1. (a) Find the vector valued function $\mathbf{x}(t)$ to describe an object in $\mathbb{R}^2$ traveling counterclockwise along the circle $x^2 + y^2 = r^2$ at constant speed $v$, with the object at $(r, 0)$ when $t = 0$.

(b) If this object represents a particle in gravitational orbit around a large body of mass $M$ at $(0, 0)$, then by Newton’s law of gravitation, the acceleration of the particle is given by $\mathbf{a} = -GM\mathbf{x}/||\mathbf{x}||^3$, where $G$ is the universal gravitational constant. Use this to derive an equation for $v$ in terms of $G$, $M$, and $r$.

(c) Let $T$ denote the period of the orbit (the amount of time it takes for the particle’s motion to repeat itself). Derive Kepler’s Third Law in this case: $T^2 = \left(\frac{4\pi^2}{GM}\right)r^3$.

2. The vector valued function $\mathbf{x}(t) = (a\cos t, b\sin t)$, $0 \leq t \leq 2\pi$, traces out an ellipse in the plane with axes (major and minor) of length $2a$ and $2b$. Write the integral for the arclength of this path (that is, the circumference of the ellipse) in terms of $a$ and $b$. Do not evaluate this integral; it has no closed form answer! This is a special case of a family of integrals now known as elliptic integrals.

3. For the path $\mathbf{x}(t) = (\cos t + t \sin t, \sin t - t \cos t, 2)$ and time $t_0 = \pi$:

(a) Calculate $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ at time $t = t_0$.

(b) Calculate $\kappa$ at time $t = t_0$ by using either formula for $\kappa$ (or, even better, use both and check your answer!).

(c) Find an equation for the osculating plane to the curve at time $t = t_0$.

(d) Find the radius of the osculating circle and the coordinates of its center, at time $t = t_0$.

4. Repeat the previous exercise for the path $\mathbf{x}(t) = (t^3/3)\mathbf{i} + (t^2/2)\mathbf{k}$ and $t_0 = 1$. 

Extra problems:

1. (a) Find the vector valued function $\mathbf{x}(t)$ to describe an object in $\mathbb{R}^2$ traveling counterclockwise along the circle $x^2 + y^2 = r^2$ at constant speed $v$, with the object at $(r, 0)$ when $t = 0$.

(b) If this object represents a particle in gravitational orbit around a large body of mass $M$ at $(0, 0)$, then by Newton’s law of gravitation, the acceleration of the particle is given by $\mathbf{a} = -GM\mathbf{x}/||\mathbf{x}||^3$, where $G$ is the universal gravitational constant. Use this to derive an equation for $v$ in terms of $G$, $M$, and $r$.

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