1. \( \pi_1(\mathbb{X}, x_0) \text{ abelian} \Rightarrow \forall \alpha, \beta, \bar{\alpha} = \bar{\beta} \):

Let \( \gamma \) be a loop at \( x_0 \): we want \( \bar{\alpha} \cdot \bar{\gamma} \cdot \alpha = \bar{\beta} \cdot \gamma \cdot \beta \).

Define \( \gamma' = \text{loop at } x_0 \text{ given by } \gamma' = \alpha \cdot \bar{\beta} \). Then \( [\gamma], [\gamma'] \in \pi_1(\mathbb{X}, x_0) \)

\( \Rightarrow \gamma \cdot \gamma' = \gamma' \cdot \gamma \Rightarrow \gamma \cdot \bar{\beta} = \alpha \cdot \bar{\beta} \cdot \gamma \)

\( \Rightarrow \alpha \cdot \bar{\gamma} \cdot \alpha \cdot \bar{\beta} \cdot \beta = \bar{\alpha} \cdot \bar{\beta} \cdot \gamma \cdot \beta = \bar{\beta} \cdot \gamma \cdot \beta \)

\( \forall \alpha, \beta, \bar{\alpha} = \bar{\beta} \Rightarrow \pi_1(\mathbb{X}, x_0) \text{ abelian} \):

Let \( \gamma, \gamma' \) be loops at \( x_0 \). Let \( \alpha \) be any path from \( x_0 \) to \( x_1 \) and define \( \beta = \gamma' \cdot \alpha \).

Then \( \bar{\alpha} \cdot \bar{\gamma} \cdot \alpha = \bar{\beta} \cdot \gamma \cdot \beta \)

\( \Rightarrow \beta \cdot (\bar{\alpha} \cdot \bar{\gamma} \cdot \alpha) = \bar{\beta} \cdot (\gamma' \cdot \alpha) \cdot \bar{\alpha} = \gamma \cdot \beta \cdot \bar{\alpha} \)

\( \Rightarrow \gamma' \cdot \gamma = \gamma' \cdot \alpha \cdot \bar{\alpha} = \beta \cdot (\bar{\alpha} \cdot \bar{\gamma} \cdot \alpha) = \gamma \cdot \beta \cdot \bar{\alpha} = \gamma (\gamma', \alpha) \cdot \alpha' \)

\( \Rightarrow \gamma' \cdot \gamma' = \gamma' \cdot \alpha \cdot \bar{\alpha} \cdot \bar{\gamma} \cdot \alpha \Rightarrow \gamma' \cdot \gamma' \)

So \([\gamma], [\gamma'] = [\gamma'], [\gamma'] \in \pi_1(\mathbb{X}, x_0)\).

2. (a) Let \( i: A \to X \) be the inclusion map \( i(a) = a \forall a \in A \). Then \( r \circ i = \text{id}_A \)

So \( \text{id} \pi_1(A, a_0) = (i_*)^{-1} \circ i_* \) where

\( \pi_1(A, a_0) \xrightarrow{i_*} \pi_1(X, a_0) \xrightarrow{r_*} \pi_1(A, a_0) \).

This means in particular that \( r_* \) is surjective since if \( [\gamma] \in \pi_1(A, a_0) \) then \( r_* (i_* [\gamma]) = [\gamma] \).

(b) Let \( X = \mathbb{D} \), the closed unit disk, and \( A = S^1 = \text{boundary of } X \).

Choose \( a_0 \in A \) and define \( r: X \to A \) by \( r(x) = a_0 \forall x \). Then \( r \) maps any loop in \( X \) to the constant loop at \( a_0 \), so \( r_*: \pi_1(X, a_0) \to \pi_1(A, a_0) \)

sends everything to the identity element. However, \( \pi_1(A, a_0) \cong \mathbb{Z} \), so \( r_* \) is not surjective.
3. Let \( h: \mathbb{R}^n \to Y \) be the inclusion map. If \( h \) is extendable to \( \tilde{h}: \mathbb{R}^n \to Y \), then \( h = \tilde{h} \circ i: A \to Y \), so \( h_* = (\tilde{h})_* \circ i_* \), where
\[
\pi_1(\mathbb{R}^n, a) \xrightarrow{i_*} \pi_1(\mathbb{R}^n, a) \xrightarrow{\tilde{h}_*} \pi_1(Y, y_a).
\]
Now \( \pi_1(\mathbb{R}^n, a) \) is the trivial group so for any \( \gamma \in \pi_1(\mathbb{R}^n, a) \),
\[
h_*([\gamma]) = (\tilde{h})_* (i_* [\gamma]) = (\tilde{h})_* (i_*) = e.
\]

Let \( n = 2 \), \( A = S^1 \) = unit circle \( \subset \mathbb{R}^2 \), \( Y = S^1 \), \( h = \) identity map \( S^1 \to S^1 \).

Then \( h_*: \pi_1(S^1, a) \to \pi_1(S^1, a) \) is the identity map \( \mathbb{Z} \to \mathbb{Z} \), not the trivial map, so \( h \) isn't extendable to \( \mathbb{R}^2 \).

Note: (9) can be solved without using \( \pi_1 \), though it's not what I had in mind. For example: \( A = \mathbb{R} \cup 0 \subset \mathbb{R} \), \( Y = \mathbb{R} \),
\[
h: \mathbb{R} \to \mathbb{R} \text{ defined by } h(x) = \frac{1}{x} \quad x \neq 0,
\]
is continuous, but there's no way to extend \( h \) to a continuous map \( \mathbb{R} \to \mathbb{R}^2 \); however one define \( h(0) \),
\( h: \mathbb{R} \to \mathbb{R} \) won't be continuous.

4. Follow the hint. If \( h \) were homotopic to \( i_A \), rel \( \partial A \),
there would be a homotopy \( F: A \times I \to A \)
with \( F(x, 0) = x \), \( F(x, 1) = h(x) \) \( \forall x \in A \)
and \( F(a, t) = a \) \( \forall a \in \partial A \).

Then the map \( G: I \times I \to A \) defined by \( G(s, t) = F(\alpha(s), t) \)
satisfies \( G(s, 0) = \alpha(s) \), \( G(s, 1) = h(\alpha(s)) = \beta(s) \),
\( G(0, t) = F(\alpha(0), t) = \alpha(0) \), \( G(1, t) = F(\alpha(1), t) = \alpha(1) \).

So \( G \) shows that \( \alpha \) is path homotopic to \( \beta \).

Thus the loops \( \beta \cdot \alpha \) and \( \alpha \cdot \alpha \) at \( (s, 0) \) are path homotopic
\[
\Rightarrow [\beta \cdot \alpha] = [\alpha \cdot \alpha] = e \in \pi_1(A, a, 0).
\]
\[
\Rightarrow p_* [\beta \cdot \alpha] = e \in \pi_1(C, 1) \Rightarrow \text{in } \mathbb{Z} = \pi_1(C, 1), \ p_* [\beta \cdot \alpha] = 0.
\]
Now $p_*[p \cdot \alpha]$ is the homology class of $(p \circ \beta) \cdot (p \cdot \alpha)$, and $p \cdot \alpha$ is the counter-clockwise loop at $1$, while $p \circ \beta$ wraps around $C$ once:

$$(p \circ \beta)(s) = p\left( h(\alpha(s)) \right) = p\left( h(st^1, 0) \right) = p\left( s+1, 2\pi s \right) \overset{\text{in polar coord.}}{=} \exp(2\pi is).$$

So in $\pi_1(C, 1) \cong \mathbb{Z}$, $p_*[p \cdot \alpha] = [p \circ \beta] \cdot [p \cdot \alpha]$ is mapped to $1 + 0 = 1 \in \mathbb{Z}$, rather than 0, contradicting...