Math 103X.02 Homework 3 Answers & Solutions

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§2.1: 14. Level curves are ellipses $4x^2 + 9y^2 = c$; the graph is an elliptic paraboloid (see Figure 2.22); 34. (a) $F(x, y, z) = x^2 + xy - xz - 2$ (other answers are possible), (b) $f(x, y) = \frac{x^2+xy-2}{x}$.

§2.2: 12. $6/5$; 28. 0; for # 13 and 23, see below.

§2.2, 13. Since $x^2 + 2xy + y^2 = (x + y)^2$, one is tempted to cancel $x + y$ from numerator and denominator to obtain $\lim_{(x,y)\to(0,0)}(x + y) = 0$, which agrees with the answer in the back of the book. This is, however, incorrect!

Recall the definition of limit: $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ means that for all $\epsilon > 0$, there is a $\delta > 0$ such that $|f(\vec{x}) - L| < \epsilon$ *for all* $\vec{x}$ satisfying the condition that $0 < \|\vec{x} - \vec{a}\| < \delta$. In particular, if the limit exists, then there must be some δ (corresponding to your favorite value of ϵ) such that $f(\vec{x})$ *exists* for all $\vec{x}$ with $0 < \|\vec{x} - \vec{a}\| < \delta$. Now for $f(x, y) = \frac{x^2+2xy+y^2}{x+y}$, $f(x, y)$ is *undefined* whenever $y = -x$. For any δ, the set of vectors (x, y) such that $0 < \|(x, y) - (0, 0)\| < \delta$ always contains a vector (x, y) with $y = -x$, for example $(\delta/2, -\delta/2)$; for such a vector, $f(x, y)$ is undefined. This contradicts a necessary condition for the limit to exist. We conclude that the desired limit *does not exist*.

§2.2, 23. Suppose that (x, y) approaches $(0, 0)$ along the straight line $y = mx$ for some fixed slope m. Then $f(x, y) = m^4x^8/(x^2 + m^4x^4)^3$ and

$$\lim_{x\to 0} \frac{m^4x^8}{(x^2 + m^4x^4)^3} = \lim_{x\to 0} \frac{m^4x^2}{(1 + m^4x^2)^3} = 0.$$

So it looks like the limit of $f(x, y)$ should be 0. In addition, if (x, y) instead approaches $(0, 0)$ along the y-axis (the line of slope ∞), then since $f(0, y) = 0$ whenever $y \neq 0$, $f(x, y)$ approaches 0 as well. In conclusion, $f(x, y) \to 0$ as (x, y) approaches $(0, 0)$ along *any* straight line.

On the other hand, $f(y^2, y) = 1/8$ for all $y \neq 0$; thus $f(x, y)$ is the constant $1/8$ as (x, y) approaches $(0, 0)$ along the parabola $x = y^2$. Since $f(x, y)$ approaches different values depending on which path to $(0, 0)$ is chosen, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

§2.3: 18. $(e^2 + 1, 2e^2 - 1)$; 24. $\begin{bmatrix} -4 & 4 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}$; 32. There are two: $4x - 12y + z = -17$ and $4x - 12y + z = 15$.

§2.4, 12:

$$\begin{aligned} f_{xx} &= \frac{y^2 \cos\sqrt{x^2+y^2} - x^2\sqrt{x^2+y^2}\sin\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}, \\ f_{xy} = f_{yx} &= \frac{-xy\sqrt{x^2+y^2}\sin\sqrt{x^2+y^2} - xy\cos\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}, \\ f_{yy} &= \frac{x^2 \cos\sqrt{x^2+y^2} - y^2\sqrt{x^2+y^2}\sin\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}. \end{aligned}$$

Extra problems:

1. We will show that $f(x,y) = c$ is continuous at an arbitrary point (a,b), that is, that $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. Let $\epsilon > 0$. We need to find $\delta > 0$ so that if (x,y) satisfies $0 < \|(x,y)-(a,b)\| < \delta$, then $|f(x,y) - f(a,b)| < \epsilon$. Since $f(x,y) = f(a,b) = c$, any δ will do. For instance, $\delta = 1$: if $0 < \|(x,y)-(a,b)\| < 1$ then $|f(x,y)-f(a,b)| = 0 < \epsilon$. Since there is a δ $(= 1)$ for any ϵ, the limit is $f(a,b)$, as desired.

2. We will show that $f(x,y) = x$ is continuous at an arbitrary point (a,b), that is, that $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. Let $\epsilon > 0$. We need to find $\delta > 0$ so that if (x,y) satisfies $0 < \|(x,y)-(a,b)\| < \delta$, then $|f(x,y)-f(a,b)| < \epsilon$. Now $|f(x,y)-f(a,b)| = |x-a|$ is the absolute value of the difference of the x coordinates of (x,y) and (a,b). Certainly if the distance between (x,y) and (a,b) is less than ϵ, then $|x-a|$ is less than ϵ as well. Hence we can set $\delta = \epsilon$: if $0 < \|(x,y)-(a,b)\| < \epsilon$ then $|f(x,y)-f(a,b)| = |x-a| < \epsilon$.

   (Why is this true? Note that

   $$\|(x,y)-(a,b)\| = \sqrt{(x-a)^2 + (y-b)^2} \geq \sqrt{(x-a)^2} = |x-a|.$$

   Thus if $\|(x,y)-(a,b)\| < \epsilon$, then $|x-a| \leq \|(x,y)-(a,b)\| < \epsilon$.)

   Since there is a δ $(= \epsilon)$ for any ϵ, the limit is $f(a,b)$, as desired.