Math 103X.02, Test 2—Solutions
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1. (10 points) Evaluate the following limit, or show that it does not exist:
\[
\lim_{(x,y)\to(0,0)} \frac{x^3 - xy + y^3}{x^2 + y^2}
\]

Solution 1. Use polar coordinates \(x = r \cos \theta, y = r \sin \theta\). Then
\[
\frac{x^3 - xy + y^3}{x^2 + y^2} = r(\cos \theta + \sin \theta) - \cos \theta \sin \theta.
\]
As \(r \to 0\), this expression approaches \(-\cos \theta \sin \theta\), which depends on \(\theta\). Hence the limit does not exist.

Solution 2. Let \(f(x, y) = \frac{x^3 - xy + y^3}{x^2 + y^2}\). Since \(f(x, 0) = x\), \(f(x, y) \to 0\) as \((x, y)\) approaches \((0, 0)\) along the \(x\) axis. Since \(f(x, x) = x - \frac{1}{2}\), \(f(x, y) \to -\frac{1}{2}\) as \((x, y)\) approaches \((0, 0)\) along the line \(y = x\). It follows that the limit does not exist.

2. (15 points) Find an approximate value for \(e^{(0.9)^2 - 1} \cos(0.2)\) by using the second order Taylor polynomial for \(e^{x^2 - 1} \cos y\) at \((1, 0)\).

Let \(f(x, y) = e^{x^2 - 1} \cos y\). One calculates \(f_x = 2xe^{x^2 - 1} \cos y, f_y = -e^{x^2 - 1} \sin y, f_{xx} = (2 + 4x^2)e^{x^2 - 1} \cos y, f_{xy} = -2xe^{x^2 - 1} \sin y, f_{yy} = -e^{x^2 - 1} \cos y\), and so \(f(1, 0) = 1, f_x(1, 0) = 2, f_y(1, 0) = 0, f_{xx}(1, 0) = 6, f_{xy}(1, 0) = 0, f_{yy}(1, 0) = -1\). The second order Taylor polynomial for \(f(x, y)\) at \((1, 0)\) is
\[
p_2(x, y) = 1 + 2(x - 1) + 3(x - 1)^2 - \frac{y^2}{2}.
\]
Plugging in \((x, y) = (0.9, 0.2)\) yields \(f(0.9, 0.2) \approx p_2(0.9, 0.2) = 0.81\). For comparison, the actual value of \(e^{(0.9)^2 - 1} \cos(0.2)\) is 0.810475 . . .

3. (25 points) Consider the function \(f(x, y) = xy - 2x - y + 1\).

(a) (5 points) Find the total differential df and the derivative matrix \(Df\).

Since \(f_x = y - 2\) and \(f_y = x - 1\), \([df = (y - 2)dx + (x - 1)dy]\) and \([Df = [y - 2 \ sbar \ x - 1]]\).

(b) (5 points) Find the maximum possible value for \(D\vec{v}f(5, 5)\), where \(\vec{v}\) ranges over all unit vectors in \(\mathbb{R}^2\).

This is \(\|\vec{v} f(5, 5)\| = \|(4, 3)\| = 5\).
(c) (5 points) Find all critical point(s) of \( f(x, y) \) in \( \mathbb{R}^2 \), if any. For each, determine whether it is a local maximum, local minimum, or saddle.

Any critical point of \( f \) satisfies \( 0 = f_x = y - 2 \) and \( 0 = f_y = x - 1 \), so the unique critical point is \((1, 2)\). The Hessian at \((1, 2)\) is \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
which has determinant \(-1\), and hence \((1, 2)\) is a saddle.

(d) (10 points) Find the maximum and minimum values for \( f(x, y) \) over the closed region in \( \mathbb{R}^2 \) bounded by the \( x \) and \( y \) axes and the line \( x + y = 7 \).

Since the one critical point of \( f(x, y) \), which lies inside the region, is a saddle, it cannot be either an absolute minimum or an absolute maximum. Thus the absolute extrema must occur on the boundary of the triangular region. On the \( x \) axis, \( f(x, 0) = -2x + 1 \) has no critical points; on the \( y \) axis, \( f(0, y) = -y + 1 \) has no critical points; on the third side of the triangle, \( f(x, 7-x) = -x^2 + 6x - 6 \) has one critical point at \( x = 3 \) (and \( y = 4 \)). So the possible points where \( f(x, y) \) could achieve an absolute extremum are \((3, 4)\) along with the vertices of the triangle, \((0, 0)\), \((7, 0)\), and \((0, 7)\). Since \( f(3, 4) = 3 \), \( f(0, 0) = 1 \), \( f(7, 0) = -13 \), and \( f(0, 7) = -6 \), the maximum and minimum values are 3 and \(-13\), respectively.

4. (25 points) Consider the surface \( S \) in \( \mathbb{R}^3 \) defined by \( x^2 + y^2 = z + z^3 \).

(a) (5 points) Find the equation for the tangent plane to \( S \) at the point \((1, -1, 1)\).

Write \( F(x, y, z) = x^2 + y^2 - z - z^3 \). The normal to \( S \) at \((1, -1, 1)\) is \( \nabla F(1, -1, 1) = (2, -2, -4) \). It follows that the tangent plane is given by \( 2(x - 1) - 2(y + 1) - 4(z - 1) = 0 \), or \( x - y - 2z = 0 \).

(b) (5 points) Explain why one can implicitly write \( z \) as a function of \( x \) and \( y \) for points \((x, y, z)\) on \( S \) near \((1, -1, 1)\).

By the Implicit Function Theorem, \( z \) is implicitly a function of \( x \) and \( y \) near \((1, -1, 1)\) if \( F_z(1, -1, 1) \neq 0 \). Since \( F_z(1, -1, 1) = -4 \), the result follows.

For the remainder of this problem, consider \( z \) as an implicit function \( z = z(x, y) \) for \((x, y, z)\) on \( S \) near \((1, -1, 1)\).

(c) (5 points) At \((1, -1, 1)\), calculate \( \frac{\partial z}{\partial y} \).

We have \( \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{2y}{1 + 3z^2} \).

At \((1, -1, 1)\), \( \frac{\partial z}{\partial y} = -\frac{1}{2} \).

(d) (10 points) At \((1, -1, 1)\), calculate \( \frac{\partial^2 z}{\partial y^2} \).
Write \( u = u(x, y, z) = \frac{\partial z}{\partial y} = \frac{2y}{1 + 3z^2}; \) we can view \( u \) as a function of only \( x \) and \( y \) by using the implicit expression \( z = z(x, y) \). We wish to calculate \( \left( \frac{\partial u}{\partial y} \right)_x \). By the Chain Rule,

\[
\left( \frac{\partial u}{\partial y} \right)_x = \left( \frac{\partial u}{\partial y} \right)_{x, z} + \left( \frac{\partial u}{\partial z} \right)_{x, y} \left( \frac{\partial z}{\partial y} \right)_x = \frac{2}{1 + 3z^2} - \frac{12yz}{(1 + 3z^2)^2} \frac{2y}{1 + 3z^2}.
\]

At \((1, -1, 1)\), we have \( \frac{\partial^2 z}{\partial y^2} = \left( \frac{\partial u}{\partial y} \right)_x = \frac{1}{8} \).

Note. An alternative solution (equivalent, but possibly easier to remember) for (c) and (d) would proceed as follows: We are given \( x^2 + y^2 = z + z^3 \). Take the partial derivative of both sides with respect to \( y \) to obtain

\[ 2y = z_y + 3z^2 z_y = (1 + 3z^2)z_y. \]

Plugging in \( y = -1, z = 1 \) yields \( z_y = -\frac{1}{2} \) at \((1, -1, 1)\). If we take a further partial derivative with respect to \( y \), we get

\[ 2 = \frac{\partial (1 + 3z^2)}{\partial y} z_y + (1 + 3z^2) z_{yy} = 6z_y^2 + (1 + 3z^2) z_{yy}. \]

Plugging in \( y = -1, z = 1 \) now yields \( z_{yy} = \frac{1}{8} \) at \((1, -1, 1)\).

5. (25 points) Let \( C \) denote the intersection of the surfaces \( x + y + z = 12 \) and \( y = x^2 + z^2 \).

(a) (15 points) Find the points on \( C \) with the maximal and minimal \( y \) coordinates. (You may assume that an absolute maximum and minimum exist.)

We wish to find the extrema for the function \( y \) given the constraints \( x + y + z = 12 \) and \( y - x^2 - z^2 = 0 \). Using Lagrange multipliers, we find that at an extremum, there exist scalars \( \lambda \) and \( \mu \), not both 0, such that \((0, 1, 0) = \lambda(1, 1, 1) + \mu(-2, 1, -2z) \). This implies that \( \lambda = 2\mu x = 2\mu z \) (and \( 1 = \lambda + \mu \), but we won’t need this relation). It follows that either \( \mu = 0 \) or \( x = z \). If \( \mu = 0 \), then \( \lambda = 2\mu x = 0 \), which is not allowed. Thus \( x = z \).

Substituting into the constraints \( x + y + z = 12 \) and \( y - x^2 - z^2 = 0 \), we find that \( y = 12 - 2x \) and \( y = 2x^2 \), so \( 2x^2 + 2x - 12 = 0 \). The roots of this equation are \( x = -3 \) and \( x = 2 \), leading to the points \((x, y, z) = (-3, 18, -3) \) and \((2, 8, 2) \). It follows that \((-3, 18, -3) \) and \((2, 8, 2) \) have the maximal and minimal \( y \) coordinates, respectively, on \( C \).

(b) (10 points) A particle is moving along \( C \), with its position at time \( t \) given by \((x(t), y(t), z(t))\), in such a way that \( x(0) = -4, y(0) = 16, \) and \( y'(0) = 1. \) Let

\[ f(t) = (x(t))^2 + (y(t))^2 + (z(t))^2 \]

be the square of the distance between the particle and the origin, at time \( t \). Find \( f'(0) \).
Solution 1. Write \( \vec{x}(t) = (x(t), y(t), z(t)) \). Note that since \((x(0), y(0), z(0))\) lies on the plane \(x + y + z = 12\), we have \(z(0) = 0\).

We next determine \( \vec{x}'(0) = (x'(0), y'(0), z'(0)) \). This vector is tangent to \(C\). Normal vectors to the surfaces \(x + y + z = 12\) and \(y - x^2 - z^2 = 0\) at \((-4, 16, 0)\) are given by the respective gradients: \((1, 1, 1)\) and \((-2x, 1, -2z) = (8, 1, 0)\), respectively. The cross product of these two vectors, \((-1, 8, -7)\), is perpendicular to both normals and thus tangent to \(C\). It follows that \(\vec{x}'(0)\) is parallel to \((-1, 8, -7)\).

Write \(f(x, y, z) = x^2 + y^2 + z^2\), so that \(f(x(t), y(t), z(t))\). Then by the Chain Rule, \(f'(0)\) is the directional derivative of \(f\) at \(\vec{x}(0)\) in the direction of \(\vec{x}'(0)\):

\[
f'(0) = \nabla f(\vec{x}(0)) \cdot \vec{x}'(0) = (-8, 32, 0) \cdot (-1/8, 1, -7/8) = 33
\]

Solution 2. As in Solution 1, write \(\vec{x}(t) = (x(t), y(t), z(t))\); then \(\vec{x}(0) = (-4, 16, 0)\). We are given 

\[
(0, 1, 0) \cdot \vec{x}'(0) = y'(0) = 1.
\]

In addition, since the path \(\vec{x}(t)\) lies entirely in the surface \(y - x^2 - z^2 = 0\), it is perpendicular at \(\vec{x}(0)\) to the normal vector to \(y - x^2 - z^2 = 0\), given by the gradient \((-2x, 1, -2z) = (8, 1, 0)\). It follows that

\[
(8, 1, 0) \cdot \vec{x}'(0) = 0.
\]

By the Chain Rule, \(f'(0) = \nabla f(\vec{x}(0)) \cdot \vec{x}'(0) = (-8, 32, 0) \cdot \vec{x}'(0)\). But \((-8, 32, 0) = 33(0, 1, 0) - (8, 1, 0)\); thus

\[
f'(0) = (-8, 32, 0) \cdot \vec{x}'(0) = 33(0, 1, 0) \cdot \vec{x}'(0) - (8, 1, 0) \cdot \vec{x}'(0) = 33.
\]