Math 103X.02—Review problems for Test 1

Instructor: Lenny Ng

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Test 1, on September 20, will cover Sections 1.1–1.5 (not torsion), 3.1 (not Kepler's Laws), and 3.2 of Colley. The test is closed book, no calculators allowed. The problems will be fairly similar to homework problems, so one good way to prepare is to make sure you understand how to do everything on the homework assignments.

1. Let ABC be a triangle, and let A', B', C' be the midpoints of BC, AC, AB, respectively. Show that AA', BB', CC' meet in a point ("the medians are concurrent") and that their common intersection point lies two-thirds of the way from A to A', from B to B', and from C to C'. For a more challenging problem, you can also use vectors to show that the three altitudes of a triangle are concurrent.

2. (a) Let $\vec{a}$, $\vec{b}$, $\vec{c}$ denote (the vectors corresponding to) three points in space. What does it mean geometrically if $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$? What if $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ instead?

   (b) Describe geometrically the direction of the vector $\vec{a} \times (\vec{b} \times \vec{c})$. Do the same for $(\vec{a} \times \vec{b}) \times \vec{c}$.

3. Let $P_1 = (3, -2, 0)$, $P_2 = (4, -3, -1)$, $P_3 = (5, 0, 1)$.

   (a) Find the area of the triangle $P_1P_2P_3$.

   (b) Calculate the lengths of P_1P_2 and of P_1P_3.

   (c) Let θ be the angle $P_1P_2P_3$. Calculate $\cos\theta$. Deduce $\sin\theta$ from this and use this to check your answer to (a).

   (d) Find an equation for the plane through P_1 perpendicular to the line segment P_2P_3. Also find a unit vector normal to this plane.

   (e) Find parametric equations for the line through P_2 and P_3. Where does this line intersect the plane from (d)?

   (f) Using (e), deduce the distance from P_1 to the line through P_2 and P_3. Now use this distance to check (once again) the answer to (a).

4. Show that an equation for the plane passing through three points (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3) is

$$\begin{vmatrix} a_1 - x & a_2 - y & a_3 - z \\ b_1 - x & b_2 - y & b_3 - z \\ c_1 - x & c_2 - y & c_3 - z \end{vmatrix} = 0.$$

   This equation looks like it should contain quadratic and cubic terms (e.g., xy and xyz), but these terms cancel out!

5. (a) Find an equation of the form $Ax+By+Cz = D$ for the plane Π_1 passing through the point $(6, 6, -3)$ and parallel to the plane Π_2 given by $x + 2y + 2z = 0$.
   (b) Find a set of parametric equations for Π_1.
   (c) Find the distance between Π_1 and Π_2. Why is this the same as the distance between the point $(6, 6, -3)$ and Π_2?

6. (a) Find parametric equations for the line perpendicular to the plane $x+2y-2z = 3$ and containing the point $(0, 1, 2)$. Write this line also in vector parametric form and symmetric form.
   (b) Find an equation for the plane perpendicular to the plane $x + 2y - 2z = 3$ and containing the points $(0, 1, 2)$ and $(-1, 0, 5)$. Show that this plane contains the line from (a).

7. (Colley §1.5 #25) Find the distance between the two lines $\mathbf{l}_1(t) = t(3, 1, 2) + (4, 0, 2)$ and $\mathbf{l}_2(t) = t(1, 2, 3)+(2, 1, 3)$. Do these lines intersect? If so, what is their intersection point?

8. Let L_1 denote the line $(x - 4)/3 = y = (z - 2)/2$ and let L_2 denote the line $x + 2 = (y - 1)/2 = (z - 3)/3$.
   (a) Find parametric equations for L_1 and L_2.
   (b) Determine whether L_1 and L_2 intersect, are parallel, or are skew.
   (c) Compute the distance between L_1 and L_2.
   (d) Find an equation (of the form $Ax + By + Cz = D$) for: the plane Π_1 containing L_1 and parallel to L_2; the plane Π_2 containing L_2 and parallel to L_1.
   (e) Find an equation for the plane Π_3 parallel to both L_1 and L_2 and equidistant (at the same distance) from both.

9. (from last year's test in Professor Witelski's section) A curve for $t \geq 0$ is given by

$$\vec{x}(t) = \sin(\pi t)\hat{\imath} + \pi t^2\hat{\jmath} + \cos(\pi t)\hat{k}.$$

   (a) Find $\vec{v}(t)$, $\vec{a}(t)$.
   (b) Write the integral for the arclength of the curve from $t = 0$ to $t = T$ in simplest form. Do not integrate.
   (c) Find the point on the curve where the curve is parallel to the line $4x - 8 = y + 1$, $z = 4$.
   (d) Evaluate the integral $\int_0^T \vec{v}(t)\, dt$.
   (e) Find $\vec{v}$, $\vec{a}$ at $t = 1$.
   (f) Find $\vec{T}$, $\vec{N}$, a_T, a_N at $t = 1$.
   (g) Find the radius and center of the osculating circle at $t = 1$.