1. (25 points) For (a) and (b): one of the vector fields \( \vec{F} \) is conservative, one is not. If \( \vec{F} \) is conservative, find a function \( f(x, y, z) \) satisfying \( \vec{F} = \nabla f \); if not, explain why not.

(a) (10 points) \( \vec{F}(x, y, z) = (z/x, z + z/y, \ln(xy)) \), with domain \( \{(x, y, z) | x > 0, y > 0\} \)

One calculates \( \nabla \times \vec{F} = (-1, 0, 0) \neq \vec{0} \), and hence \( \vec{F} \) is not conservative. (Note that the domain of \( \vec{F} \) is simply connected, but this isn’t relevant!)

(b) (10 points) \( \vec{F}(x, y, z) = (2xe^y \cos(z + x^2), ze^y \sin(z + x^2), z + ze^y \cos(z + x^2) + e^y \sin(z + x^2)) \)

One calculates \( \nabla \times \vec{F} = \vec{0} \); since the domain of \( \vec{F}, \mathbb{R}^3 \), is simply connected, \( \vec{F} \) is conservative.

We now solve \( \vec{F} = \nabla f \) for \( f \). Integrate the \( x \) component of \( \vec{F} \) with respect to \( x \) to obtain

\[
f(x, y, z) = ze^y \sin(z + x^2) + f_1(y, z);
\]

now differentiate \( f \) with respect to \( y \) to obtain \( \partial f_1 / \partial y = 0 \), and so \( f_1(y, z) \) is just a function of \( z \); finally, differentiate \( f \) with respect to \( z \) to obtain \( df_1 / dz = z \), so \( f_1 = z^2/2 \). The final answer is \( f(x, y, z) = ze^y \sin(z + x^2) + z^2/2 \) (of course, an arbitrary constant can be added to this).

(c) (5 points) For \( \vec{F} \) from (b), evaluate \( \int_C \vec{F} \cdot d\vec{s} \), where \( C \) is the helix \((\frac{\pi}{2} \sin t, t, \frac{\pi}{2} \cos t)\), \( 0 \leq t \leq \pi/2 \).

Since \( \vec{F} = \nabla f \),

\[
\int_C \vec{F} \cdot d\vec{s} = f(C(\pi/2)) - f(C(0)) = f(\pi/2, \pi/2, 0) - f(0, 0, \pi/2) = \frac{\pi}{2} - \frac{\pi^2}{8}.
\]

2. (20 points) Let \( C \) be the piecewise smooth closed curve which traverses the boundary of the square \([0, 2] \times [-2, 0] \subset \mathbb{R}^2 \) clockwise. Let

\[
\vec{F}(x, y) = (5x - 3y, x + y).
\]

(a) (10 points) Evaluate \( \oint_C (\vec{F} \cdot \hat{n}) \, ds \).

Let \( C' \) be the same curve as \( C \) but oriented counterclockwise, and let \( D = [0, 2] \times [-2, 0] \). Then by the Divergence Theorem in the plane,

\[
\oint_{C'} (\vec{F} \cdot \hat{n}) \, ds = \iint_D (\nabla \cdot \vec{F}) \, dA = \iint_D 6 \, dA = 24.
\]
Since the unit normal vectors \( \hat{n} \) for \( C \) and \( C' \) differ by a sign, we conclude that
\[
\oint_C (\vec{F} \cdot \hat{n}) \, ds = -\oint_{C'} (\vec{F} \cdot \hat{n}) \, ds = -24.
\]

Note. For a scalar line integral like \( \oint_C (\vec{F} \cdot \hat{n}) \, ds \), the orientation of \( C \) doesn’t matter, so we don’t get an extra \(-\) sign from that. The \(-\) sign comes from the fact that \( \hat{n} \) switches sign when the orientation of \( C \) is reversed.

(b) (10 points) Evaluate \( \oint_C (\vec{F} \cdot \vec{T}) \, ds \).

Note that \( \oint_C (\vec{F} \cdot \vec{T}) \, ds = \oint_C \vec{F} \cdot d\vec{s} = -\oint_{C'} \vec{F} \cdot d\vec{s} \). By Green’s Theorem,
\[
\oint_{C'} \vec{F} \cdot d\vec{s} = \oint_{C'} (5x - 3y) \, dx + (x + y) \, dy = \iint_D 4 \, dA = 16.
\]

It follows that \( \oint_C (\vec{F} \cdot \vec{T}) \, ds = -16 \).

3. (10 points) The parametric equations
\[
x = \sin^2 t, \quad y = \sin t \cos t
\]
for \( 0 \leq t \leq \pi \) determine a simple closed curve in \( \mathbb{R}^2 \). Find the area of the region bounded by this curve.

(Random fact that you doubtless already know: \( \int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C. \))

Let \( C \) be the closed curve given by the parametric equations, and let \( D \) be the region that it bounds. (It turns out that \( C \) is oriented the wrong way to calculate area, but that won’t matter.) Then
\[
\oint_C (-y \, dx + x \, dy) = \int_0^\pi (-\sin^2 t \cos^2 t - \sin^4 t) \, dt = -\int_0^\pi \sin^2 t \, dt = -\frac{\pi}{2}.
\]

The area of \( D \) is \( \frac{1}{2} \oint_C (-y \, dx + x \, dy) = \frac{\pi}{4} \).

Remark: the curve is actually just the circle \((x - 1/2)^2 + y^2 = 1/4\), which has radius 1/2 and thus area \( \pi/4 \).

4. (30 points) Let \( S \) denote the surface (a frustum of a cone) given by the part of \( z^2 = x^2 + y^2 \) satisfying \( 1 \leq z \leq 2 \). Orient \( S \) upwards (in the positive \( z \) direction).

(a) (10 points) Find the surface area of \( S \).

Parametrize \( S \) by \( x = s \cos t, y = s \sin t, z = s \), with \( 1 \leq s \leq 2 \) and \( 0 \leq t \leq 2\pi \).

The normal vector is \( \vec{N} = (-s \cos t, -s \sin t, s) \). Hence the surface area of \( S \) is
\[
\iint_S dS = \int_1^2 \int_0^{2\pi} \| \vec{N} \| \, dt \, ds = \int_1^2 \int_0^{2\pi} s \sqrt{2} \, dt \, ds = 3\pi \sqrt{2}.
\]
(b) (10 points) Find the average $z$-coordinate of points in $S$.

We calculate

\[
\int_S z \, dS = \int_1^2 \int_0^{2\pi} s |\vec{N}| \, dt \, ds = \int_1^2 \int_0^{2\pi} s^2 \sqrt{2} \, dt \, ds = \frac{14\pi \sqrt{2}}{3}.
\]

Thus $\bar{z} = \left( \frac{\int_S z \, dS}{\int_S dS} \right) = \frac{14}{19}$. 

(c) (10 points) Let $S'$ be the surface $z = 2$, $x^2 + y^2 \leq 4$; note that $S'$ shares a circle boundary with $S$. Orient $S'$ so as to be consistent with the orientation of $S$. Finally, let $S''$ be the union of $S$ and $S'$. Find the flux through $S''$ of \[ \vec{F} = x \hat{i} + y \hat{j} + \hat{k}. \]

Note that our parametrization of $S$ has the correct orientation; hence

\[
\iint_S \vec{F} \cdot d\vec{S} = \int_1^2 \int_0^{2\pi} (s \cos t, s \sin t, 1) \cdot \vec{N} \, dt \, ds = \int_1^2 \int_0^{2\pi} (s - s^2) \, dt \, ds = -\frac{5\pi}{3}.
\]

Viewed from above, the orientation on $S$ induces a counterclockwise orientation on the boundary circle $z = 2$, $x^2 + y^2 = 4$. The compatible orientation on $S'$ must induce the opposite orientation on this circle; it follows that $S'$ is oriented downwards.

A parametrization $X$ for $S'$ is given by $x = s, y = t, z = 2$ for $(s, t)$ in the disk $s^2 + t^2 \leq 4$. The normal vector for this parametrization is $\hat{k}$, which points the wrong way. Nevertheless, for this parametrization, we get

\[
\iint_X \vec{F} \cdot d\vec{S} = \iint_X \vec{F} \cdot \hat{n} \, dS = \iint_X dS = \text{Area}(S') = 4\pi.
\]

With $S'$ oriented downwards, we thus have $\iint_{S'} \vec{F} \cdot d\vec{S} = -4\pi$. Adding everything together gives

\[
\iint_{S''} \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot d\vec{S} + \iint_{S'} \vec{F} \cdot d\vec{S} = -\frac{17\pi}{3}.
\]

5. (15 points) Let $C$ be the “castle-shaped” path in $\mathbb{R}^2$ consisting of straight line segments connecting the following points, in order: $(0, 0), (0, 2), (1, 2), (1, 1), (2, 1), (2, 2), (3, 2), (3, 0)$. (Thus $C$ begins at $(0, 0)$ and ends at $(3, 0)$.) Compute

\[
\int_C y \sin(xy) \, dx + (x \sin(xy) + 5x) \, dy.
\]
(Hint: there’s an easier solution than computing seven line integrals.)

Let $C'$ be the straight-line path from $(0, 0)$ to $(3, 0)$. The union of $C$ and $C'$ bounds a rectilinear region $D$ with area 5, and the counterclockwise-oriented boundary of $D$ consists of $C'$ along with the orientation reverse of $C$.

Write $M = y \sin(xy)$ and $N = x \sin(xy) + 5x$. By Green’s Theorem,

$$\int_{C'} M \, dx + N \, dy - \int_C M \, dx + N \, dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA = \iint_D 5 \, dA = 25.$$

Parametrize $C'$ as $\vec{x}(t) = (t, 0)$, $0 \leq t \leq 3$; then

$$\int_{C'} M \, dx + N \, dy = \int_0^3 ((0)(1) + (5t)(0)) \, dt = 0.$$

It follows that $\int_C M \, dx + N \, dy = -25$. 

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