The augmentation category of a Legendrian knot

Lenny Ng

Duke University

Session on contact geometry and low-dimensional topology
AMS Western Sectional Meeting
April 19, 2015

A report on part of arXiv:1502.04939, joint with:
Dan Rutherford (Ball State), Vivek Shende (UC Berkeley),
Steven Sivek (Princeton), and Eric Zaslow (Northwestern).

One direction of motivation: $W$ exact symplectic manifold with convex contact boundary $V$.

Vague question: construct a Fukaya category from exact Lagrangians $L$ in $W$, cylindrical near boundary, in terms of just the boundary data $\Lambda \subset V$.

Perhaps fix the boundary condition $\Lambda \subset V$. 
The setting

Let $M$ be a smooth manifold of dimension $n$. We’ll work with the contact manifold

$$(V, \xi) = (J^1(M), \ker \alpha),$$

where $J^1(M) = T^*M \times \mathbb{R}$ and $\alpha = dz - \lambda$, with $\lambda$ the canonical Liouville 1-form on $T^*M$.

For $M = \mathbb{R}^n$,

$$V = J^1(\mathbb{R}^n) = \mathbb{R}^{2n+1}_{x_1, \ldots, x_n, y_1, \ldots, y_n, z}$$

$$\alpha = dz - \sum_{i=1}^{n} y_i \, dx_i.$$

A submanifold $\Lambda \subset V$ is **Legendrian** if $\alpha|_{\Lambda} \equiv 0$ and $\dim(\Lambda) = n$.

Particular case of interest: $n = 1$, $(V, \alpha) = (\mathbb{R}^3, dz - y \, dx)$, and $\Lambda$ is a Legendrian knot or link.
Legendrian contact homology (Eliashberg–Hofer, Chekanov, late '90s): associated to $\Lambda \subset V$, part of the Symplectic Field Theory package.

Denote by $R_{\alpha}$ the Reeb vector field on $V$, defined by

$$\nu_{R_{\alpha}} d\alpha = 0, \quad \alpha(R_{\alpha}) = 1$$

for $(V, \alpha) = (J^1(M), dz - \lambda)$, this is $R_{\alpha} = \partial/\partial z$. Assume $\Lambda$ has finitely many Reeb chords (integral curves for $R_{\alpha}$ with endpoints on $\Lambda$) and write

$$\mathcal{R} = \{\text{Reeb chords of } \Lambda\} = \{a_1, \ldots, a_p\}.$$
The DGA for LCH

We associate to $\Lambda$ the **Chekanov–Eliashberg differential graded algebra** $(A, \partial)$: here $A$ is the tensor algebra over $R = \mathbb{Z}[H_1(\Lambda)]$ generated by $R = \{a_1, \ldots, a_p\}$,

$$A = R\langle a_1, \ldots, a_p \rangle,$$

with grading induced by the Conley–Zehnder indices of $a_1, \ldots, a_p$. The differential $\partial : A \to A$ is defined by

$$\partial(a_i) = \sum_{j_1, \ldots, j_k; \ k \geq 0} \sum_{\dim \mathcal{M}(a_i; a_{j_1}, \ldots, a_{j_k}) = 1} \text{sgn}(u) e[u] a_{j_1} \cdots a_{j_k};$$

extend to $A$ by the Leibniz rule $\partial(xy) = (\partial x)y + (-1)^{|x|}x(\partial y)$. Here $\mathcal{M}(a_i; a_{j_1}, \ldots, a_{j_k}) (k \geq 0)$ is a moduli space to be defined next.
### The moduli space

Let \((\mathbb{R}_t \times V, d(e^t\alpha))\) be the symplectization of \((V, \alpha)\), with a compatible almost complex structure \(J\).

\[M(a_i; a_{j_1}, \ldots, a_{j_k})\] is the space of all \(J\)-holomorphic maps

\[u : (\Delta - \{p^+, p_1^-, \ldots, p_k^\}, \partial \Delta) \rightarrow (\mathbb{R} \times V, \mathbb{R} \times \Lambda)\]

sending a neighborhood of \(p^+\) to \(a_i\) at \(t = +\infty\) and a neighborhood of \(p^-_{\ell}\) to \(a_{j_\ell}\) at \(t = -\infty\):

![Diagram](image-url)
Theorem (Ekholm–Etnyre–Sullivan 2005)

Let \( \Lambda \) be Legendrian in \( V = J^1(M) \). Then for the DGA \((\mathcal{A}, \partial)\) associated to \( \Lambda \):

- \( \text{deg}(\partial) = -1; \)
- \( \partial^2 = 0; \)
- the homology \( H_*(\mathcal{A}, \partial) \) is invariant under Legendrian isotopy of \( \Lambda \).

This homology is the Legendrian contact homology of \( \Lambda \).
Legendrian knots in $\mathbb{R}^3$

In $(\mathbb{R}^3, \ker(dz - y \, dx))$, there are two useful projections: the front projection $\mathbb{R}^3_{xyz} \rightarrow \mathbb{R}^2_{xz}$ and the Lagrangian projection $\mathbb{R}^3_{xyz} \rightarrow \mathbb{R}^2_{xy}$.

- The front projection completely determines a Legendrian knot, via $y = dz/dx$.
- In the Lagrangian projection, Reeb chords correspond to crossings.
- There is a procedure called “resolution” for passing from front projection to Lagrangian projection.
Holomorphic disks in Lagrangian projection

Chekanov: in the $xy$ projection, the holomorphic disks are given by immersed disks $u$ with:

- boundary of $u$ on $\pi_{xy}(\Lambda)$
- convex corners at $a_i, a_{j_1}, \ldots, a_{j_k}$, with “Reeb sign” $+$ at $a_i$ and $-$ at the rest.

$$\partial(a_i) = a_{j_1} \cdots a_{j_k} + \cdots$$
Example: DGA for the Legendrian trefoil

\[ A = (\mathbb{Z}[t^{\pm 1}])\langle a_1, a_2, a_3, a_4, a_5 \rangle \]

\[ |a_1| = |a_2| = 1, \quad |a_3| = |a_4| = |a_5| = |t| = 0 \]

\[ \partial(a_1) = t + a_3 + a_5 + a_5a_4a_3 \]
\[ \partial(a_2) = 1 - a_3 - a_5 - a_3a_4a_5 \]
\[ \partial(a_3) = \partial(a_4) = \partial(a_5) = 0. \]
Example: DGA for the Legendrian trefoil

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Example: DGA for the Legendrian trefoil

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\[ \partial(a_2) = 1 - a_3 - a_5 - a_3 a_4 a_5 \]

\[ \partial(a_3) = \partial(a_4) = \partial(a_5) = 0. \]
**Theorem (Chekanov, late ’90s)**

Let $\Lambda$ be Legendrian in $\mathbb{R}^3$. Then for the DGA $(\mathcal{A}, \partial)$ associated to $\Lambda$:

- $\deg(\partial) = -1$;
- $\partial^2 = 0$;
- the homology $H_*(\mathcal{A}, \partial)$ is invariant under Legendrian isotopy of $\Lambda$.

In fact, the DGA $(\mathcal{A}, \partial)$ is invariant under an equivalence relation called **stable tame isomorphism**, and stable tame isomorphism implies quasi-isomorphism.
Let \((\mathcal{A}, \partial)\) be the DGA for a Legendrian \(\Lambda\). Let \(\mathbb{k}\) be a field (actually also works for a unital commutative ring).

**Definition**

An **augmentation** of \((\mathcal{A}, \partial)\) is a (graded) DGA map

\[
\epsilon : (\mathcal{A}, \partial) \to (\mathbb{k}, 0);
\]

that is, \(\epsilon \circ \partial = 0\), \(\epsilon(1) = 1\), and \(\epsilon(a) = 0\) if \(|a| \neq 0\).

**Theorem (Leverson 2014)**

Let \((\mathcal{A}, \partial)\) be the DGA of a Legendrian knot in \(\mathbb{R}^3\) over \(R = \mathbb{Z}[t^{\pm 1}]\). Any (graded) augmentation \(\epsilon\) of \((\mathcal{A}, \partial)\) must satisfy \(\epsilon(t) = -1\).
Let $\epsilon : \mathcal{A} \to \mathbb{k}$ be an augmentation. We can use $\epsilon$ to linearize the differential, as follows:

Write $\mathcal{A}_k = \mathcal{A} \otimes_R \mathbb{k} = \mathbb{k}\langle a_1, \ldots, a_p \rangle$. Define the $\mathbb{k}$-algebra automorphism $\phi_\epsilon : \mathcal{A}_k \to \mathcal{A}_k$ by $\phi_\epsilon(a_i) = a_i + \epsilon(a_i)$. Then

$$\partial_\epsilon := \phi_\epsilon \circ \partial \circ \phi_\epsilon^{-1} : \mathcal{A}_k \to \mathcal{A}_k$$

is a filtered differential w.r.t. the wordlength filtration on $\mathcal{A}_k$:

$$\mathcal{A}_k = \mathcal{F}^0 \mathcal{A}_k \supset \mathcal{F}^1 \mathcal{A}_k \supset \mathcal{F}^2 \mathcal{A}_k \supset \cdots,$$

where $\mathcal{F}^m \mathcal{A}_k$ is generated by words of length $\geq m$. So $\partial_\epsilon$ descends to a map on the $\mathbb{k}$-vector space

$$\mathcal{F}^1 \mathcal{A}_k / \mathcal{F}^2 \mathcal{A}_k = \mathbb{k}\langle a_1, \ldots, a_p \rangle.$$ 

The homology of this is linearized contact homology $LCH_*(\epsilon)$. 
Example: trefoil

\[ A = (\mathbb{Z}[t^{\pm}])\langle a_1, a_2, a_3, a_4, a_5 \rangle, \ |a_1| = |a_2| = 1, \ |a_3| = |a_4| = |a_5| = 0 \]

\[ \partial(a_1) = t + a_3 + a_5 + a_5 a_4 a_3 \]
\[ \partial(a_2) = 1 - a_3 - a_5 - a_3 a_4 a_5 \]

Five augmentations \( \epsilon : A \to \mathbb{F}_2 \): \( \epsilon(t) = 1, \ \epsilon(a_1) = \epsilon(a_2) = 0 \), and
\( (\epsilon(a_3), \epsilon(a_4), \epsilon(a_5)) = (0, 0, 1), \ (0, 1, 1), \ (1, 0, 0), \ (1, 1, 0), \text{ or } (1, 1, 1). \)
Example: trefoil

\[ A = (\mathbb{Z}[t^{\pm}])[a_1, a_2, a_3, a_4, a_5], \quad |a_1| = |a_2| = 1, \quad |a_3| = |a_4| = |a_5| = 0 \]

\[ \partial_\epsilon(a_1) = 1 + a_3 + (a_5 + 1) + (a_5 + 1)a_4a_3 \]
\[ \partial_\epsilon(a_2) = 1 + a_3 + (a_5 + 1) + a_3a_4(a_5 + 1) \]

Five augmentations \( \epsilon : A \to \mathbb{F}_2 \): \( \epsilon(t) = 1, \quad \epsilon(a_1) = \epsilon(a_2) = 0 \), and
\( (\epsilon(a_3), \epsilon(a_4), \epsilon(a_5)) = (0, 0, 1), \quad (0, 1, 1), \quad (1, 0, 0), \quad (1, 1, 0), \quad \text{or} \quad (1, 1, 1). \)

\[ \implies \partial_\epsilon = \phi_\epsilon \circ \partial \circ \phi_\epsilon^{-1} \quad \text{where} \quad \phi_\epsilon(a_1, \ldots, a_4) = 0, \quad \phi_\epsilon(a_5) = 1 \]
Example: trefoil

\[ A = (\mathbb{Z}[t^\pm]) \langle a_1, a_2, a_3, a_4, a_5 \rangle, \ |a_1| = |a_2| = 1, \ |a_3| = |a_4| = |a_5| = 0 \]

\[ \partial_\epsilon(a_1) = a_3 + a_5 + a_4 a_3 + a_5 a_4 a_3 \]
\[ \partial_\epsilon(a_2) = a_3 + a_5 + a_3 a_4 + a_3 a_4 a_5 \]

Five augmentations \( \epsilon : A \to \mathbb{F}_2 \): \( \epsilon(t) = 1, \epsilon(a_1) = \epsilon(a_2) = 0 \), and

\( (\epsilon(a_3), \epsilon(a_4), \epsilon(a_5)) = (0, 0, 1), \ (0, 1, 1), \ (1, 0, 0), \ (1, 1, 0), \) or \( (1, 1, 1). \)

\[ \implies \partial_\epsilon = \phi_\epsilon \circ \partial \circ \phi_\epsilon^{-1} \text{ where } \phi_\epsilon(a_1, \ldots, a_4) = 0, \ \phi_\epsilon(a_5) = 1 \]
Example: trefoil

\[ A = (\mathbb{Z}[t^{\pm}])\langle a_1, a_2, a_3, a_4, a_5 \rangle, \quad |a_1| = |a_2| = 1, \quad |a_3| = |a_4| = |a_5| = 0 \]

\[ \partial_\epsilon(a_1) = a_3 + a_5 + a_4 a_3 + a_5 a_4 a_3 \]
\[ \partial_\epsilon(a_2) = a_3 + a_5 + a_3 a_4 + a_3 a_4 a_5 \]

Five augmentations \( \epsilon : A \to \mathbb{F}_2 \): \( \epsilon(t) = 1, \epsilon(a_1) = \epsilon(a_2) = 0 \), and
\( \epsilon(a_3), \epsilon(a_4), \epsilon(a_5) \) = (0, 0, 1), (0, 1, 1), (1, 0, 0), (1, 1, 0), or (1, 1, 1).

\[ \implies \partial_\epsilon = \phi_\epsilon \circ \partial \circ \phi_\epsilon^{-1} \text{ where } \phi_\epsilon(a_1, \ldots, a_4) = 0, \quad \phi_\epsilon(a_5) = 1 \]
\[ \partial_\epsilon : \mathbb{F}_2\langle a_1, a_2, a_3, a_4, a_5 \rangle \to \mathbb{F}_2\langle a_1, a_2, a_3, a_4, a_5 \rangle \]
Linearized contact cohomology

Write $C_* = \mathbb{k}\langle a_1, \ldots, a_p \rangle$. Dualize to $C^* = \mathbb{k}\langle a_1^\vee, \ldots, a_p^\vee \rangle$ with $|a_i^\vee| = |a_i| + 1$. Then the adjoint of $\partial_\epsilon : C_* \to C_{*-1}$ is $\partial_*^\epsilon : C^* \to C^*+1$.

The linearized Legendrian contact cohomology is $LCH^*(\epsilon) = H^*(C^*, \partial_*^\epsilon)$. This counts augmented holomorphic disks:

$$
\partial_*^\epsilon(a_j^\vee) = a_i^\vee + \ldots
$$

For the trefoil, $LCH^2(\epsilon) = \mathbb{F}_2\langle a_1^\vee \rangle$, $LCH^1(\epsilon) = \mathbb{F}_2\langle a_3^\vee, a_4^\vee \rangle$:

$$
\partial_\epsilon(a_1) = a_3 + a_5 \quad \partial_\epsilon(a_2) = a_3 + a_5
$$

$\rightarrow$

$$
\partial_*^\epsilon(a_3^\vee) = a_1^\vee + a_2^\vee \quad \partial_*^\epsilon(a_5^\vee) = a_1^\vee + a_2^\vee.
$$
Some (but not all) augmentations have a geometric interpretation.

Let \((W, \omega = d\theta)\) be an exact symplectic filling of \((V, \alpha)\): near \(\partial W = V\), \(W\) looks like the symplectization of \(V\): \(\theta = e^{t\alpha}\).

A Lagrangian \(L \subset W\) is an exact Lagrangian filling of \(\Lambda \subset V\) if \(\partial L = \Lambda\) and \(\theta|_L\) is exact.

By “functoriality of LCH”, this produces an augmentation

\[\varepsilon_L : (\mathcal{A}, \partial) \to (\mathbb{F}_2, 0)\].

Theorem (Ekholm–Honda–Kálmán 2012)

An exact Lagrangian filling

\[ L \subset \mathbb{R}^4 = (-\infty, 0] \times \mathbb{R}^3 \]

for a Legendrian \( \Lambda \subset \mathbb{R}^3 \) induces an augmentation

\[ \epsilon_L : (\mathcal{A}, \partial) \to (\mathbb{F}_2, 0). \]

Ekholm–Honda–Kálmán: the Legendrian trefoil has 5 exact Lagrangian fillings, inducing the 5 augmentations of the DGA \((\mathcal{A}, \partial)\). Furthermore, these fillings are pairwise nonisotopic.

Fillings that are isotopic should induce augmentations that are “equivalent” in some sense. In what sense?
\( \Lambda \) Legendrian. The 2-copy of \( \Lambda \) is \( \Lambda^2 = \Lambda_1 \sqcup \Lambda_2 \), where \( \Lambda_1, \Lambda_2 \) are copies of \( \Lambda \) pushed off in the Reeb \( \frac{\partial}{\partial z} \) direction, with \( \Lambda_1 \) above \( \Lambda_2 \).
Mishachev link grading

Let $\mathcal{R} = \{\text{Reeb chords of } \Lambda^2\}$. Then

$$\mathcal{R} = \mathcal{R}_{11} \sqcup \mathcal{R}_{12} \sqcup \mathcal{R}_{21} \sqcup \mathcal{R}_{22}$$

where $\mathcal{R}_{ij} = \{\text{Reeb chords } \Lambda_i \xleftarrow{} \Lambda_j\}$: $\mathcal{R}_{11}, \mathcal{R}_{22}$ are Reeb chords of $\Lambda_1, \Lambda_2$ ("pure chords"), while $\mathcal{R}_{12}, \mathcal{R}_{21}$ are "mixed chords".

An augmentation $\epsilon$ of $(A_{\Lambda^2}, \partial)$ is pure if $\epsilon = 0$ on mixed chords $\mathcal{R}_{12}, \mathcal{R}_{21}$.
Link grading splits the differential

Given a pure augmentation \((\epsilon_1, \epsilon_2)\) of the 2-copy DGA \(A_{\Lambda^2}\), the linearized differential \(\partial(\epsilon_1, \epsilon_2)\) splits: if we write \(C_{ij} = \mathbb{k}\langle R_{ij}\rangle\), then

\[
\partial(\epsilon_1, \epsilon_2) : C_{ij} \to C_{ij}.
\]

In particular, the differential \(\partial(\epsilon_1, \epsilon_2) : C_{12} \to C_{12}\) can be pictured as follows: if \(a \in R_{12}\), then

\[
\partial(\epsilon_1, \epsilon_2)(a) = \sum_{\dim M(a; b) = 1} \#(M(a; b) / \mathbb{R}) b
\]

\[M(a; b) = \{ \]

\[\begin{array}{cccccccc}
+ & + & + & + & + & + & + & + \\
\end{array} \]
Example: 2-copy of the trefoil

For $\epsilon_1(a_5) = \epsilon_2(a_5) = 1$: differentials $\partial(\epsilon_1,\epsilon_2)$ on $C_{12}$ and $C_{21}$:

- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = 0$
- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = 0$
- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = 0$
- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = a_{12}^{12} + a_{21}^{12}$
- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = 0$
- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = 0$
- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = 0$
- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = 0$
- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = a_{21}^{21} + a_{5}^{21}$
- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = a_{21}^{21} + a_{5}^{21}$
- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = a_{21}^{21} + a_{5}^{21}$
- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = a_{21}^{21}$
- $\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) = 0$. 
Example: 2-copy of the trefoil

For $\epsilon_1(a_5) = \epsilon_2(a_5) = 1$: differentials $\partial(\epsilon_1,\epsilon_2)$ on $C_{12}$ and $C_{21}$:

\[
\begin{align*}
\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) &= 0 \\
\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) &= 0 \\
\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) &= 0 \\
\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) &= 0 \\
\partial(\epsilon_1,\epsilon_2)(a_{12}^{12}) &= a_{12}^{12} + a_{22}^{12} \\
\partial(\epsilon_1,\epsilon_2)(a_{21}^{21}) &= a_{31}^{21} + a_{51}^{21} \\
\partial(\epsilon_1,\epsilon_2)(a_{21}^{21}) &= a_{31}^{21} + a_{51}^{21} \\
\partial(\epsilon_1,\epsilon_2)(a_{31}^{21}) &= 0 \\
\partial(\epsilon_1,\epsilon_2)(a_{31}^{21}) &= 0 \\
\partial(\epsilon_1,\epsilon_2)(a_{31}^{21}) &= 0 \\
\partial(\epsilon_1,\epsilon_2)(a_{31}^{21}) &= 0.
\end{align*}
\]
Example: 2-copy of the trefoil

For $\epsilon_1(a_5) = \epsilon_2(a_5) = 1$: differentials $\partial(\epsilon_1, \epsilon_2)$ on $C_{12}$ and $C_{21}$:

\[
\begin{align*}
\partial(\epsilon_1, \epsilon_2)(a_{12}^{12}) &= 0 \\
\partial(\epsilon_1, \epsilon_2)(a_{12}^{12}) &= 0 \\
\partial(\epsilon_1, \epsilon_2)(a_{12}^{12}) &= 0 \\
\partial(\epsilon_1, \epsilon_2)(a_{12}^{12}) &= 0 \\
\partial(\epsilon_1, \epsilon_2)(a_{12}^{12}) &= a_{12}^{12} + a_{12}^{12} \\
\partial(\epsilon_1, \epsilon_2)(a_{21}^{21}) &= a_{21}^{21} + a_{21}^{21} \\
\partial(\epsilon_1, \epsilon_2)(a_{21}^{21}) &= a_{21}^{21} + a_{21}^{21} \\
\partial(\epsilon_1, \epsilon_2)(a_{21}^{21}) &= 0 \\
\partial(\epsilon_1, \epsilon_2)(a_{21}^{21}) &= 0 \\
\partial(\epsilon_1, \epsilon_2)(a_{21}^{21}) &= 0.
\end{align*}
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Example: 2-copy of the trefoil

For $\epsilon_1(a_5) = \epsilon_2(a_5) = 1$: differentials $\partial_{(\epsilon_1,\epsilon_2)}$ on $C_{12}$ and $C_{21}$:

- $\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^{11}) = 0$
- $\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^{12}) = 0$
- $\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^{31}) = 0$
- $\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^{41}) = 0$
- $\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^{51}) = a_{12}^{11} + a_{12}^{12}$

- $\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^{21}) = a_{21}^{21} + a_{21}^{51}$
- $\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^{22}) = a_{21}^{21} + a_{21}^{21}$
- $\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^{23}) = 0$
- $\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^{24}) = 0$
- $\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^{25}) = 0$. 
Example: 2-copy of the trefoil

For $\epsilon_1(a_5) = \epsilon_2(a_5) = 1$: differentials $\partial_{(\epsilon_1,\epsilon_2)}$ on $C_{12}$ and $C_{21}$:

\[
\begin{align*}
\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^{11}) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^{21}) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^{31}) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^{41}) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^{21}) &= a_1^{12} + a_2^{12} \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^{21}) &= a_3^{21} + a_5^{21} \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^{21}) &= a_3^{21} + a_5^{21} \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^{21}) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^{21}) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^{21}) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^{21}) &= 0.
\end{align*}
\]
Example: 2-copy of the trefoil

For $\epsilon_1(a_5) = \epsilon_2(a_5) = 1$: differentials $\partial_{(\epsilon_1, \epsilon_2)}$ on $C_{12}$ and $C_{21}$:

\[
\begin{align*}
\partial_{(\epsilon_1, \epsilon_2)}(a_{12}^{12}) &= 0 \\
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\partial_{(\epsilon_1, \epsilon_2)}(a_{12}^{12}) &= 0 \\
\partial_{(\epsilon_1, \epsilon_2)}(a_{12}^{12}) &= 0 \\
\partial_{(\epsilon_1, \epsilon_2)}(a_{12}^{12}) &= a_{12}^{12} + a_{12}^{12} \\
\partial_{(\epsilon_1, \epsilon_2)}(a_{21}^{21}) &= a_{3}^{21} + a_{5}^{21} \\
\partial_{(\epsilon_1, \epsilon_2)}(a_{21}^{21}) &= a_{3}^{21} + a_{5}^{21} \\
\partial_{(\epsilon_1, \epsilon_2)}(a_{21}^{21}) &= 0 \\
\partial_{(\epsilon_1, \epsilon_2)}(a_{21}^{21}) &= 0 \\
\partial_{(\epsilon_1, \epsilon_2)}(a_{21}^{21}) &= 0.
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\partial_{(\epsilon_1, \epsilon_2)}(a_{12}^{12}) &= 0 \\
\partial_{(\epsilon_1, \epsilon_2)}(a_{12}^{12}) &= 0 \\
\partial_{(\epsilon_1, \epsilon_2)}(a_{12}^{12}) &= a_{12}^{12} + a_{12}^{12}
\end{align*}
\]
Example: 2-copy of the trefoil

For $\epsilon_1(a_5) = \epsilon_2(a_5) = 1$: differentials $\partial_{(\epsilon_1,\epsilon_2)}$ on $C_{12}$ and $C_{21}$:

\[
\begin{align*}
\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^1) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^2) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^3) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^4) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{12}^5) &= a_{12}^1 + a_{12}^2 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^1) &= a_{21}^1 + a_{21}^2 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^2) &= a_{21}^1 + a_{21}^2 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^3) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^4) &= 0 \\
\partial_{(\epsilon_1,\epsilon_2)}(a_{21}^5) &= 0.
\end{align*}
\]
Augmentation category

Summary: augmentations $\epsilon_1, \epsilon_2$ of $(A_\Lambda, \partial)$ yield a map $\partial_{(\epsilon_1,\epsilon_2)}$ on $C_{12} = k\langle \text{Reeb chords } \Lambda_1 \leftarrow \Lambda_2 \rangle$ with $\partial^2_{(\epsilon_1,\epsilon_2)} = 0$. Dually, if $C^{12} = (C_{12})^*$:

$$\partial^*_{(\epsilon_1,\epsilon_2)} : C^{12} \to C^{12}.$$

Now construct an $A_{\infty}$ category out of augmentations of $\Lambda$, as follows:

- **objects** are augmentations of $(A_\Lambda, \partial)$;
- **morphisms** are elements of the graded vector spaces

$$\text{Hom}(\epsilon_1, \epsilon_2) = C^{12}.$$

The morphism spaces come equipped with the differential

$$m_1 = \partial^*_{(\epsilon_1,\epsilon_2)} : \text{Hom}(\epsilon_1, \epsilon_2) \to \text{Hom}(\epsilon_1, \epsilon_2).$$
\( A_\infty \) operations

In fact, \( m_1 : \text{Hom}(\epsilon_1, \epsilon_2) \to \text{Hom}(\epsilon_1, \epsilon_2) \) is the first in a sequence of \( A_\infty \) operations

\[
m_k : \text{Hom}(\epsilon_k, \epsilon_{k+1}) \otimes \text{Hom}(\epsilon_{k-1}, \epsilon_k) \otimes \cdots \otimes \text{Hom}(\epsilon_1, \epsilon_2) \to \text{Hom}(\epsilon_1, \epsilon_{k+1})
\]

satisfying the \( A_\infty \) relations

\[
m_1(m_1(a_1)) = 0
\]
\[
m_1(m_2(a_1, a_2)) = m_2(m_1(a_1), a_2) \pm m_2(a_1, m_1(a_2))
\]
\[
m_2(a_1, m_2(a_2, a_3)) - m_2(m_1(a_1, a_2), a_3) = m_1(m_3(a_1, a_2, a_3)) + m_3(m_1(a_1), a_2, a_3)
\]
\[
\pm m_3(a_1, m_1(a_2), a_3) \pm m_3(a_1, a_2, m_1(a_3))
\]
\[\vdots\]

In particular, if we pass to cohomology with respect to \( m_1 \), then we get an honest category where composition is given by \( m_2 \).
Definition of $m_2$

Let $\Lambda^3 = \Lambda_1 \sqcup \Lambda_2 \sqcup \Lambda_3$ be the 3-copy of $\Lambda$ ($\Lambda_1$ lies above $\Lambda_2$, which lies above $\Lambda_3$, in the $z$ direction).

$$C_{ij} = \mathbb{k}\langle \text{Reeb chords } \Lambda_i \leftarrow \Lambda_j \rangle, \quad C^{ij} = C_{ij}^*.$$  

Then

$$m_2 : \text{Hom}(\epsilon_2, \epsilon_3) \otimes \text{Hom}(\epsilon_1, \epsilon_2) \to \text{Hom}(\epsilon_1, \epsilon_3)$$

counts holomorphic disks

à la multiplication in Lagrangian intersection Floer theory.
The augmentation category

Theorem (NRSSZ 2015)

Let $\Lambda \subset \mathbb{R}^3$ be a Legendrian knot or link. The $m_k$ operations satisfy the $A_\infty$ relations, and so we get an $A_\infty$ category, the augmentation category $\text{Aug}_+(\Lambda, \mathbb{L})$. The corresponding cohomology category $H^*\text{Aug}_+(\Lambda, \mathbb{L})$ is an ordinary category.

This follows (modulo some details) by dualizing the fact that the DGA of the $n$-copy $\Lambda^n$ satisfies $\partial^2 = 0$.

Theorem (NRSSZ 2015)

Up to $A_\infty$ equivalence, the augmentation category $\text{Aug}_+(\Lambda, \mathbb{L})$ is an invariant of $\Lambda$ under Legendrian isotopy (and the choice of perturbation needed to define the category).
The Bourgeois–Chantraine augmentation category

We can construct another $A_\infty$ category, with the same objects as $\text{Aug}_+(\Lambda, k)$, by ordering the components in the $n$-copies from bottom to top instead of from top to bottom.

(Equivalently, use $C_{21}$ instead of $C_{12}$ as the hom spaces.)

This produces a category introduced by Bourgeois and Chantraine in 2012: the original “augmentation category”. We call this $\text{Aug}_-(\Lambda, k)$.

For clarity, write $\text{Hom}_+, \text{Hom}_-$ for the hom spaces in $\text{Aug}_+, \text{Aug}_-$. The cohomology of $\text{Hom}_-$ is linearized contact cohomology:

$$H^* \text{Hom}_-(\epsilon, \epsilon) \cong LCH^*(\epsilon).$$
Theorem

Let $\Lambda$ be a Legendrian link and $\epsilon_1, \epsilon_2$ two augmentations of $(A_{\Lambda}, \partial_{\Lambda})$. There is a short exact sequence of chain complexes

$$0 \to \text{Hom}_-(\epsilon_1, \epsilon_2) \to \text{Hom}_+(\epsilon_1, \epsilon_2) \to C^*(\Lambda) \to 0$$

resulting in a long exact sequence

$$\cdots \to H^{i-1}(\Lambda) \to H^i \text{Hom}_-(\epsilon_1, \epsilon_2) \to H^i \text{Hom}_+(\epsilon_1, \epsilon_2) \to H^i(\Lambda) \to \cdots .$$
Unitality

**Theorem (NRSSZ)**

\( \text{Aug}_+(\Lambda, \mathbb{I}) \) is unital: for \( \epsilon \in \text{Ob} \text{Aug}_+ \), there is \( e_\epsilon \in \text{Hom}_+(\epsilon, \epsilon) \) that composes under \( m_2 \) as the identity.

In the front projection, \( e_\epsilon = \sum_c c^\vee \) where the sum is over all the Reeb chords \( c \) at left cusps of the 2-copy.

By contrast, \( \text{Aug}_-(\Lambda, \mathbb{I}) \) is not unital.
Isomorphism of augmentations

In a unital \((A_\infty)\) category, there is a natural notion of isomorphism of objects.

**Definition**

Two augmentations \(\epsilon_1, \epsilon_2\) of \(\Lambda\) are isomorphic if there exist cocycles \(x \in \text{Hom}_+(\epsilon_1, \epsilon_2), y \in \text{Hom}_+(\epsilon_2, \epsilon_1)\) such that

\[
\begin{align*}
[m_2(y, x)] &= [e_{\epsilon_1}] \in H^* \text{Hom}(\epsilon_1, \epsilon_1) \\
[m_2(x, y)] &= [e_{\epsilon_2}] \in H^* \text{Hom}(\epsilon_2, \epsilon_2).
\end{align*}
\]

This notion turns out to coincide with the notion of “DGA homotopy” of maps \((A_{\Lambda}, \partial) \to (\mathbb{K}, 0)\).
Isomorphism and isotopic fillings

In the setting of fillings, we can reinterpret several results from the literature in terms of $\text{Aug}_+$. 

**Result** (Ekholm–Honda–Kálmán): isotopic fillings induce augmentations that are DGA homotopic. 

**Reinterpretation:**

**Theorem**

$L_1, L_2$ exact Lagrangian fillings of $\Lambda$ with corresponding augmentations $\epsilon_{L_1}, \epsilon_{L_2}$ of $(\mathcal{A}_\Lambda, \partial)$. If $L_1, L_2$ are isotopic, then

$$\epsilon_{L_1} \simeq \epsilon_{L_2}.$$

Can use this (as in EHK) to show that the five fillings of the Legendrian trefoil are non-isotopic.
**Result** ("Sabloff duality", Ekholm–Etnyre–Sabloff): given an augmentation $\epsilon$ of $\Lambda$, there is a long exact sequence relating linearized contact homology and cohomology:

$$\cdots \to H^{i-1}(\Lambda) \to LCH^i(\epsilon) \to LCH_{1-i}(\epsilon) \to H^i(\Lambda) \to \cdots .$$

Compare to

$$\to H^{i-1}(\Lambda) \to H^i \text{Hom}_-(\epsilon_1, \epsilon_2) \to H^i \text{Hom}_+(\epsilon_1, \epsilon_2) \to H^i(\Lambda) \to .$$

**Reinterpretation:**

**Theorem**

$\text{Hom}_+(\epsilon_1, \epsilon_2)$ and $\text{Hom}_-(\epsilon_2, \epsilon_1)$ are dual complexes (up to quasi-isomorphism).
Let $L$ be an exact Lagrangian filling of $\Lambda$, with augmentation $\epsilon_L$.

**Result** (Seidel; see also Ekholm, Ekholm–Honda–Kálmán, Dimitroglou Rizell, Bourgeois–Chantraine):

$$LCH_\ast(\epsilon_L) \cong H_\ast(L).$$

**Reinterpretation:**

**Theorem**

$$H^i \text{Hom}_+(\epsilon_L, \epsilon_L) \cong H^i(L)$$

$$H^i \text{Hom}_-(\epsilon_L, \epsilon_L) \cong H^i(L, \Lambda)$$

and the long exact sequence

$$\rightarrow H^{i-1}(\Lambda) \rightarrow H^i \text{Hom}_-(\epsilon_L, \epsilon_L) \rightarrow H^i \text{Hom}_+(\epsilon_L, \epsilon_L) \rightarrow H^i(\Lambda) \rightarrow$$

is the standard long exact sequence in relative cohomology.
Cohomology and compactly supported cohomology

\[ H^i \text{Hom}_+(\epsilon, \epsilon) \cong H^i(L) \]
\[ H^i \text{Hom}_-(\epsilon, \epsilon) \cong H^i(L, \Lambda) \]

- The augmentation category \( \text{Aug}_+ \) measures cohomology and is unital;
- the BC augmentation category \( \text{Aug}_- \) measures compactly supported cohomology and is non-unital.

In our trefoil example, we have

\[ H^0 \text{Hom}_+(\epsilon, \epsilon) = \mathbb{F}_2 \]
\[ H^1 \text{Hom}_+(\epsilon, \epsilon) = \mathbb{F}_2 \]
\[ H^1 \text{Hom}_-(\epsilon, \epsilon) = \mathbb{F}_2 \]
\[ H^2 \text{Hom}_-(\epsilon, \epsilon) = \mathbb{F}_2. \]

in agreement with \( H^*, H^*_c \) for the punctured torus.
Augmentations are sheaves

Nadler–Zaslow correspondence ("microlocalization")

\[ Sh(M; \mathbb{k}) \xrightarrow{\sim} Fuk_\epsilon(T^*M; \mathbb{k}) : \]

equivalence between a category of sheaves on a manifold \( M \) and the infinitesimally wrapped Fukaya category of \( T^*M \).

The augmentation category is some flavor of the RHS.

The LHS in this setting is \( STZ(\Lambda, \mathbb{k}) \), the Shende–Treumann–Zaslow dg category of rank 1 constructible sheaves (2014).

**Theorem (NRSSZ)**

Let \( \Lambda \) be a Legendrian knot or link in \( \mathbb{R}^3 \). Then we have an equivalence of \( A_\infty \) categories

\[ STZ(\Lambda, \mathbb{k}) \cong Aug_+(\Lambda, \mathbb{k}). \]