1. (a) ⇒: If \( X \) is contractible, then for some \( p \in X \), the constant map \( c_p : X \to X \), \( c_p(x) = p \ \forall x \), is homotopic to \( \text{id}_X \). Then define \( X \xrightarrow{\cong} \{ p \} \) by \( f(x) = p \ \forall x \), \( g(p) = p \); then \( f \circ g = \text{id}_X \circ p \) and \( g \circ f = c_p = \text{id}_X \).

⇐: Given \( X \xrightarrow{\cong} \{ p \} \) with \( f \circ g = \text{id}_X \circ p \) and \( g \circ f = c_p = \text{id}_X \), let \( q = g(p) \).
Then \( f(x) = p \ \forall x \) and \( g \circ f = c_q \), the constant map \( c_q(x) = q \ \forall x \), so \( c_q = \text{id}_X \).

(b) Suppose \( X \) is contractible, and let \( q \in X \) with \( \text{id}_X = c_q \), constant map \( q \).
Let \( F: X \times I \to X \) be a homotopy with \( F(x, 0) = x \), \( F(x, 1) = p \ \forall x \in X \).
Suppose \( r: X \to A \) is a retraction. Define \( G: A \times I \to A \) by \( G(a, t) = r(F(a, t)) \). Then \( G(a, 0) = r(a) = a \) and \( G(a, 1) = r(p) \ \forall a \in A \).
So \( G \) is a homotopy between \( \text{id}_A \) and \( c_{r(p)} \), constant map at \( r(p) \).
⇒ \( A \) is contractible.

2. (a) First consider the case \( x_0 = b_0 \). Fix a path \( \alpha_0 \) in \( S^1 \) from \( b_0 \) to \( h(b) \).

If \( d \) is the degree for \( x_0 = b_0 \), then by definition

\[ h^o \gamma = (\alpha_0^{-1} \cdot \gamma \cdot \alpha_0)^d \sim \alpha_0^{-1} \cdot \gamma^d \cdot \alpha_0 \] (path homotopic as loops at \( h(b_0) \)).

Now let \( x_0 \) be arbitrary. Let \( \alpha_0 \) be a path from \( b_0 \) to \( x_0 \). Then

\[ \gamma(x_0) = \alpha_0^{-1} \cdot \gamma \cdot \alpha_0. \]

Also for \( x_1 = h(x_0) \), define \( \alpha_2 \) to be the path from \( b_0 \) to \( x_1 \), given by

\[ \alpha_2 = \alpha_0 \cdot (h^o \alpha_1) \]  (goes from \( b_0 \) to \( h(b_0) \), \( h^o \alpha_1 \) goes from \( h(b_0) \) to \( h(b_1) \)).

Then

\[ \gamma(x_1) = \alpha_2^{-1} \cdot \gamma \cdot \alpha_2. \]

Now let \( d' \) be the degree for \( x_0 \); by definition,

\[ h^o (\gamma(x_0)) = (\gamma(x_1))^{d'}. \]
2. (c) Then

\[
\begin{align*}
\text{h}_0(\gamma(x_0)) & \equiv (\text{h}_0\alpha_1)^\circ \cdot (\text{h}_0\gamma) \cdot (\text{h}_0\alpha) \\
& \equiv (\text{h}_0\alpha_1)^\circ \cdot \alpha_0^{-1} \cdot \gamma_1 \cdot \alpha_0 \cdot (\text{h}_0\alpha_1) \\
& \equiv \alpha_0^{-1} \cdot \gamma_1 \cdot \alpha_0.
\end{align*}
\]

while

\[
\begin{align*}
(\gamma(x))^{\circ} & \equiv (\alpha_2^{-1} \cdot \gamma_1 \cdot \alpha_2) \cdot (\alpha_2^{-1} \cdot \gamma_2 \cdot \alpha_2) \cdot \ldots \cdot (\alpha_2^{-1} \cdot \gamma_1 \cdot \alpha_2) \\
& \equiv \alpha_2^{-1} \cdot \gamma_1 \cdot \alpha_2.
\end{align*}
\]

Finally, \(\gamma^{d_1} \sim \alpha_2 \cdot \text{h}_0(\gamma(x)) \cdot \alpha_2^{-1} \equiv \alpha_2 \cdot (\gamma(x))^{d_1} \cdot \alpha_2^{-1} \equiv \gamma^{d_1}\).

Since in \(\pi_1(S', b_0) \cong \mathbb{Z}\) we have \(\gamma_1 = 1\) and \(\gamma_1' = 1\), this implies \(d = d_1\), as desired.

(5) Choose \(x_0 \in S'\) and define \(h(x_0) = x_1, k(x_0) = x_2\). By Lemma 58.4, there is a path \(\alpha'\) from \(x_1\) to \(x_2\) such that the following diagram commutes:

\[
\begin{array}{ccc}
\pi_1(S', x_0) & \xrightarrow{k_\ast} & \pi_1(S', x_1) \\
\downarrow{k_\ast} & & \downarrow{k_\ast} \\
\pi_1(S', x_0) & \xrightarrow{\gamma} & \pi_1(S', x_2)
\end{array}
\]

IF \(h_\ast (\gamma(x)) = (\gamma(x))^{\circ}\) (we use multiplicative notation) then

\[
\begin{align*}
k_\ast (\gamma(x)) & = \alpha' \cdot (\gamma(x))^{\circ} = (\alpha')^{-1} \cdot (\gamma(x))^{\circ} \cdot \alpha'.
\end{align*}
\]

On the other hand, we can choose paths \(\alpha_1\) from \(p_0\) to \(x_1\), and \(\alpha_2\) from \(p_0\) to \(x_2\) such that \(\gamma(x_1) \sim (\alpha_1^{-1} \cdot \gamma(x) \cdot \alpha_1): \) for any \(\alpha_1\), let \(x_2 = \alpha_2 \cdot x_1\).

Then IF \(k_\ast (\gamma(x_0)) = (\gamma(x_0))^{d_1}\), the

\[
\begin{align*}
k_\ast (\gamma(x_0)) & = (\gamma(x_0))^{d_1} = (\alpha')^{-1} \cdot (\gamma(x_0))^{d_1} \cdot (\alpha').
\end{align*}
\]

and so \(d = d_1\) as before.
2. (c) Suppose $x_0 \in S$, $x_1 = k(x_0)$, $x_2 = h(x_1)$. Write $\deg h = d$, $\deg k = d'$.  

Then by definition,  
\begin{align*}
h*(\gamma(x_1)) &= (\gamma(x_2))^d \\
k*(\gamma(x_0)) &= (\gamma(x_1))^d'
\end{align*}

so  
\begin{align*}
(h \circ k)^*(\gamma(x_0)) &= h^*k^*(\gamma(x_0)) \\
&= h^*((\gamma(x_1))^d') \\
&= (h^* (\gamma(x_1)))^{d'} \\
&= (\gamma(x_1))^{dd'}
\end{align*}

since $h^*$ is a homomorphism.

and it follows by definition that $\deg (h \circ k) = dd'$.

(d) Note each of these maps sends $e$ to $e$, so it's convenient to choose $x_0 = e$ in each case: then for each map $h$, we need to find $d$ such that $h^* \gamma = \gamma^d$.

$h = \text{const.}$: $h^* \gamma = \text{constant loop}$ so $d = 0$.

$h = (\partial)$: clearly $d = 1$.

$h = p$: $h^* \gamma = \gamma^{-1}$ so $d = -1$.

$h(2) = \exp$: $h^* \gamma$ is the loop $\gamma^n$ (sending $t$ to $\exp(2\pi i nt)$ for $t \in [0,1]$) so $d = n$. 