Math 103X.02 Homework 2 Answers & Solutions

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§3.1, 24(a): The equation for projectile motion is $\vec{x}(t) = -\frac{1}{2}gt^2\hat{\jmath} + t\vec{v_0} + \vec{x_0}$. If Gertrude's feet are at the origin, then $\vec{x_0} = \hat{\jmath}$, while $\vec{v_0} = (7/\sqrt{2})(\hat{\imath} + \hat{\jmath})$ because of the initial speed and angle of the water. With $g = 9.8m/s^2$, the height of the water when it reaches Egbert ($x = 5$) is 1 meter, so Egbert does get wet. (Incidentally, the answer to (b) is between $11.2°$ and $34.2°$, or between $62.6°$ and $67.5°$.)

§3.2: 2. $\int_0^4 \sqrt{(2t)^2 + 4(2t+1)}\,dt = \int_0^4 (t+1)\,dt = 24$; 28. The speed is $s' = 1$ and so $s'' = 0$; also $\kappa = 1$. Tangential component $= 0$; normal component $= 1$.

Extra problems:

1. (a) The obvious path is $\vec{x}(\tilde{t}) = (r\cos\tilde{t}, r\sin\tilde{t}, 0)$, but this has speed r. To get speed v, we need to reparametrize. If t is another parameter then $d\vec{x}/dt = (d\vec{x}/d\tilde{t})(d\tilde{t}/dt)$ so we need $d\tilde{t}/dt = v/r$, or $\tilde{t} = vt/r$. (Note that we want $t = 0$ when $\tilde{t} = 0$.) So the desired function is $\vec{x}(t) = (r\cos(vt/r), r\sin(vt/r), 0)$.
   Note that this is actually one of two possible functions, the other one traveling clockwise rather than counterclockwise and given by $\vec{x}(t) = (r\cos(vt/r), -r\sin(vt/r), 0)$.
   (b) $v = \sqrt{GM/r}$.
   (c) The object completes an orbit of circumference $2\pi r$ in time $2\pi r/v = T$ since it travels at speed v. Thus $T = 2\pi r\sqrt{\frac{r}{GM}}$, which implies Kepler's Third Law.
2. The arclength is $\int_{t=0}^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$.
3. (a) $\vec{v} = (t\cos t, t\sin t, 0)$ and so $\vec{T} = (\cos t, \sin t, 0)$, $\vec{N} = (d\vec{T}/dt)/\|d\vec{T}/dt\| = (-\sin t, \cos t, 0)$, $\vec{B} = \vec{T} \times \vec{N} = (0, 0, 1)$. At $t = \pi$, $\vec{T} = (-1, 0, 0)$, $\vec{N} = (0, -1, 0)$, $\vec{B} = (0, 0, 1)$.
   Note that if $t < 0$, the expressions for $\vec{T}$ and $\vec{N}$ are different (why?) but $\vec{B}$ is still $(0, 0, 1)$. What happens at $t = 0$?
   (b) $\kappa = \|d\vec{T}/dt\|/\|\vec{v}\| = \|\vec{v} \times \vec{a}\|/\|\vec{v}\|^3 = 1/|t|$; thus at $t = \pi$, the curvature is $\kappa = 1/\pi$.
   (c) The osculating plane passes through $\vec{x}(\pi) = (-1, \pi, 2)$ and is normal to $\vec{B} = (0, 0, 1)$, so it is given by the equation $z = 2$.
   (d) The radius of the osculating circle is $1/\kappa = \pi$; its center is at $\vec{x} + \vec{N}/\kappa = (-1, 0, 2)$.
4. (a) $\vec{T} = (\frac{t}{\sqrt{1+t^2}}, 0, \frac{1}{\sqrt{1+t^2}}) = (1/\sqrt{2}, 0, 1/\sqrt{2})$, $\vec{N} = (\frac{1}{\sqrt{1+t^2}}, 0, -\frac{t}{\sqrt{1+t^2}}) = (1/\sqrt{2}, 0, -1/\sqrt{2})$, $\vec{B} = (0, 1, 0)$. Again note that the general formulas depend on t being positive; what happens if $t < 0$?
   (b) $\kappa = 1/(t(1+t^2)^{3/2}) = 1/(2\sqrt{2})$.
   (c) $y = 0$.
   (d) The radius is $2\sqrt{2}$ and the center is $(7/3, 0, -3/2)$.