Math 411 HW 4 — Outline of Solutions

1. Note \( U \subset \mathbb{R}^2 \) (the usual limit topology) is open \( \Rightarrow \)
   \( \forall x \in U, \) there is \( I(a,b) \) with \( x \in [a,b] \subset U \)
   \( \Rightarrow \)
   \( \exists \varepsilon > 0 \) such that \( \varepsilon \cdot [x, b] = [a, b] \).

   Thus \( f \) is continuous \( \mathbb{R} \rightarrow \mathbb{R}^2 \) \( \Rightarrow \)
   for any basic open set \( (c, d) \) in \( \mathbb{R}^2, \)
   \( f^{-1}(c, d) \) is open in \( \mathbb{R}^2 \)
   \( \Rightarrow \)
   for any \( c < d \) and any \( x, y \in f^{-1}(c, d), \)
   there is \( b > a \) with \( f(x, y) \in (c, d) \)
   \( \Rightarrow \)
   for any \( c < d \) and any \( y \) with \( f(y) \in (c, d), \)
   there is \( b > a \) with \( f(x, y) \in (c, d) \)
   for all \( x \) with \( a \leq x < b \).

   Now \( \varepsilon \) given \( a, b, \) \( \varepsilon, \)
   choose \( c = f(a) - \varepsilon, \)
   \( d = f(a) + \varepsilon \)
   \( \Rightarrow \)
   there is \( b > a \) with \( |f(x) - f(a)| < \varepsilon \)
   whenever \( a \leq x < b \)
   choose \( \delta = b - a \Rightarrow \) lower semicontinuous.

   Conversely, lower semicontinuous \( \Rightarrow \) suppose \( c < d \) and \( f(a) \in (c, d). \)
   Then there is \( \varepsilon \) with \( f(a) - \varepsilon, f(a) + \varepsilon \in (c, d) \)
   \( \Rightarrow \) there is \( \delta \) such that whenever \( a \leq x < a + \delta, \)
   \( f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon) \Rightarrow f(x) \in (c, d) \)
   \( \Rightarrow \) choose \( b = a + \delta > a \).

2. Define \( d \) by \( d(x, y) = \sqrt{1 + \left| y - y' \right|^2} \) if \( x = x' \) and \( |y - y'| \leq 1 \)
   \( 1 \) otherwise. (Note \( d \) is \( 1 \).

   Note this is a metric: for triangle inequality \( d(x, y, x', y') \leq d(x, y, x', y') + d(x', y', x''') \),
   this is satisfied whenever either term on the right = \( 1, \) since left hand side \( \leq 1. \)

   When both are \( 1, \) then \( x = x' = x'' \) and this follows from \( \min(1, |y - y'|, 1) \leq |y' - y'| \leq |y' - y''| + |y'' - y'|.

   Claim: the metric topology and the order topology are the same.

   Write \( \mathcal{B} \) = basic open sets in metric topology = \{ \mathcal{B}(x, y) \}
   \( \mathcal{B}' \) = \{ \mathcal{B} \}.

   Note \( \mathcal{B}(x, y) = \{ (x, y), (x, y), (x, y), (x, y) \} \)
   \( \mathcal{B} \) is \( \subseteq \) \( \mathcal{B} \) for everything in \( \mathbb{R} \) except \( \mathbb{R}^2 \)
   if \( r > 1 \) so everything in \( \mathcal{B} \) except \( \mathbb{R}^2 \)
2. The order is finer than metric. Conversely, if \((x'y', y') \in \mathcal{B}'\) and \(x''y'' \in (x'y', x'y')\), then there is some \(\varepsilon > 0\) such that
\[
x''y'' \in (x''y'' - \varepsilon, x''y'' + \varepsilon) = (x'y', x'y'),
\]
so metric is finer than order.

3. (a) If \(x \in \overline{A}\) then for any \(\varepsilon > 0\), there is \(y \in \overline{A} \cap B(x, \varepsilon)\) so \(d_A(y) \leq d(x, y) < \varepsilon\).

This can only happen if \(d_A(x) = 0\).

If \(d_A(x) > 0\), then for any \(\varepsilon > 0\), there is \(y \in \overline{A}\) with \(d(x, y) < \varepsilon\) or \(y \not\in \overline{A} \cap B(x, \varepsilon)\).

Thus every neighborhood of \(x\) intersects \(A\), so \(x \in \overline{A}\).

(b) Lemma: If \(x, y \in \overline{A}\) then \(d_A(x) \leq d_A(y) + d(x, y)\).

\[d_A(x) \leq d_A(y) + d(x, y)\]
\[d_A(x) - d_A(y) \text{ is a lower bound for } \{d_A(z) \mid z \in \overline{A}\} \Rightarrow d_A(x) - d_A(y) \leq d_A(y).\]

Main proof: Suppose \(x \notin \overline{A}\). Then \(d_A(x) > 0\) so there is \(\varepsilon > 0\) with \(d_A(x) - \varepsilon, d_A(x) + \varepsilon \in (a-e, a+e)\).

Now if \(y \notin \mathcal{B}(x, \varepsilon)\) then by lemma, intersect \(x\).
\[d_A(x) - \varepsilon < d_A(x) - d_A(y) < d_A(y) \leq d_A(x) + d(x, y) < d_A(x) + \varepsilon\]
\[d_A(y) \in (a-e, a+e)\]. Thus \(x \notin \mathcal{B}(x, \varepsilon) \cup A\), so \(U\) is open.

4. (a) Finite complement: yes.

Suppose \(F\) is an open cover and choose any nonempty \(U \in F\); write \(U = X \setminus \bigcup_{i \neq j} X_i\).

Since \(F\) covers \(X\), there are \(U_1, \ldots, U_n \in F\) with \(x \in U_i\) for each \(i\).

Some \(U_i\) must be the same. Then \(X = U \cup U_1 \cup \cdots \cup U_n\).

Lower-limit: no.

The open cover \(\{[n, n+1) \mid n \in \mathbb{Z}\}\) consists of disjoint sets, thus has no finite subcover (a nontrivial subcover at all).
Consider the open (in $\mathbb{R}^n$) cover given by $\{U_n\}_{n \in \mathbb{N}}$, $U_n = \left[\frac{1}{n}, \frac{1}{n} - \frac{1}{m}\right)$, along with $U_0 = \left[1, 2\right)$. Any finite subset $U_{n_1}, \ldots, U_{n_k}$ and possibly $U_0$ doesn't cover any $x \in \left[\frac{1}{m}, 1\right)$ where $m = \max(n_1, \ldots, n_k)$.

Note any set that's open in $\mathbb{R}^n$ is also open in $\mathbb{R}_0$.

Suppose $A$ is compact in $\mathbb{R}_n$.

Let $F$ be any open cover of $A$, consisting of open sets in $\mathbb{R}_0$.

Then this is also an open cover of $A$, consisting of open sets in $\mathbb{R}_0$.

Since $A$ is compact in $\mathbb{R}_0$, $F$ has a finite subcover.

Thus $A$ is compact in $\mathbb{R}_0$. 