Test 1, on September 20, will cover Sections 1.1–1.5 (not torsion), 3.1 (not Kepler’s Laws), and 3.2 of Colley. The test is closed book, no calculators allowed. The problems will be fairly similar to homework problems, so one good way to prepare is to make sure you understand how to do everything on the homework assignments.

1. Let $ABC$ be a triangle, and let $A', B', C'$ be the midpoints of $BC$, $AC$, $AB$, respectively. Show that $AA'$, $BB'$, $CC'$ meet in a point (“the medians are concurrent”) and that their common intersection point lies two-thirds of the way from $A$ to $A'$, from $B$ to $B'$, and from $C$ to $C'$. For a more challenging problem, you can also use vectors to show that the three altitudes of a triangle are concurrent.

2. (a) Let $\vec{a}$, $\vec{b}$, $\vec{c}$ denote (the vectors corresponding to) three points in space. What does it mean geometrically if $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$? What if $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ instead?

(b) Describe geometrically the direction of the vector $\vec{a} \times (\vec{b} \times \vec{c})$. Do the same for $(\vec{a} \times \vec{b}) \times \vec{c}$.

3. Let $P_1 = (3, -2, 0)$, $P_2 = (4, -3, -1)$, $P_3 = (5, 0, 1)$.

(a) Find the area of the triangle $P_1P_2P_3$.

(b) Calculate the lengths of $P_1P_2$ and of $P_1P_3$.

(c) Let $\theta$ be the angle $P_1P_2P_3$. Calculate $\cos \theta$. Deduce $\sin \theta$ from this and use this to check your answer to (a).

(d) Find an equation for the plane through $P_1$ perpendicular to the line segment $P_2P_3$. Also find a unit vector normal to this plane.

(e) Find parametric equations for the line through $P_2$ and $P_3$. Where does this line intersect the plane from (d)?

(f) Using (e), deduce the distance from $P_1$ to the line through $P_2$ and $P_3$. Now use this distance to check (once again) the answer to (a).

4. Show that an equation for the plane passing through three points $(a_1, a_2, a_3)$, $(b_1, b_2, b_3)$, and $(c_1, c_2, c_3)$ is

$$\begin{vmatrix}
    a_1 - x & a_2 - y & a_3 - z \\
    b_1 - x & b_2 - y & b_3 - z \\
    c_1 - x & c_2 - y & c_3 - z
\end{vmatrix} = 0.$$

This equation looks like it should contain quadratic and cubic terms (e.g., $xy$ and $xyz$), but these terms cancel out!
5. (a) Find an equation of the form $Ax + By + Cz = D$ for the plane $\Pi_1$ passing through the point $(6, 6, -3)$ and parallel to the plane $\Pi_2$ given by $x + 2y + 2z = 0$.

(b) Find a set of parametric equations for $\Pi_1$.

(c) Find the distance between $\Pi_1$ and $\Pi_2$. Why is this the same as the distance between the point $(6, 6, -3)$ and $\Pi_2$?

6. (a) Find parametric equations for the line perpendicular to the plane $x + 2y - 2z = 3$ and containing the point $(0, 1, 2)$. Write this line also in vector parametric form and symmetric form.

(b) Find an equation for the plane perpendicular to the plane $x + 2y - 2z = 3$ and containing the points $(0, 1, 2)$ and $(-1, 0, 5)$. Show that this plane contains the line from (a).

7. (Colley §1.5 #25) Find the distance between the two lines $l_1(t) = t(3, 1, 2) + (4, 0, 2)$ and $l_2(t) = t(1, 2, 3) + (2, 1, 3)$. Do these lines intersect? If so, what is their intersection point?

8. Let $L_1$ denote the line $(x - 4)/3 = y = (z - 2)/2$ and let $L_2$ denote the line $x + 2 = (y - 1)/2 = (z - 3)/3$.

(a) Find parametric equations for $L_1$ and $L_2$.

(b) Determine whether $L_1$ and $L_2$ intersect, are parallel, or are skew.

(c) Compute the distance between $L_1$ and $L_2$.

(d) Find an equation (of the form $Ax + By + Cz = D$) for: the plane $\Pi_1$ containing $L_1$ and parallel to $L_2$; the plane $\Pi_2$ containing $L_2$ and parallel to $L_1$.

(e) Find an equation for the plane $\Pi_3$ parallel to both $L_1$ and $L_2$ and equidistant (at the same distance) from both.

9. (from last year’s test in Professor Witelski’s section) A curve for $t \geq 0$ is given by

$$\vec{x}(t) = \sin(\pi t)\hat{i} + \pi t^2\hat{j} + \cos(\pi t)\hat{k}.$$  

(a) Find $\vec{v}(t)$, $\vec{a}(t)$.

(b) Write the integral for the arclength of the curve from $t = 0$ to $t = T$ in simplest form. Do not integrate.

(c) Find the point on the curve where the curve is parallel to the line $4x - 8 = y + 1$, $z = 4$.

(d) Evaluate the integral $\int_0^T \vec{v}(t) \, dt$.

(e) Find $\vec{v}$, $\vec{a}$ at $t = 1$.

(f) Find $\vec{T}$, $\vec{N}$, $a_T$, $a_N$ at $t = 1$.

(g) Find the radius and center of the osculating circle at $t = 1$. 