.

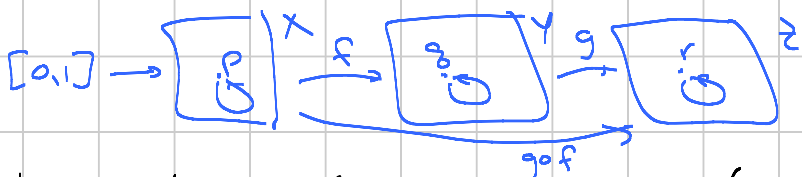
This is a <u>homomorphism</u>: $\qquad [\alpha] \longmapsto [f\circ\alpha]$

$$f_*([\alpha]\cdot[\beta]) = f_*([\alpha\cdot\beta]) = f\circ(\alpha\cdot\beta)$$

$$f_*[\alpha]\cdot f_*[\beta] = (f\circ\alpha)\cdot(f\circ\beta) \quad //$$

So: $f: X \to Y$ gives $f_*: \pi_1(X,p) \to \pi_1(Y,q)$ homomorphism.

Furthermore: Suppose we have $\quad X \xrightarrow{f} Y \xrightarrow{g} Z$.



write $q = f(p)$, $r = g(q)$

If $\gamma = $ loop in X at p then $\quad (g\circ f)\circ\gamma = g\circ(f\circ\gamma)$ So:

$$\begin{tikzcd} \pi_1(X,p) \arrow[r, "f_*"] \arrow[rr, bend right, "(g\circ f)_*"'] & \pi_1(Y,q) \arrow[r, "g_*"] & \pi_1(Z,r) \end{tikzcd}$$

$$[\gamma] \qquad [f\circ\gamma] \qquad [g\circ f\circ\gamma]$$

$$\boxed{(g\circ f)_* = g_*\circ f_*}.$$

Special case: suppose X, Y are homeomorphic, $X \underset{g}{\overset{f}{\rightleftarrows}} Y$, $p \in X$, $q = f(p) \in Y$.

This gives homomorphisms $\pi_1(X,p) \underset{g_*}{\overset{f_*}{\rightleftarrows}} \pi_1(Y,q)$.

$g\circ f = \mathrm{id}_X \Rightarrow g_*\circ f_* = (\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,p)} \Rightarrow f_*$ is an isomorphism

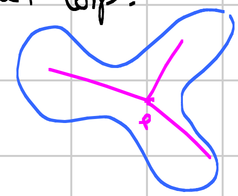
$\qquad f_*\circ g_* = (\mathrm{id}_Y)_* = \mathrm{id}_{\pi_1(Y,q)} \qquad \pi_1(X,p) \to \pi_1(Y,q)$.

<u>Homeomorphic spaces have isomorphic fundamental groups.</u>

Def If X is path connected and $\pi_1(X,p) \cong \{e\}$ then X is <u>simply connected</u>.

Notes:
- Doesn't depend on the point p
- this means: any loop at p is homotopic to the constant loop.

Ex. $\mathbb{R}^n$, or any convex or star-convex subset of $\mathbb{R}^n$:

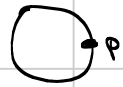


Use straight-line homotopy.

What isn't simply connected?

Thm $\pi_1(S^1, p) \cong \mathbb{Z}$ ← $\mathbb{Z}$ with "multiplication" given by $+$.

take $p = 1$:



For $n \in \mathbb{Z}$, define $\tilde{\gamma}_n : [0,1] \to \mathbb{R}$, $\tilde{\gamma}_n(s) = ns$.

$\gamma_n : [0,1] \to S^1$, $\gamma_n = \pi \circ \tilde{\gamma}_n$

$\pi : \mathbb{R} \to S^1$, $\pi(x) = \exp(2\pi i x)$.

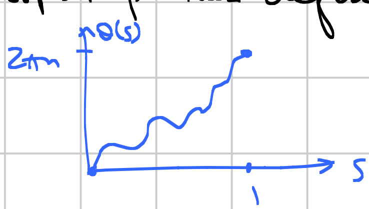
Then define $\phi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1)$. Thm: this is an isomorphism.

$n \longmapsto [\gamma_n]$.

3 parts:
1. ϕ is a homomorphism.
2. ϕ is onto
3. ϕ is one-to-one.

Idea: given a loop $\gamma : [0,1] \to S^1$, write $\gamma(s) = \exp(i\theta(s))$ and plot $\theta(s)$.

If γ is a loop at p then define $\theta(0) = 0$: $\theta(2\pi) = 2\pi n$ for some $n \in \mathbb{Z}$.

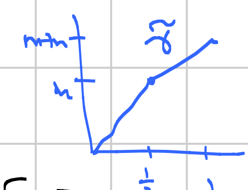


n is called the <u>degree</u> of γ, and the inverse of ϕ maps $[\gamma]$ to n.

11/21

1. ϕ = <u>homom.</u>: need $[\gamma_m] \cdot [\gamma_n] = [\gamma_{m+n}]$ i.e. $\gamma_m \cdot \gamma_n \simeq \gamma_{m+n}$.

Define $\tilde{\gamma} : [0,1] \to \mathbb{R}$ by



so $\gamma_m \cdot \gamma_n = \pi \circ \tilde{\gamma}$.

Now $\tilde{\gamma}$ and $\tilde{\gamma}_{m+n}$ are both $[0,1] \to \mathbb{R}$ and agree on endpoints, so $\tilde{\gamma} \simeq \tilde{\gamma}_{m+n}$ rel $\{0,1\} \Rightarrow \gamma_m \cdot \gamma_n = \pi \circ \tilde{\gamma} \simeq \pi \circ \tilde{\gamma}_{m+n} = \gamma_{m+n}$ rel $\{0,1\}$. $\square$

2. ϕ is onto: Say γ = loop in S^1.

Path-lifting lemma: γ = path in S^1, $\gamma(0)=1$. Then $\exists!$ path $\tilde{\gamma}:[0,1]\to\mathbb{R}$ such that $\tilde{\gamma}(0)=0$ and $\gamma=\pi\circ\tilde{\gamma}$.

Given this: lift γ to $\tilde{\gamma}:[0,1]\to\mathbb{R}$.

Then $\tilde{\gamma}(1)=n$ for some $n\in\mathbb{Z}$, and $\tilde{\gamma},\tilde{\gamma}_n:[0,1]\to\mathbb{R}$ have same endpoints so $\tilde{\gamma}\simeq\tilde{\gamma}_n$ (path homotopic) $\Rightarrow \gamma\simeq\gamma_n$.

[Lemma If $\gamma,\gamma':[0,1]\to X$ are path homotopic and $f:X\to Y$, then $f\circ\gamma,\ f\circ\gamma':[0,1]\to Y$ are path homotopic.]

Pf of path-lifting lemma: (uses Lebesgue lemma) See Munkres Lemma 54.1.

3. ϕ is one-to-one:

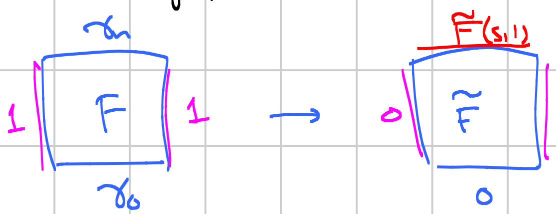
Homotopy-Lifting Lemma $F:I\times I\to S^1$ htpy of loops in S^1: $F(0,t)=F(1,t)=1\ \forall t$. Then $\exists!\ \tilde{F}:I\times I\to\mathbb{R}$ such that $\tilde{F}(0,t)=0\ \forall t$ and $\pi\circ\tilde{F}=F$.

(See Munkres Lemma 54.2 for proof.)

Then: if $\phi(n)=\phi(n')\Rightarrow\phi(n-n')=\phi(n)\cdot\phi(-n')=\phi(n)\cdot\phi(n')^{-1}=e$ = constant loop $[\gamma_0]$.

So it suffices to show if $\gamma_n\simeq\gamma_0$ then $n=0$.

A homotopy F between γ_0 and γ_n lifts to a homotopy $\tilde{F}:I\times I\to\mathbb{R}$.



Note $\{\tilde{F}(1,t)\}\subset\mathbb{Z}\subset\mathbb{R}$ must be connected, and $\tilde{F}(1,0)=0\Rightarrow\tilde{F}(1,t)=0$ for all t.

Now $\tilde{F}(s,1)$ is a lift of $\gamma_n \Rightarrow$ by path lifting, it's equal to $\tilde{\gamma}_n$, so $\tilde{F}(s,1)=ns \Rightarrow n=0$. $\square$

Other facts about π_1 (we won't prove).

Prop For $n \geq 2$, S^n is simply connected.
(Special case of Van Kampen's Thm).

Prop X, Y path-connected. Then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.
(G, H groups $\Rightarrow G \times H = \{(g,h) \mid g \in G, h \in H\}$, mult. given by
$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot g_2, h_1 \cdot h_2)$, $e = (e, e)$)

Applications:

$T^2 = S^1 \times S^1 \Rightarrow \pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.

$$\pi_1(\mathbb{R}^n \setminus \{0\}) \cong \pi_1(S^{n-1} \times \mathbb{R}) \cong \pi_1(S^{n-1}) \cong \begin{cases} \{e\} & n > 2 \\ \mathbb{Z} & n = 2 \end{cases}.$$

$\mathbb{R}^n \setminus \{0\}$ homeo to $S^{n-1} \times \mathbb{R}$

Homotopy Type

π_1 is invariant under not just homeomorphism but something more general.

Def X, Y have the <u>same homotopy type</u> if there are maps

$X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$. Write $\underline{X \simeq Y}$.

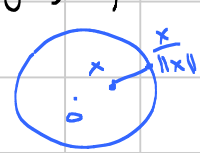
Ex. 1. homeomorphic $\Rightarrow$ same htpy type

2. $X \subset \mathbb{R}^n$ convex or star-convex $\Rightarrow X \simeq \{pt\}$: X is "<u>contractible</u>".

$X \underset{g}{\overset{f}{\rightleftarrows}} Y = \{q\}$ define $f(x) = q \ \forall x \in X$; $g(q) = p$ for some $p \in X$.
Then $f \circ g = id_Y$, and $g \circ f \simeq id_X$ since any maps to X are homotopic.

3. $\mathbb{R}^n - \{0\} \simeq S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$. $f(x) = \frac{x}{\|x\|}$, $g(y) = y$.

$\underset{g}{\overset{f}{\rightleftarrows}}$ $f \circ g = id$, $g \circ f(x) = \frac{x}{\|x\|}$.



$g \circ f \simeq id_X$ on $X = \mathbb{R}^n - \{0\}$: homotopy $F: X \times I \to X$

11/28 $F(x,t) = (1-t)x + t\frac{x}{\|x\|}$ so $F(x,0) = x$, $F(x,1) = \frac{x}{\|x\|}$.

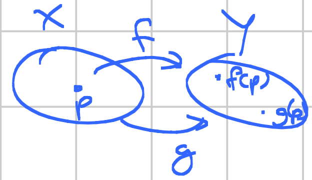
Prop $\simeq$ is an equivalence relation.

Pf. $X \underset{g_1}{\overset{f_1}{\rightleftarrows}} Y \underset{g_2}{\overset{f_2}{\rightleftarrows}} Z \quad \rightsquigarrow \quad X \underset{g_1 \circ g_2}{\overset{f_2 \circ f_1}{\rightleftarrows}} Z$.

$$\underbrace{f_2 \circ f_1 \circ g_1}_{\simeq\, id_Y} \circ g_2 : Z \to Z \quad \text{is} \quad \simeq f_2 \circ id_Y \circ g_2 = f_2 \circ g_2 \simeq id_Z$$

and similarly $g_1 \circ g_2 \circ f_2 \circ f_1 \simeq id_X$. □

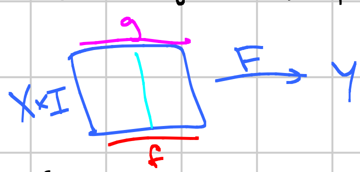
Prop Suppose $f, g : X \to Y$ are homotopic maps giving



$$\begin{tikzcd} & \pi_1(Y, f(p)) \arrow[d, "\hat{\gamma}", "\cong"'] \\ \pi_1(X,p) \arrow[ur, "f_*"] \arrow[r, "g_*"'] & \pi_1(Y, g(p)) \end{tikzcd}$$

Then there is an isom. $\hat{\gamma} : \pi_1(Y, f(p)) \to \pi_1(Y, g(p))$ such that $g_* = \hat{\gamma} \circ f_*$.

More precisely:



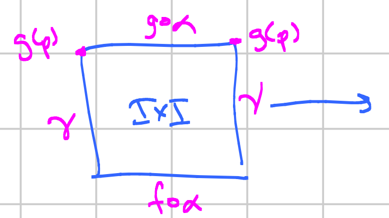
gives a path γ in Y from $f(p)$ to $g(p)$: $\gamma(t) = F(p,t)$.

Then $\hat{\gamma}$ is as before: $\hat{\gamma}([\alpha]) = [\bar{\gamma} \cdot \alpha \cdot \gamma]$.

Pf Let α = loop at p. Want $[g \circ \alpha] = g_*[\alpha] \overset{?}{=} \hat{\gamma} f_*[\alpha] = [\bar{\gamma} \cdot (f \circ \alpha) \cdot \gamma]$.

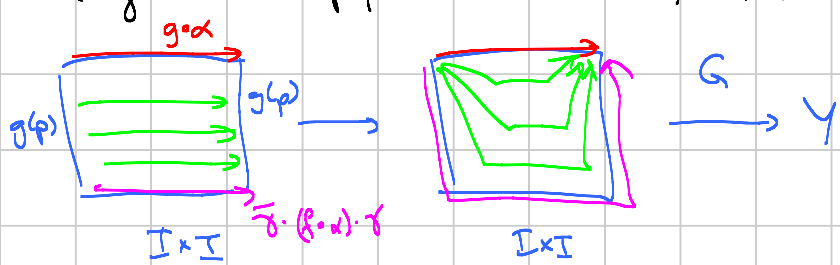
Define $G : I \times I \to Y$ by

$$G(s,t) = F(\alpha(s), t)$$



$$G(s,0) = F(\alpha(s), 0) = f(\alpha(s))$$
$$G(s,1) = F(\alpha(s), 1) = g(\alpha(s))$$
$$G(0,t) = G(1,t) = F(p,t) = \gamma(t)$$

Then we get a homotopy $H : I \times I \to Y$ from $\bar{\gamma} \cdot (f \circ \alpha) \cdot \gamma$ to $g \circ \alpha$ as desired. □



Prop If X, Y are path connected and $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$.

Pf $\underset{p}{X} \underset{g}{\overset{f}{\rightleftarrows}} \underset{f(p)}{Y}$ claim: $f_* : \pi_1(X,p) \to \pi_1(Y, f(p))$ is $\cong$.

$$\begin{tikzcd} \underset{p}{X} \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & \underset{f(p)}{Y} \arrow[r, "g"] & \underset{gf(p)}{X} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \pi_1(X,p) \arrow[r, "f_*"] \arrow[rr, bend right, "(g\circ f)_* = g_* \circ f_*"'] & \pi_1(Y, f(p)) \arrow[r, "g_*"] & \pi_1(X, gf(p)) \end{tikzcd}$$

$g\circ f$, id_X are homotopic maps $X \to X$

$\Rightarrow$ by previous prop, $\exists\, \hat{\gamma}$ w/ $\begin{tikzcd} \pi_1(X,p) \arrow[r, "id"] \arrow[dr, "(g\circ f)_*"'] & \pi_1(X,p) \arrow[d, "\hat{\gamma}", "\cong"'] \\ & \pi_1(X, gf(p)) \end{tikzcd}$

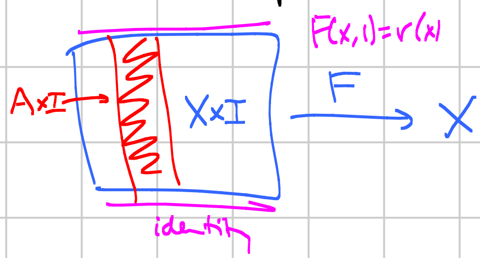
So $g_* \circ f_* = (g\circ f)_* = \hat{\gamma}$ is an isom $\Rightarrow$ f_* is one-to-one.

Similarly $f_* \circ g_*$ is an isom $\Rightarrow$ f_* is onto. $\square$

Important example of homotopy equiv: deformation retract.

Def $A \subset X$. A deformation retract of X onto A is a homotopy rel A, between $id: X \to X$ and a retraction $r: X \to A$: i.e., a map $F: X \times I \to X$ such that

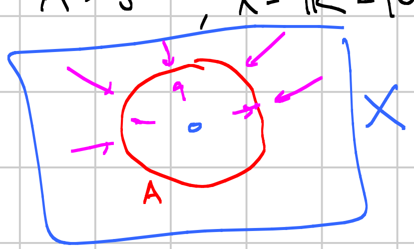
$$\begin{cases} F(x,0) = x \\ F(x,1) \in A \quad \forall x \in X \quad (= r(x)) \\ F(a,t) = a \quad \forall a \in A, t \in I. \end{cases}$$



Observation: if $\exists$ def. retract then $X \simeq A$: $A \underset{r}{\overset{i \text{ (inclusion)}}{\rightleftarrows}} X$

$r \circ i = id_A$; $\underset{x \mapsto F(x,1)}{i \circ r : X \to X}$ is htpic to $\underset{x \mapsto F(x,0)}{id : X \to X}$.

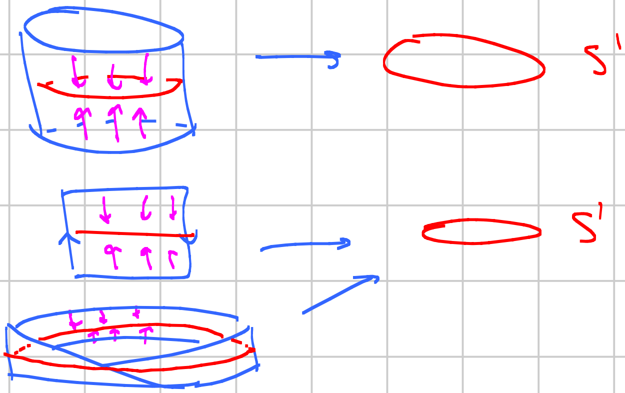
Ex $A = S^{n-1}$, $X = \mathbb{R}^n - \{0\}$ $\qquad F: X \times I \to X \quad F(x,t) = (1-t)x + t\dfrac{x}{\|x\|}$.



Other deformation retracts:

cylinder $S^1 \times I$

Möbius band



$\Rightarrow \pi_1(\text{cylinder}) \cong \pi_1(\text{Möbius}) \cong \mathbb{Z}$.

11/30

Application: Fundamental Theorem of Algebra.

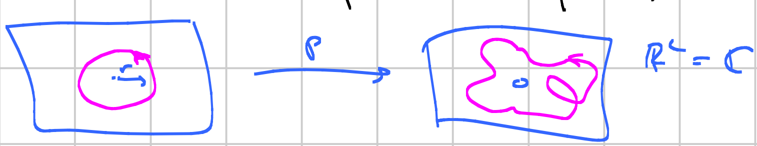
FTA Any nonconstant polynomial with complex coefficients has at least one root in $\mathbb{C}$.

($\Rightarrow$ any poly of degree n has n roots, with multiplicity).

Pf Suffices to consider monic poly, $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$.

Assume $p(z) \neq 0 \ \forall z$. For any $r \geq 0$, define

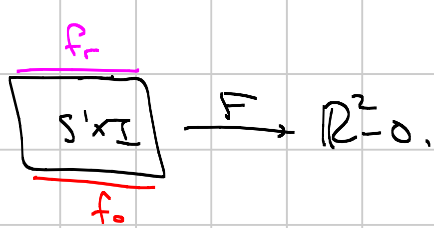
$f_r : S^1 \to \mathbb{R}^2 \setminus \{0\}$ by $f_r(z) = p(rz)$. ($z \in S^1$ = unit circle in $\mathbb{C}$; $p(rz) \in \mathbb{C} \setminus \{0\}$)



Note $f_0(z) = p(0)$ is a constant loop.

Claim 1 $f_0 \simeq f_r$ for any r.

Pf. $F(z,t) = p(rtz)$ is a homotopy



Claim 2 Define $g: S^1 \to \mathbb{R}^2 - 0$ by $g(z) = z^n$.

For $r \gg 0$, $f_r \simeq g$.

Pf Define $g_r : S^1 \to \mathbb{R}^2 - 0$ by $g_r(z) = (rz)^n$; then $g_r \simeq g$ by straight-line homotopy. Need $f_r \simeq g_r$. Define

$G : S^1 \times I \to \mathbb{R}^2 - 0$ $\quad \overset{f_r}{\underset{g_r}{\boxed{S^1 \times I}}} \xrightarrow{G} \mathbb{R}^2 - 0 \quad$ $G(z,t) = (rz)^n + t\left(a_{n-1}(rz)^{n-1} + \dots + a_1(rz) + a_0\right)$

For this to work, need G to miss 0.

Choose r sufficiently large that $\quad \left|\frac{a_{n-1}}{r}\right| + \dots + \left|\frac{a_1}{r^{n-1}}\right| + \left|\frac{a_0}{r^n}\right| < 1$. Then

$$\left| t\left(a_{n-1}(rz)^{n-1} + \dots + a_1(rz) + a_0\right)\right| = |tr^n| \left| \frac{a_{n-1}}{r} z^{n-1} + \dots + \frac{a_1}{r^{n-1}} z + \frac{a_0}{r^n} \right|$$

$$t \leq 1,\ |z| = 1 \longrightarrow \leq r^n \left(\left|\frac{a_{n-1}}{r}\right| + \dots + \left|\frac{a_1}{r^{n-1}}\right| + \left|\frac{a_0}{r^n}\right| \right)$$

$$< r^n = |(rz)^n|$$

$\Rightarrow G(z,t) \neq 0$ for all $t \leq 1$, $|z| = 1$.

Claim 3 $g \not\simeq$ constant map.

Pf.

$$\begin{tikzcd} S^1 \arrow[r, shift left, "g"] \arrow[r, shift right, "\text{const}"'] \arrow[rr, bend left, "h"] \arrow[rr, bend right, "c"'] & \mathbb{R}^2 - 0 \arrow[r, "p"] & S^1 \end{tikzcd}$$

$\left(p(z) = \frac{z}{|z|}\right)$

Define $h = p \circ g$, $h(z) = z^n$: in polar coordinates, $h(\theta) = n\theta$.

If $g \simeq \text{const}$ then $h = p \circ g \simeq p \circ \text{const} = c$ (constant map). So:

$$\begin{tikzcd} \mathbb{Z} \cong \pi_1(S^1) \arrow[r, "h_*"] \arrow[dr, "c_*"'] & \pi_1(S^1) \cong \mathbb{Z} \arrow[d, "\cong"] \\ & \pi_1(S^1) \cong \mathbb{Z} \end{tikzcd}$$

But c_* is the zero map while $h_*(1) = [h \circ \gamma_1] = [\gamma_n] = n$. $\square$

12/5 2

∅ (we didn't do this)

Borsuk-Ulam Thm

B-U Thm $f: S^n \to \mathbb{R}^n$ continuous, $n \geq 1$. Then there is $x \in S^n$ with $f(x) = f(-x)$.

(for $n=1$ this was on the midterm)

We'll prove for $n=2$.

Def $f: S^n \to S^m$ is antipode-preserving if $f(-x) = -f(x)$ $\forall\, x \in S^n$.
(f sends antipodal pts to antipodal pts)

Lemma Any antipode-preserving, continuous map $S^1 \to S^1$ is not null homotopic.

Pf Say $h: S^1 \to S^1$ is antipode-preserving. Fix $p \in S^1$; can assume $h(p) = p$ (or else compose h with a rotation). Define $q: S^1 \to S^1$, $q(z) = z^2$: note $q(z) = q(-z)$.

$$\begin{tikzcd} S^1 \arrow[r, "h"] \arrow[d, "q"'] & S^1 \arrow[d, "q"] \\ S^1 \arrow[r, "k"] & S^1 \end{tikzcd}$$

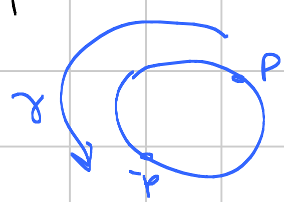
Then h induces a well-defined map k with $q \circ h = k \circ q$. ↳ since h is antipode-preserving.

This induces

$$\begin{tikzcd} \pi_1(S^1, p) \arrow[r, "h_*"] \arrow[d, "q_*"'] & \pi_1(S^1, p) \arrow[d, "q_*"] \\ \pi_1(S^1, p^2) \arrow[r, "k_*"] & \pi_1(S^1, p^2) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[r, "0"] \arrow[d, "\times 2"'] & \mathbb{Z} \arrow[d, "\times 2"] \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

q has degree 2 (see HW) so q_* maps n to $2n$.

If h is null-homotopic then $h_* =$ trivial (0) map $\Rightarrow k_* =$ trivial map.



Now let $\gamma =$ path in S^1 from p to $-p$.

$\Rightarrow h \circ \gamma =$ path in S^1 from p to $-p$

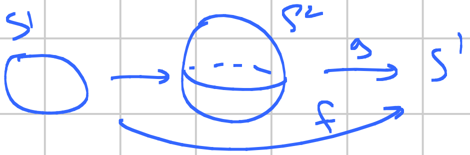
$\Rightarrow q \circ \gamma$, $q \circ (h \circ \gamma) =$ loops in S^1 at p^2, winding around S^1 an odd # of times.

$$\pi_1(S^1, p^2) \xrightarrow{k_*} \pi_1(S^1, p^2) \cong \mathbb{Z}$$
$$[q \circ \gamma] \longrightarrow [k \circ q \circ \gamma] = [q \circ h \circ \gamma]$$ → this is odd & $k_* \neq 0$. ~~⇒⇐~~ ☐

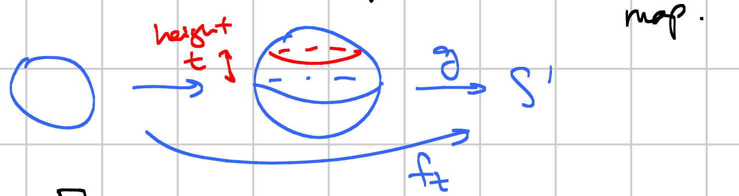
Lemma There is no continuous, antipode-preserving map $S^2 \to S^1$.

Pf Suppose $\exists\ g: S^2 \to S^1$ continuous, antipode-preserving.

Equator $= S^1$ so g restricts to antipode preserving $f: S^1 \to S^1$.



Now f is nullhomotopic: $f_0 = f$, $f_1 =$ constant map.



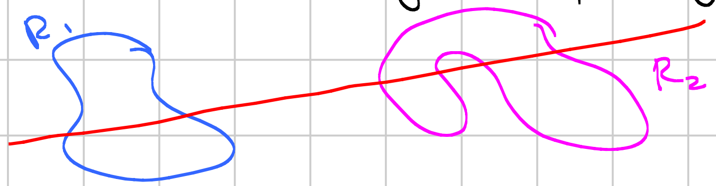
This contradicts previous lemma. $\square$

Pf of B-U for $n=2$.

Suppose $\exists\ f: S^2 \to \mathbb{R}^2$ with $f(x) \neq f(-x)\ \forall x$. Define
$g: S^2 \to S^1$ by $\quad g(x) = \dfrac{f(x) - f(-x)}{|f(x) - f(-x)|}$. Note $g(-x) = -g(x)$:
So g is a continuous, antipode-preserving map $S^2 \to S^1 \Rightarrow\Leftarrow$ $\square$

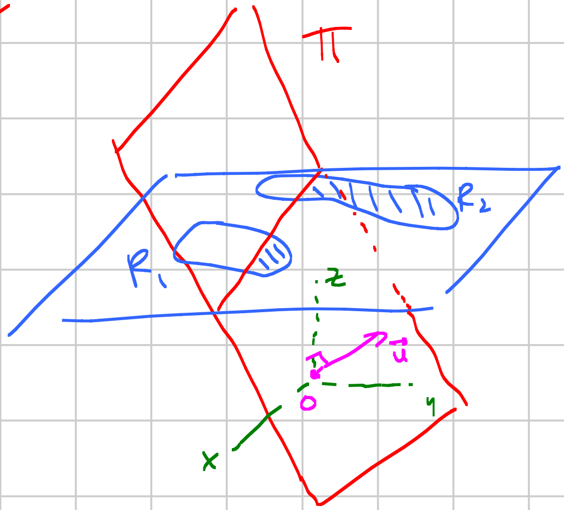
Consequences 1. At any time, there are two antipodal points on earth with same temp + pressure.

2. Bisection theorem: Given two regions $R_1, R_2 \subset \mathbb{R}^2$ of finite area, there is some line that bisects each region into pieces of equal area.



Pf View $\mathbb{R}^2$ as $\{z=1\} \subset \mathbb{R}^3$. Any point in S^2 is a unit vector $\vec{u}$ in $\mathbb{R}^3$.

Let $\Pi =$ plane through O $\perp$ to u. Define $f_i: S^2 \to \mathbb{R}$, $i=1,2$:
$f_i(\vec{u}) =$ area of the portion of R_i lying on the side of Π that $\vec{u}$ lies in.



$f = (f_1, f_2) : S^2 \to \mathbb{R}^2$ so by B-U, $\exists$ $\vec{u} \in S^2$ with $f(\vec{u}) = f(-\vec{u})$

$\Rightarrow f_1(u) = f_1(-u), \quad f_2(u) = f_2(-u)$.

For this u, π intersects $\{z=1\}$ in a line and this is the line we want. $\square$

<u>Rmk</u> Borsuk-Ulam for $n=3$ similarly gives:

<u>Ham Sandwich Thm</u> For three regions R_1, R_2, R_3 in $\mathbb{R}^3$, there is a plane bisecting each region into regions of equal volume.

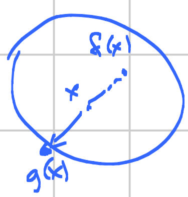
<u>Brouwer Fixed Point Thm</u> Every continuous function $f : \bar{B}^n \to \bar{B}^n$, $n \geq 1$, must have a fixed point (x s.t. $f(x) = x$).

(important for differential equations etc.)

$n=1$: $\bar{B}^1 = [-1,1]$: this was done in H/W.

We'll prove <u>$n=2$</u>. Suppose $f : \bar{B}^2 \to \bar{B}^2$ has no fixed points, $S^1 = \operatorname{Bd} \bar{B}^2$.

Define $g : \bar{B}^2 \to S^1$ by: $x \in S^2 \Rightarrow$ draw ray from $f(x)$ to x; this intersects S^1 in $g(x)$.



- g is continuous (write in coordinates)
- if $x \in S^1$ then $g(x) = x$.

Thus g is a retraction from $\bar{B}^2$ to S^1, contradicting:

<u>Lemma</u> There is no retraction from $\bar{B}^2$ to S^1.

<u>Pf</u> $r : \bar{B}^2 \to S^1 \Rightarrow r_* : \pi_1(\bar{B}^2) \to \pi_1(S^1)$ is a surjection: but $\pi_1(\bar{B}^2) \cong \{e\}$, $\pi_1(S^1) \cong \mathbb{Z}$. $\square$