# Math 103X.02, Test 1—Solutions

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1. (20 points) Let $P = (2, 0, 1)$, $Q = (2, 1, 2)$, $R = (1, -2, 2)$, and $S = (1, -1, 0)$ be four points in $\mathbb{R}^3$.

   (a) (5 points) Find the coordinates of the point $U$ such that $PQRU$ is a parallelogram (with the vertices in that order!).

   We need $\overrightarrow{QP} = \overrightarrow{RU}$ and so $\vec{U} = \vec{P} - \vec{Q} + \vec{R} = \boxed{(1, -3, 1)}$.

   (b) (5 points) Find the coordinates of the midpoint of the line segment $\overline{PR}$.

   The midpoint is $\frac{1}{2}(\vec{P} + \vec{R}) = \boxed{(3/2, -1, 3/2)}$.

   (c) (5 points) Find the angle $\angle QPR$, i.e., the angle (somewhere between $0$ and $\pi$) between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

   Since $\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\| \cos(\angle QPR) = \overrightarrow{PQ} \cdot \overrightarrow{PR}$, it follows that $\angle QPR = \boxed{\cos^{-1}\left(-\frac{1}{\sqrt{12}}\right)}$.

   (d) (5 points) Do $P$, $Q$, $R$, and $S$ lie in a plane? If so, find an equation for the plane (in the form $Ax + By + Cz = D$). If not, find the volume of the parallelepiped generated by $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$.

   $P$, $Q$, $R$, and $S$ are coplanar precisely if the parallelepiped generated by $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ has volume $0$. The volume of this parallelepiped is the absolute value of the triple product

   $$(\overrightarrow{PQ} \times \overrightarrow{PR}) \cdot \overrightarrow{PS} = \begin{vmatrix} 0 & 1 & 1 \\ -1 & -2 & 1 \\ -1 & -1 & -1 \end{vmatrix} = -3.$$

   It follows that $\boxed{P, Q, R, \text{ and } S \text{ are not coplanar}}$ and the volume of the parallelepiped is $\boxed{3}$.

2. (10 points) An object $\vec{x}(t)$ moves in $\mathbb{R}^2$ in such a way that its acceleration satisfies

   $$\vec{a}(t) = (6, 6).$$

   At $t = 0$, the object is at $P_0 = (0, 0)$; at $t = 1$, the object is at $P_1 = (3, 1)$.

   (a) (5 points) Find $\vec{x}(t)$ for all $t$.

   The velocity of the object is $\vec{v}(t) = \int \vec{a}(t)\,dt = (6t, 6t) + \vec{v}_0$ for some $\vec{v}_0$, and the position is $\vec{x}(t) = \int \vec{v}(t)\,dt = (3t^2, 3t^2) + t\vec{v}_0 + \vec{x}_0$ for some $\vec{x}_0$. Since $\vec{x}(0) = (0, 0)$ and $\vec{x}(1) = (3, 1)$, it follows that $\vec{x}_0 = 0$ and $\vec{v}_0 = (0, -2)$; hence $\boxed{\vec{x}(t) = (3t^2, 3t^2 - 2t)}$.

(b) (5 points) It is a consequence of the Mean Value Theorem that, somewhere along the path of the object between $P_0$ and $P_1$, the velocity of the object is parallel to the vector $\overrightarrow{P_0P_1}$. Find this point.

We want to find the point at which the velocity vector $\vec{v}(t) = (6t, 6t-2)$ is proportional to $\overrightarrow{P_0P_1} = (3,1)$, that is, $(6t, 6t-2) = (3k, k)$ for some $k$. This happens when $6t = 3(6t-2)$, or $t = 1/2$. The point is $\vec{x}(1/2) = \boxed{(3/4, -1/4)}$.

3. (25 points) Consider the lines $\ell_1$ given by $\vec{x}(t) = t(-2,2,1) + (2,2,2)$ and $\ell_2$ given by $\vec{x}(t) = t(0,1,1) + (3,-1,4)$.

(a) (5 points) Let $\mathbf{\Pi}_1$ be the plane perpendicular to $\ell_1$ and passing through $(3,-1,4)$. Find an equation of the form $Ax + By + Cz = D$ for $\mathbf{\Pi}_1$.

A normal vector to $\mathbf{\Pi}_1$ is given by the direction vector of $\ell_1$, $(-2,2,1)$. The equation is $\boxed{-2x + 2y + z = -4}$.

(b) (5 points) Let $\mathbf{\Pi}_2$ be the plane parallel to $\ell_2$ and passing through the points $(1,1,1)$ and $(3,0,6)$. Find a set of *parametric* equations for $\mathbf{\Pi}_2$.

$\mathbf{\Pi}_2$ passes through $(1,1,1)$ and is parallel to the vectors $(0,1,1)$ (since it is parallel to $\ell_2$) and $(3,0,6) - (1,1,1) = (2,-1,5)$. Points on the plane are of the form $(x,y,z) = (1,1,1) + s(0,1,1) + t(2,-1,5)$, or $\boxed{x = 1 + 2t,\ y = 1 + s - t,\ z = 1 + s + 5t}$. (Other answers are possible.)

(c) (10 points) Calculate the distance between $\ell_1$ and $\ell_2$.

A normal vector to both $\ell_1$ and $\ell_2$ is $\vec{n} = (-2,2,1) \times (0,1,1) = (1,2,-2)$. Points $P_1 = (2,2,2)$ and $P_2 = (3,-1,4)$ lie on $\ell_1$ and $\ell_2$, respectively, and the desired distance is $\|\mathrm{proj}_{\vec{n}} \overrightarrow{P_1P_2}\| = \frac{|(1,2,-2)\cdot(1,-3,2)|}{\|(1,2,-2)\|} = \boxed{3}$.

(d) (5 points) Are the lines $\ell_1$ and $\ell_2$ intersecting, parallel, or skew? Explain.

The distance between the two lines is nonzero so they don't intersect; they point in different directions so they aren't parallel. Thus they are $\boxed{\text{skew}}$.

4. (30 points) Consider the path $\vec{x}(t) = (3\sin(t^2), -4t^2, 3\cos(t^2))$.

(a) (5 points) Calculate $\vec{v}$ at time $t = \sqrt{\pi}$.

$\vec{v} = (6t\cos t^2, -8t, -6t\sin t^2)$ so the answer is $\boxed{(-6\sqrt{\pi}, -8\sqrt{\pi}, 0)}$.

(b) (5 points) Calculate the arclength of the path between times $t = 0$ and $t = \sqrt{\pi}$.

$\|\vec{v}\| = 10t$ (for $t > 0$). The arclength is $\int_0^{\sqrt{\pi}} 10t\,dt = \boxed{5\pi}$.

(c) (5 points) Calculate $\vec{T}$ and $\vec{N}$ at time $t = \sqrt{\pi}$.

$\vec{T} = \vec{v}/\|\vec{v}\| = (\frac{3}{5}\cos t^2, -\frac{4}{5}, -\frac{3}{5}\sin t^2)$; $d\vec{T}/dt = (-\frac{6t}{5}\sin t^2, 0, -\frac{6t}{5}\cos t^2)$; $\|d\vec{T}/dt\| = \frac{6t}{5}$; $\vec{N} = \frac{d\vec{T}/dt}{\|d\vec{T}/dt\|} = (-\sin t^2, 0, -\cos t^2)$. At $t = \sqrt{\pi}$, $\boxed{\vec{T} = (-3/5, -4/5, 0) \text{ and } \vec{N} = (0,0,1)}$.

(d) (5 points) Find the curvature $\kappa$ at time $t = \sqrt{\pi}$.

$\kappa = \frac{\|d\vec{T}/dt\|}{\|\vec{v}\|} = \boxed{3/25}$.

(e) (5 points) Find the radius of the osculating circle at time $t = \sqrt{\pi}$.

The radius is $1/\kappa = \boxed{25/3}$.

(f) (5 points) The **normal plane** to a path $\vec{x}(t)$ at time $t_0$ is the plane through the path at $\vec{x}(t_0)$ parallel to the vectors $\vec{N}$ and $\vec{B}$. Find an equation of the form $Ax + By + Cz = D$ for the normal plane to the given path $\vec{x}(t)$ at time $t = \sqrt{\pi}$.

The normal plane passes through $(0, -4\pi, -3)$ and is normal to $\vec{T} = (-3/5, -4/5, 0)$ (since it is parallel to $\vec{N}$ and $\vec{B}$). Its equation is $\boxed{3x + 4y = -16\pi}$.

5. (15 points)

(a) (10 points) Suppose that the vector $\vec{a} \in \mathbb{R}^3$ satisfies

$$\vec{a} \times (0, 0, -2) = (-4, 3, 0).$$

Find the minimum possible value for $\|\vec{a}\|$. Justify your answer.

If $\theta$ is the angle between $\vec{a}$ and $(0, 0, -2)$, then

$$5 = \|(-4, 3, 0)\| = \|\vec{a} \times (0, 0, -2)\| = 2\|\vec{a}\| \sin\theta \leq 2\|\vec{a}\|$$

and hence $\|\vec{a}\| \geq 5/2$. Equality is attained when $\theta = \pi/2$, and so the minimum value is $\boxed{5/2}$.

*Note 1:* The locus of possible vectors $\vec{a}$ satisfying $\vec{a} \times (0, 0, -2) = (-4, 3, 0)$ is a line perpendicular to $(0, 0, -2)$ in the plane through the origin perpendicular to $(-4, 3, 0)$; more precisely, it is given by $\{(2, -3/2, 0) + t(0, 0, 1) \mid t \in \mathbb{R}\}$.

*Note 2:* The problem as originally stated used $(1, 0, -2)$ instead of $(0, 0, -2)$. In this case, no such vector $\vec{a}$ exists because $(1, 0, -2)$ is not orthogonal to $(-4, 3, 0)$.

(b) (5 points) Suppose that $\vec{a}$ and $\vec{b}$ are vectors in $\mathbb{R}^2$ such that

$$\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$$

for all vectors $\vec{c} \in \mathbb{R}^2$. Prove that $\vec{a} = \vec{b}$.

*Solution 1.* Write $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$. Successively substituting $\vec{c} = \vec{\imath}$, $\vec{c} = \vec{\jmath}$, and $\vec{c} = \vec{k}$ into the given equation yields $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$. The result follows.

*Solution 2.* Substituting $\vec{c} = \vec{a} - \vec{b}$ into the given equation, we find that $\|\vec{a} - \vec{b}\|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$. It follows that $\vec{a} - \vec{b} = \vec{0}$, as desired.

*Note:* The same result, with a similar proof, applies if $\mathbb{R}^2$ is replaced by $\mathbb{R}^n$ for any $n$.