The augmentation category of a Legendrian knot

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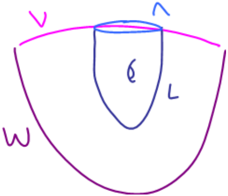
A report on part of arXiv:1502.04939, joint with:
Dan Rutherford (Ball State), Vivek Shende (UC Berkeley), Steven Sivek (Princeton), and Eric Zaslow (Northwestern).

These slides available at `http://www.math.duke.edu/~ng/unlv.pdf` .

Motivation

One direction of motivation: W exact symplectic manifold with convex contact boundary V.



Vague question: construct a Fukaya category from exact Lagrangians L in W, cylindrical near boundary, in terms of just the boundary data $\Lambda \subset V$.

Perhaps fix the boundary condition $\Lambda \subset V$.

The setting

Let M be a smooth manifold of dimension n. We'll work with the contact manifold

$$(V, \xi) = (J^1(M), \ker \alpha),$$

where $J^1(M) = T^*M \times \mathbb{R}$ and $\alpha = dz - \lambda$, with λ the canonical Liouville 1-form on T^*M.

For $M = \mathbb{R}^n$,

$$V = J^1(\mathbb{R}^n) = \mathbb{R}^{2n+1}_{x_1,\ldots,x_n,y_1,\ldots,y_n,z}$$
$$\alpha = dz - \sum_{i=1}^{n} y_i \, dx_i.$$

A submanifold $\Lambda \subset V$ is *Legendrian* if $\alpha|_\Lambda \equiv 0$ and $\dim(\Lambda) = n$.
Particular case of interest: $n = 1$, $(V, \alpha) = (\mathbb{R}^3, dz - y\,dx)$, and Λ is a Legendrian knot or link.

Legendrian contact homology

Legendrian contact homology (Eliashberg–Hofer, Chekanov, late '90s): associated to $\Lambda \subset V$, part of the Symplectic Field Theory package.
Denote by R_α the Reeb vector field on V, defined by

$$\iota_{R_\alpha} d\alpha = 0, \quad \alpha(R_\alpha) = 1 :$$

for $(V, \alpha) = (J^1(M), dz - \lambda)$, this is $R_\alpha = \partial/\partial z$. Assume Λ has finitely many Reeb chords (integral curves for R_α with endpoints on Λ) and write

$$\mathcal{R} = \{\text{Reeb chords of } \Lambda\} = \{a_1, \ldots, a_p\}.$$

The DGA for LCH

We associate to Λ the Chekanov–Eliashberg differential graded algebra $(\mathcal{A}, \partial)$: here $\mathcal{A}$ is the tensor algebra over $R = \mathbb{Z}[H_1(\Lambda)]$ generated by $\mathcal{R} = \{a_1, \ldots, a_p\}$,

$$\mathcal{A} = R\langle a_1, \ldots, a_p \rangle,$$

with grading induced by the Conley–Zehnder indices of $a_1, \ldots, a_p$. The differential $\partial : \mathcal{A} \to \mathcal{A}$ is defined by

$$\partial(a_i) = \sum_{\substack{j_1,\ldots,j_k;\ k \geq 0 \\ \dim \mathcal{M}(a_i; a_{j_1},\ldots,a_{j_k}) = 1}} \ \sum_{u \in \mathcal{M}/\mathbb{R}} \operatorname{sgn}(u) e^{[u]} a_{j_1} \cdots a_{j_k};$$

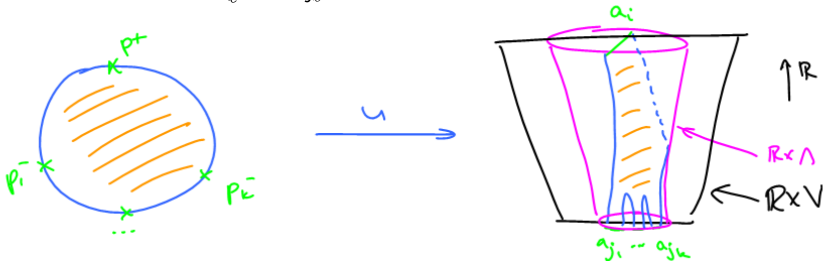
extend to $\mathcal{A}$ by the Leibniz rule $\partial(xy) = (\partial x)y + (-1)^{|x|} x(\partial y)$. Here $\mathcal{M}(a_i; a_{j_1}, \ldots, a_{j_k})$ $(k \geq 0)$ is a moduli space to be defined next.

The moduli space

Let $(\mathbb{R}_t \times V, d(e^t\alpha))$ be the symplectization of (V, α), with a compatible almost complex structure J.
$\mathcal{M}(a_i; a_{j_1}, \ldots, a_{j_k})$ is the space of all J-holomorphic maps

$$u : (\Delta - \{p^+, p_1^-, \ldots, p_k^-\}, \partial\Delta) \to (\mathbb{R} \times V, \mathbb{R} \times \Lambda)$$

sending a neighborhood of p^+ to a_i at $t = +\infty$ and a neighborhood of p_ℓ^- to a_{j_ℓ} at $t = -\infty$:



LCH invariance

Theorem (Ekholm–Etnyre–Sullivan 2005)

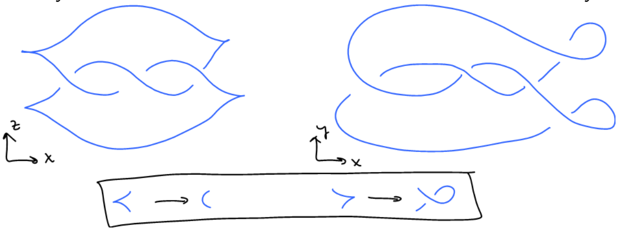
Let Λ be Legendrian in $V = J^1(M)$. Then for the DGA $(\mathcal{A}, \partial)$ associated to Λ:

- $\deg(\partial) = -1$;
- $\partial^2 = 0$;
- *the homology $H_*(\mathcal{A}, \partial)$ is invariant under Legendrian isotopy of Λ.*

This homology is the Legendrian contact homology of Λ.

Legendrian knots in $\mathbb{R}^3$

In $(\mathbb{R}^3, \ker(dz - y\,dx))$, there are two useful projections: the front projection $\mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xz}$ and the Lagrangian projection $\mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xy}$.

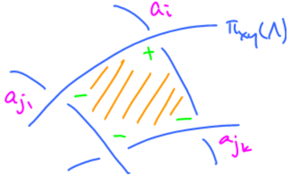


- The front projection completely determines a Legendrian knot, via $y = dz/dx$.
- In the Lagrangian projection, Reeb chords correspond to crossings.
- There is a procedure called "resolution" for passing from front projection to Lagrangian projection.

Holomorphic disks in Lagrangian projection

Chekanov: in the xy projection, the holomorphic disks are given by immersed disks u with:

- boundary of u on $\pi_{xy}(\Lambda)$
- convex corners at $a_i, a_{j_1}, \ldots, a_{j_k}$, with "Reeb sign" $+$ at a_i and $-$ at the rest.

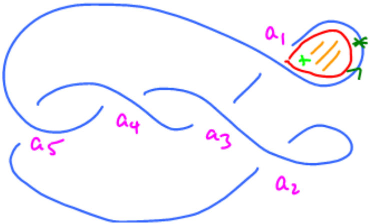


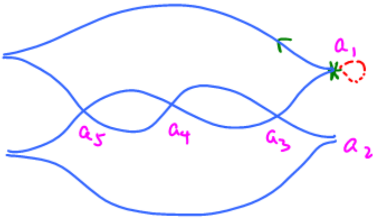
$$\partial(a_i) = a_{j_1} \cdots a_{j_k} + \cdots$$

Example: DGA for the Legendrian trefoil

$$\mathcal{A} = (\mathbb{Z}[t^{\pm 1}])\langle a_1, a_2, a_3, a_4, a_5 \rangle$$
$$|a_1| = |a_2| = 1, \ |a_3| = |a_4| = |a_5| = |t| = 0$$

$$\partial(a_1) = t + a_3 + a_5 + a_5 a_4 a_3$$
$$\partial(a_2) = 1 - a_3 - a_5 - a_3 a_4 a_5$$
$$\partial(a_3) = \partial(a_4) = \partial(a_5) = 0.$$





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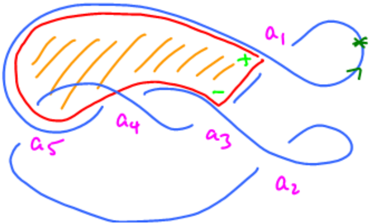

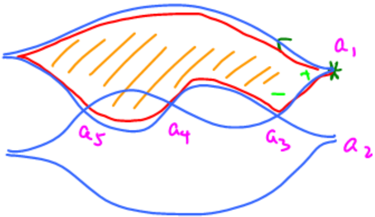

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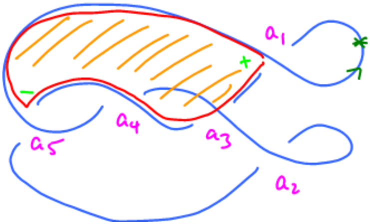

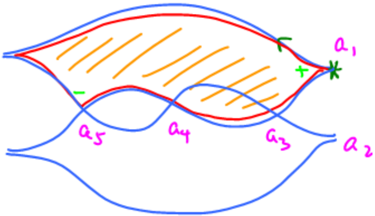

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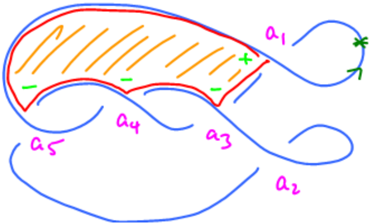

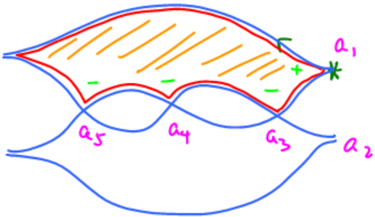

LCH invariance in $\mathbb{R}^3$

Theorem (Chekanov, late '90s)

Let Λ be Legendrian in $\mathbb{R}^3$. Then for the DGA $(\mathcal{A}, \partial)$ associated to Λ:

- $\deg(\partial) = -1$;
- $\partial^2 = 0$;
- *the homology $H_*(\mathcal{A}, \partial)$ is invariant under Legendrian isotopy of Λ.*

In fact, the DGA $(\mathcal{A}, \partial)$ is invariant under an equivalence relation called stable tame isomorphism, and stable tame isomorphism implies quasi-isomorphism.

Augmentations

Let $(\mathcal{A}, \partial)$ be the DGA for a Legendrian Λ. Let $\Bbbk$ be a field (actually also works for a unital commutative ring).

Definition

An augmentation of $(\mathcal{A}, \partial)$ is a (graded) DGA map

$$\epsilon : (\mathcal{A}, \partial) \to (\Bbbk, 0);$$

that is, $\epsilon \circ \partial = 0$, $\epsilon(1) = 1$, and $\epsilon(a) = 0$ if $|a| \neq 0$.

Theorem (Leverson 2014)

Let $(\mathcal{A}, \partial)$ be the DGA of a Legendrian knot in $\mathbb{R}^3$ over $R = \mathbb{Z}[t^{\pm 1}]$. Any (graded) augmentation ϵ of $(\mathcal{A}, \partial)$ must satisfy $\epsilon(t) = -1$.

Linearized LCH

Let $\epsilon : \mathcal{A} \to \mathbb{k}$ be an augmentation. We can use ϵ to linearize the differential, as follows:

Write $\mathcal{A}_{\mathbb{k}} = \mathcal{A} \otimes_R \mathbb{k} = \mathbb{k}\langle a_1, \ldots, a_p \rangle$. Define the $\mathbb{k}$-algebra automorphism $\phi_\epsilon : \mathcal{A}_{\mathbb{k}} \to \mathcal{A}_{\mathbb{k}}$ by $\phi_\epsilon(a_i) = a_i + \epsilon(a_i)$. Then

$$\partial_\epsilon := \phi_\epsilon \circ \partial \circ \phi_\epsilon^{-1} : \mathcal{A}_{\mathbb{k}} \to \mathcal{A}_{\mathbb{k}}$$

is a *filtered* differential w.r.t. the wordlength filtration on $\mathcal{A}_{\mathbb{k}}$:

$$\mathcal{A}_{\mathbb{k}} = \mathcal{F}^0\mathcal{A}_{\mathbb{k}} \supset \mathcal{F}^1\mathcal{A}_{\mathbb{k}} \supset \mathcal{F}^2\mathcal{A}_{\mathbb{k}} \supset \cdots,$$

where $\mathcal{F}^m\mathcal{A}_{\mathbb{k}}$ is generated by words of length $\geq m$. So ∂_ϵ descends to a map on the $\mathbb{k}$-vector space

$$\mathcal{F}^1\mathcal{A}_{\mathbb{k}}/\mathcal{F}^2\mathcal{A}_{\mathbb{k}} = \mathbb{k}\langle a_1, \ldots, a_p \rangle.$$

The homology of this is linearized contact homology $LCH_*(\epsilon)$.

Example: trefoil

$$\mathcal{A} = (\mathbb{Z}[t^{\pm}])\langle a_1, a_2, a_3, a_4, a_5 \rangle, \ |a_1| = |a_2| = 1, \ |a_3| = |a_4| = |a_5| = 0$$

$$\partial(a_1) = t + a_3 + a_5 + a_5 a_4 a_3$$
$$\partial(a_2) = 1 - a_3 - a_5 - a_3 a_4 a_5$$

Five augmentations $\epsilon : \mathcal{A} \to \mathbb{F}_2$: $\epsilon(t) = 1$, $\epsilon(a_1) = \epsilon(a_2) = 0$, and

$(\epsilon(a_3), \epsilon(a_4), \epsilon(a_5)) = (0,0,1), \ (0,1,1), \ (1,0,0), \ (1,1,0)$, or $(1,1,1)$.

Example: trefoil

$$\mathcal{A} = (\mathbb{Z}[t^{\pm}])\langle a_1, a_2, a_3, a_4, a_5 \rangle,\ |a_1| = |a_2| = 1,\ |a_3| = |a_4| = |a_5| = 0$$

$$\partial_\epsilon(a_1) = 1 + a_3 + (a_5 + 1) + (a_5 + 1)a_4 a_3$$
$$\partial_\epsilon(a_2) = 1 + a_3 + (a_5 + 1) + a_3 a_4 (a_5 + 1)$$

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$$\implies \partial_\epsilon = \phi_\epsilon \circ \partial \circ \phi_\epsilon^{-1} \text{ where } \phi_\epsilon(a_1, \ldots, a_4) = 0,\ \phi_\epsilon(a_5) = 1$$

Example: trefoil

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Five augmentations $\epsilon : \mathcal{A} \to \mathbb{F}_2$: $\epsilon(t) = 1$, $\epsilon(a_1) = \epsilon(a_2) = 0$, and

$(\epsilon(a_3), \epsilon(a_4), \epsilon(a_5)) = (0, 0, 1)$, $(0, 1, 1)$, $(1, 0, 0)$, $(1, 1, 0)$, or $(1, 1, 1)$.

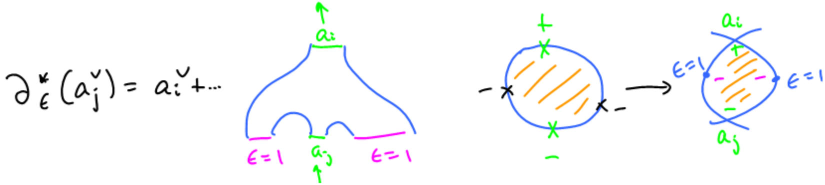
$$\Longrightarrow \partial_\epsilon = \phi_\epsilon \circ \partial \circ \phi_\epsilon^{-1} \text{ where } \phi_\epsilon(a_1, \ldots, a_4) = 0, \ \phi_\epsilon(a_5) = 1$$

$$\partial_\epsilon : \mathbb{F}_2\langle a_1, a_2, a_3, a_4, a_5 \rangle \to \mathbb{F}_2\langle a_1, a_2, a_3, a_4, a_5 \rangle$$

Linearized contact cohomology

Write $C_* = \mathbb{k}\langle a_1, \ldots, a_p \rangle$. Dualize to $C^* = \mathbb{k}\langle a_1^\vee, \ldots, a_p^\vee \rangle$ with $|a_i^\vee| = |a_i| + 1$. Then the adjoint of $\partial_\epsilon : C_* \to C_{*-1}$ is $\partial_\epsilon^* : C^* \to C^{*+1}$.

The linearized Legendrian contact cohomology is $LCH^*(\epsilon) = H^*(C^*, \partial_\epsilon^*)$. This counts augmented holomorphic disks:



For the trefoil, $LCH^2(\epsilon) = \mathbb{F}_2\langle a_1^\vee \rangle$, $LCH^1(\epsilon) = \mathbb{F}_2\langle a_3^\vee, a_4^\vee \rangle$:

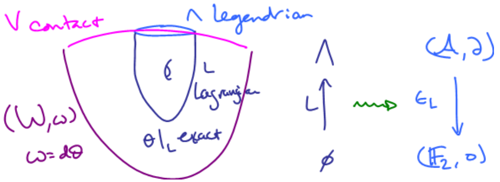
$$\begin{aligned} \partial_\epsilon(a_1) &= a_3 + a_5 \\ \partial_\epsilon(a_2) &= a_3 + a_5 \end{aligned} \longrightarrow \begin{aligned} \partial_\epsilon^*(a_3^\vee) &= a_1^\vee + a_2^\vee \\ \partial_\epsilon^*(a_5^\vee) &= a_1^\vee + a_2^\vee . \end{aligned}$$

Geometric augmentations

Some (but not all) augmentations have a geometric interpretation.

Let $(W, \omega = d\theta)$ be an exact symplectic filling of (V, α): near $\partial W = V$, W looks like the symplectization of V: $\theta = e^t \alpha$.
A Lagrangian $L \subset W$ is an exact Lagrangian filling of $\Lambda \subset V$ if $\partial L = \Lambda$ and $\theta|_L$ is exact.
By "functoriality of LCH", this produces an augmentation

$$\epsilon_L : (\mathcal{A}, \partial) \to (\mathbb{F}_2, 0).$$



Geometric augmentations, continued

Theorem (Ekholm–Honda–Kálmán 2012)

An exact Lagrangian filling

$$L \subset \mathbb{R}^4_- = (-\infty, 0] \times \mathbb{R}^3$$

for a Legendrian $\Lambda \subset \mathbb{R}^3$ *induces an augmentation*

$$\epsilon_L : (\mathcal{A}, \partial) \to (\mathbb{F}_2, 0).$$

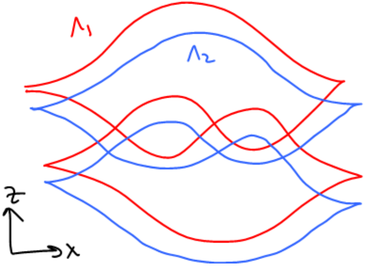
Ekholm–Honda–Kálmán: the Legendrian trefoil has 5 exact Lagrangian fillings, inducing the 5 augmentations of the DGA $(\mathcal{A}, \partial)$. Furthermore, these fillings are pairwise nonisotopic.

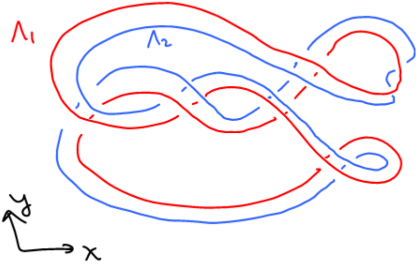
Fillings that are isotopic should induce augmentations that are "equivalent" in some sense. In what sense?

2-copy

Definition

Λ Legendrian. The 2-copy of Λ is $\Lambda^2 = \Lambda_1 \sqcup \Lambda_2$, where Λ_1, Λ_2 are copies of Λ pushed off in the Reeb $(\partial/\partial z)$ direction, with Λ_1 above Λ_2.



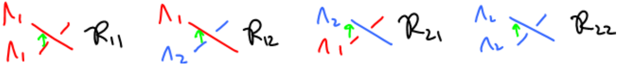


Mishachev link grading

Let $\mathcal{R} = \{\text{Reeb chords of } \Lambda^2\}$. Then

$$\mathcal{R} = \mathcal{R}_{11} \sqcup \mathcal{R}_{12} \sqcup \mathcal{R}_{21} \sqcup \mathcal{R}_{22}$$

where $\mathcal{R}_{ij} = \{\text{Reeb chords } \Lambda_i \leftarrow \Lambda_j\}$: $\mathcal{R}_{11}, \mathcal{R}_{22}$ are Reeb chords of Λ_1, Λ_2 ("pure chords"), while $\mathcal{R}_{12}, \mathcal{R}_{21}$ are "mixed chords".



An augmentation ϵ of $(\mathcal{A}_{\Lambda^2}, \partial)$ is pure if $\epsilon = 0$ on mixed chords $\mathcal{R}_{12}, \mathcal{R}_{21}$.

Pure augmentation ϵ of $\mathcal{A}_{\Lambda^2}$, the DGA of the 2-copy

$\updownarrow$

augmentations ϵ_1, ϵ_2 of $\mathcal{A}_{\Lambda_1}, \mathcal{A}_{\Lambda_2} \cong \mathcal{A}_\Lambda$, the DGA of Λ.

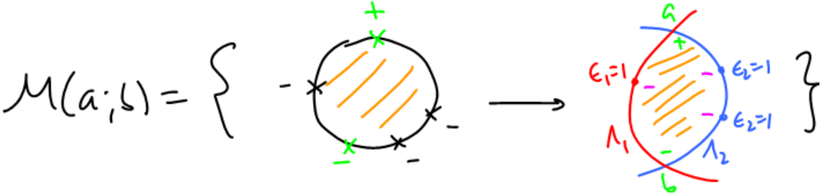
Link grading splits the differential

Given a pure augmentation (ϵ_1, ϵ_2) of the 2-copy DGA $\mathcal{A}_{\Lambda^2}$, the linearized differential $\partial_{(\epsilon_1,\epsilon_2)}$ splits: if we write $C_{ij} = \Bbbk\langle \mathcal{R}_{ij} \rangle$, then

$$\partial_{(\epsilon_1,\epsilon_2)} : C_{ij} \to C_{ij}.$$

In particular, the differential $\partial_{(\epsilon_1,\epsilon_2)} : C_{12} \to C_{12}$ can be pictured as follows: if $a \in \mathcal{R}_{12}$, then

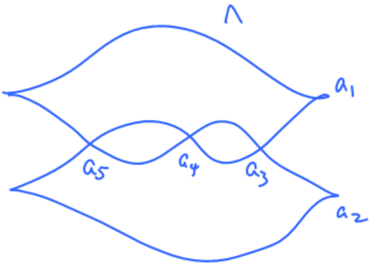
$$\partial_{(\epsilon_1,\epsilon_2)}(a) = \sum_{\dim \mathcal{M}(a;b)=1} \#(\mathcal{M}(a;b)/\mathbb{R})\, b$$

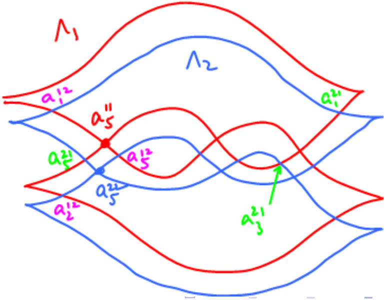


Example: 2-copy of the trefoil

For $\epsilon_1(a_5) = \epsilon_2(a_5) = 1$: differentials $\partial_{(\epsilon_1,\epsilon_2)}$ on C_{12} and C_{21}:

$$\partial_{(\epsilon_1,\epsilon_2)}(a_1^{12}) = 0 \qquad \partial_{(\epsilon_1,\epsilon_2)}(a_1^{21}) = a_3^{21} + a_5^{21}$$
$$\partial_{(\epsilon_1,\epsilon_2)}(a_2^{12}) = 0 \qquad \partial_{(\epsilon_1,\epsilon_2)}(a_2^{21}) = a_3^{21} + a_5^{21}$$
$$\partial_{(\epsilon_1,\epsilon_2)}(a_3^{12}) = 0 \qquad \partial_{(\epsilon_1,\epsilon_2)}(a_3^{21}) = 0$$
$$\partial_{(\epsilon_1,\epsilon_2)}(a_4^{12}) = 0 \qquad \partial_{(\epsilon_1,\epsilon_2)}(a_4^{21}) = 0$$
$$\partial_{(\epsilon_1,\epsilon_2)}(a_5^{12}) = a_1^{12} + a_2^{12} \qquad \partial_{(\epsilon_1,\epsilon_2)}(a_5^{21}) = 0.$$





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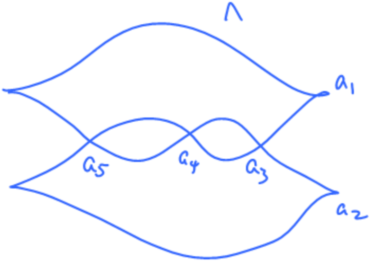

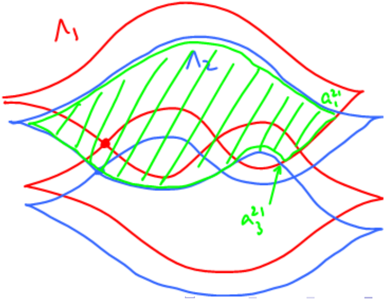

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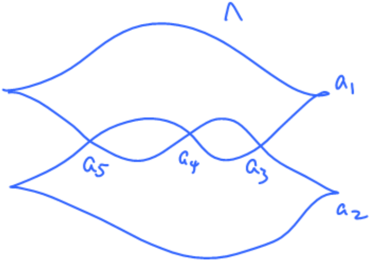

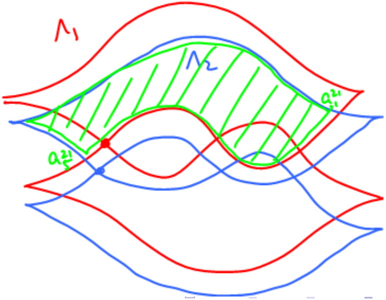

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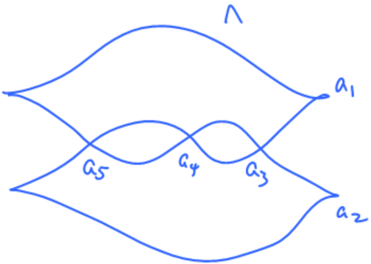

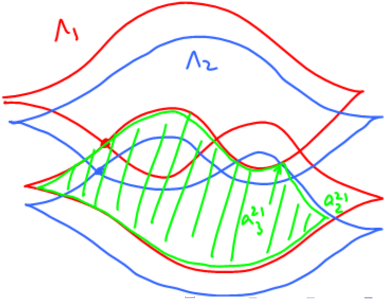

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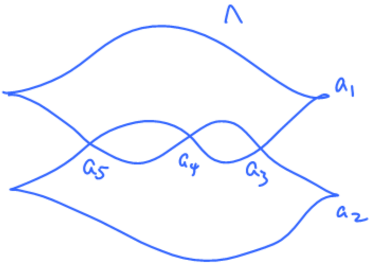

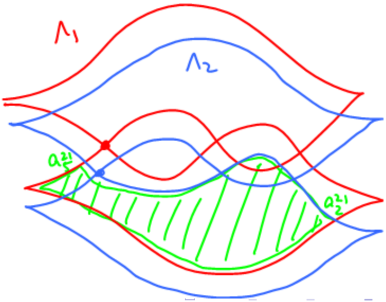

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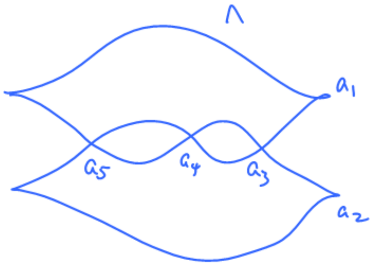

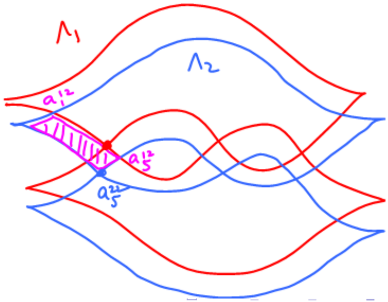

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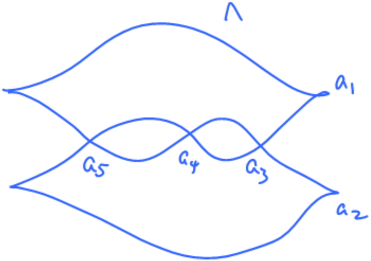

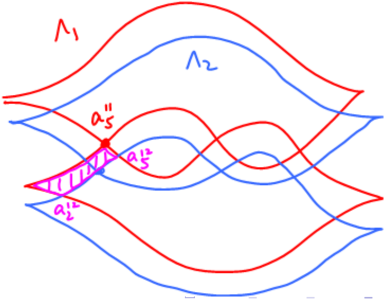

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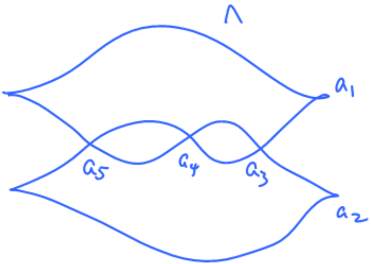
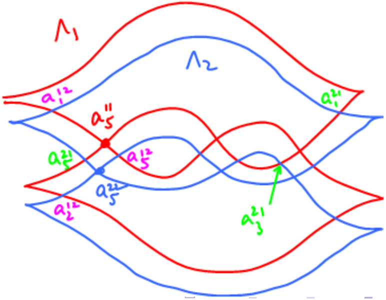




Augmentation category

Summary: augmentations ϵ_1, ϵ_2 of $(\mathcal{A}_\Lambda, \partial)$ yield a map $\partial_{(\epsilon_1,\epsilon_2)}$ on $C_{12} = \Bbbk\langle\text{Reeb chords } \Lambda_1 \leftarrow \Lambda_2\rangle$ with $\partial^2_{(\epsilon_1,\epsilon_2)} = 0$. Dually, if $C^{12} = (C_{12})^*$:

$$\partial^*_{(\epsilon_1,\epsilon_2)} : C^{12} \to C^{12}.$$

Now construct an A_∞ category out of augmentations of Λ, as follows:

- objects are augmentations of $(\mathcal{A}_\Lambda, \partial)$;
- morphisms are elements of the graded vector spaces

$$\mathrm{Hom}(\epsilon_1, \epsilon_2) = C^{12}.$$

The morphism spaces come equipped with the differential

$$m_1 = \partial^*_{(\epsilon_1,\epsilon_2)} : \mathrm{Hom}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}(\epsilon_1, \epsilon_2).$$

A_∞ operations

In fact, $m_1 : \mathrm{Hom}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}(\epsilon_1, \epsilon_2)$ is the first in a sequence of A_∞ operations

$$\begin{aligned} m_k : \mathrm{Hom}(\epsilon_k, \epsilon_{k+1}) \otimes \mathrm{Hom}(\epsilon_{k-1}, \epsilon_k) \otimes \cdots \otimes \mathrm{Hom}(\epsilon_1, \epsilon_2) \\ \to \mathrm{Hom}(\epsilon_1, \epsilon_{k+1}) \end{aligned}$$

satisfying the A_∞ relations

$$\begin{aligned} m_1(m_1(a_1)) &= 0 \\ m_1(m_2(a_1, a_2)) &= m_2(m_1(a_1), a_2) \pm m_2(a_1, m_1(a_2)) \\ m_2(a_1, m_2(a_2, a_3)) - m_2(m_1(a_1, a_2), a_3) &= m_1(m_3(a_1, a_2, a_3)) + m_3(m_1(a_1), a_2, a_3) \\ &\quad \pm m_3(a_1, m_1(a_2), a_3) \pm m_3(a_1, a_2, m_1(a_3)) \\ &\vdots \end{aligned}$$

In particular, if we pass to cohomology with respect to m_1, then we get an honest category where composition is given by m_2.

Definition of m_2

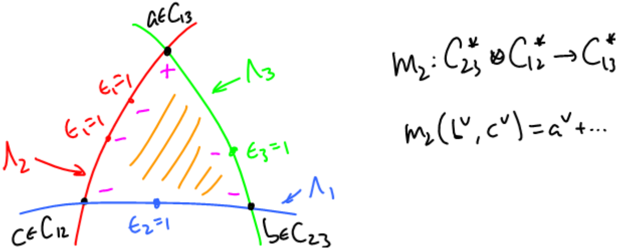
Let $\Lambda^3 = \Lambda_1 \sqcup \Lambda_2 \sqcup \Lambda_3$ be the 3-copy of Λ (Λ_1 lies above Λ_2, which lies above Λ_3, in the z direction).

$$C_{ij} = \Bbbk\langle \text{Reeb chords } \Lambda_i \leftarrow \Lambda_j \rangle, \qquad C^{ij} = C_{ij}^*.$$

Then

$$m_2 : \operatorname{Hom}(\epsilon_2, \epsilon_3) \otimes \operatorname{Hom}(\epsilon_1, \epsilon_2) \to \operatorname{Hom}(\epsilon_1, \epsilon_3)$$

counts holomorphic disks



à la multiplication in Lagrangian intersection Floer theory.

The augmentation category

Theorem (NRSSZ 2015)

Let $\Lambda \subset \mathbb{R}^3$ be a Legendrian knot or link. The m_k operations satisfy the A_∞ relations, and so we get an A_∞ category, the augmentation category $\mathcal{A}ug_+(\Lambda, \Bbbk)$. The corresponding cohomology category $H^*\mathcal{A}ug_+(\Lambda, \Bbbk)$ is an ordinary category.

This follows (modulo some details) by dualizing the fact that the DGA of the n-copy Λ^n satisfies $\partial^2 = 0$.

Theorem (NRSSZ 2015)

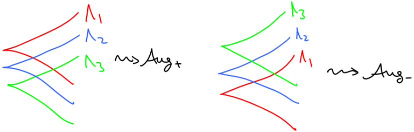
Up to A_∞ equivalence, the augmentation category $\mathcal{A}ug_+(\Lambda, \Bbbk)$ is an invariant of Λ under Legendrian isotopy (and the choice of perturbation needed to define the category).

The Bourgeois–Chantraine augmentation category

We can construct another A_∞ category, with the same objects as $\mathcal{A}ug_+(\Lambda, \Bbbk)$, by ordering the components in the n-copies *from bottom to top* instead of from top to bottom.

(Equivalently, use C_{21} instead of C_{12} as the hom spaces.)

This produces a category introduced by Bourgeois and Chantraine in 2012: the original "augmentation category". We call this $\mathcal{A}ug_-(\Lambda, \Bbbk)$.



For clarity, write $\mathrm{Hom}_+, \mathrm{Hom}_-$ for the hom spaces in $\mathcal{A}ug_+, \mathcal{A}ug_-$. The cohomology of Hom_- is linearized contact cohomology:

$$H^* \,\mathrm{Hom}_-(\epsilon, \epsilon) \cong LCH^*(\epsilon).$$

Hom_+ and Hom_-

Theorem

Let Λ be a Legendrian link and ϵ_1, ϵ_2 two augmentations of $(\mathcal{A}_\Lambda, \partial_\Lambda)$. There is a short exact sequence of chain complexes

$$0 \to \mathrm{Hom}_-(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_+(\epsilon_1, \epsilon_2) \to C^*(\Lambda) \to 0$$

resulting in a long exact sequence

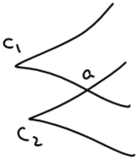
$$\cdots \to H^{i-1}(\Lambda) \to H^i \mathrm{Hom}_-(\epsilon_1, \epsilon_2) \to H^i \mathrm{Hom}_+(\epsilon_1, \epsilon_2) \to H^i(\Lambda) \to \cdots .$$

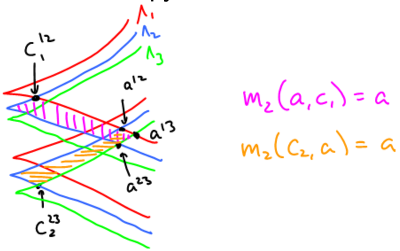
Unitality

Theorem (NRSSZ)

$\mathcal{A}ug_+(\Lambda, \Bbbk)$ is unital: for $\epsilon \in \mathrm{Ob}\, \mathcal{A}ug_+$, there is $e_\epsilon \in \mathrm{Hom}_+(\epsilon, \epsilon)$ that composes under m_2 as the identity.

In the front projection, $e_\epsilon = \sum_c c^\vee$ where the sum is over all the Reeb chords c at left cusps of the 2-copy.





By contrast, $\mathcal{A}ug_-(\Lambda, \Bbbk)$ is *not* unital.

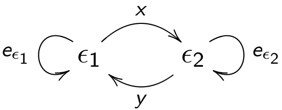
Isomorphism of augmentations

In a unital (A_∞) category, there is a natural notion of isomorphism of objects.

Definition

Two augmentations ϵ_1, ϵ_2 of Λ are isomorphic if there exist cocycles $x \in \mathrm{Hom}_+(\epsilon_1, \epsilon_2)$, $y \in \mathrm{Hom}_+(\epsilon_2, \epsilon_1)$ such that

$$[m_2(y, x)] = [e_{\epsilon_1}] \in H^* \mathrm{Hom}(\epsilon_1, \epsilon_1)$$
$$[m_2(x, y)] = [e_{\epsilon_2}] \in H^* \mathrm{Hom}(\epsilon_2, \epsilon_2).$$



This notion turns out to coincide with the notion of "DGA homotopy" of maps $(\mathcal{A}_\Lambda, \partial) \to (\mathbb{k}, 0)$.

Isomorphism and isotopic fillings

In the setting of fillings, we can reinterpret several results from the literature in terms of $\mathcal{A}ug_+$.

Result (Ekholm–Honda–Kálmán): isotopic fillings induce augmentations that are DGA homotopic.

Reinterpretation:

Theorem

L_1, L_2 exact Lagrangian fillings of Λ with corresponding augmentations $\epsilon_{L_1}, \epsilon_{L_2}$ of $(\mathcal{A}_\Lambda, \partial)$. If L_1, L_2 are isotopic, then

$$\epsilon_{L_1} \cong \epsilon_{L_2}.$$

Can use this (as in EHK) to show that the five fillings of the Legendrian trefoil are non-isotopic.

Sabloff duality

Result ("Sabloff duality", Ekholm–Etnyre–Sabloff): given an augmentation ϵ of Λ, there is a long exact sequence relating linearized contact homology and cohomology:

$$\cdots \to H^{i-1}(\Lambda) \to LCH^{i}(\epsilon) \to LCH_{1-i}(\epsilon) \to H^{i}(\Lambda) \to \cdots .$$

Compare to

$$\to H^{i-1}(\Lambda) \to H^{i}\,\mathrm{Hom}_{-}(\epsilon_1, \epsilon_2) \to H^{i}\,\mathrm{Hom}_{+}(\epsilon_1, \epsilon_2) \to H^{i}(\Lambda) \to .$$

Reinterpretation:

Theorem

$\mathrm{Hom}_{+}(\epsilon_1, \epsilon_2)$ *and* $\mathrm{Hom}_{-}(\epsilon_2, \epsilon_1)$ *are dual complexes (up to quasi-isomorphism).*

Topology of fillings

Let L be an exact Lagrangian filling of Λ, with augmentation ϵ_L.
Result (Seidel; see also Ekholm, Ekholm–Honda–Kálmán, Dimitroglou Rizell, Bourgeois–Chantraine):

$$LCH_*(\epsilon_L) \cong H_*(L).$$

Reinterpretation:

Theorem

$$H^i \operatorname{Hom}_+(\epsilon_L, \epsilon_L) \cong H^i(L)$$
$$H^i \operatorname{Hom}_-(\epsilon_L, \epsilon_L) \cong H^i(L, \Lambda)$$

and the long exact sequence

$$\to H^{i-1}(\Lambda) \to H^i \operatorname{Hom}_-(\epsilon_L, \epsilon_L) \to H^i \operatorname{Hom}_+(\epsilon_L, \epsilon_L) \to H^i(\Lambda) \to$$

is the standard long exact sequence in relative cohomology.

Cohomology and compactly supported cohomology

$$H^i \operatorname{Hom}_+(\epsilon_L, \epsilon_L) \cong H^i(L)$$
$$H^i \operatorname{Hom}_-(\epsilon_L, \epsilon_L) \cong H^i(L, \Lambda)$$

- The augmentation category $\mathcal{A}ug_+$ measures cohomology and is unital;
- the BC augmentation category $\mathcal{A}ug_-$ measures compactly supported cohomology and is non-unital.

In our trefoil example, we have

$$H^0 \operatorname{Hom}_+(\epsilon, \epsilon) = \mathbb{F}_2 \qquad H^1 \operatorname{Hom}_+(\epsilon, \epsilon) = \mathbb{F}_2^2$$
$$H^1 \operatorname{Hom}_-(\epsilon, \epsilon) = \mathbb{F}_2^2 \qquad H^2 \operatorname{Hom}_-(\epsilon, \epsilon) = \mathbb{F}_2.$$

in agreement with H^*, H_c^* for the punctured torus.

Augmentations are sheaves

Nadler–Zaslow correspondence ("microlocalization")

$$Sh(M; \Bbbk) \xrightarrow{\sim} Fuk_\epsilon(T^*M; \Bbbk):$$

equivalence between a category of sheaves on a manifold M and the infinitesimally wrapped Fukaya category of T^*M.

The augmentation category is some flavor of the RHS.

The LHS in this setting is $STZ(\Lambda, \Bbbk)$, the Shende–Treumann–Zaslow dg category of rank 1 constructible sheaves (2014).

Theorem (NRSSZ)

Let Λ be a Legendrian knot or link in $\mathbb{R}^3$. Then we have an equivalence of A_∞ categories

$$STZ(\Lambda, \Bbbk) \cong \mathcal{A}ug_+(\Lambda, \Bbbk).$$