From holomorphic curves to knot invariants via the cotangent bundle

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Based on joint work with: Tobias Ekholm (Uppsala University), John Etnyre (Georgia Tech), and Michael Sullivan (University of Massachusetts); and Mina Aganagic (UC Berkeley), Tobias Ekholm, and Cumrun Vafa (Harvard University).
Outline

1. Topological motivation
2. The cotangent bundle
3. Knot contact homology
4. Relation to physics
Classification of manifolds

Motivating question in low-dimensional topology: classify or characterize topological/smooth manifolds in 3 and 4 dimensions, up to equivalence.

Three types of equivalence of manifolds:

- homotopy equivalence
- homeomorphism (topological equivalence)
- diffeomorphism (smooth equivalence).

We have

\[ \text{diffeomorphic} \Rightarrow \text{homeomorphic} \Rightarrow \text{homotopy equivalent}. \]

In three dimensions, diffeomorphic $\Leftrightarrow$ homeomorphic.
Let $M$ be a closed topological 3-manifold such that

\[ \pi_1(M) = 1. \]

Then $M$ is homeomorphic to $S^3$.

Poincaré conjecture famously proven by Perelman about a decade ago.
Poincaré conjecture

Let $M$ be a closed topological 3-manifold such that

$$\pi_1(M) = 1.$$ 

Then $M$ is homeomorphic to $S^3$.

Poincaré conjecture famously proven by Perelman about a decade ago.

$n$-dimensional Poincaré conjecture

Any topological manifold homotopy equivalent to $S^n$ is homeomorphic to $S^n$.

True in all dimensions (Smale $n \geq 5$; Freedman $n = 4$; Perelman $n = 3$).
Smooth $n$-dimensional Poincaré conjecture

Any smooth manifold homotopy equivalent to $S^n$ is diffeomorphic to $S^n$.

True for $n \leq 3$; resolved for $n \geq 5$ (e.g., false for $n = 7$: Milnor’s exotic $S^7$’s).
The smooth Poincaré conjecture

**Smooth $n$-dimensional Poincaré conjecture**

Any smooth manifold homotopy equivalent to $S^n$ is **diffeomorphic** to $S^n$.

True for $n \leq 3$; resolved for $n \geq 5$ (e.g., false for $n = 7$: Milnor’s exotic $S^7$’s).

Number of smooth structures on $S^n$:

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Kervaire–Milnor (1963): count for $n \geq 5$ using homotopy theory.
The smooth Poincaré conjecture

### Smooth $n$-dimensional Poincaré conjecture

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$n = 4$: open!
Smooth 4-dimensional Poincaré conjecture

*If a smooth manifold $M$ is homotopy equivalent (or homeomorphic) to $S^4$, then it is diffeomorphic to $S^4$.*

There are a number of possible counterexamples to this conjecture: proposed “exotic $S^4$’s”.

One stumbling block: a lack of good invariants of smooth 4-manifolds that apply to this setting.
Smooth 4-dimensional Poincaré

Smooth 4-dimensional Poincaré conjecture

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Cotangent bundles to the rescue?
Phase space

Particle in $\mathbb{R}^3$:
- position $q = (q_1, q_2, q_3)$
- momentum $p = (p_1, p_2, p_3)$

The *phase space* of the particle is

$$\mathbb{R}^6 = \mathbb{R}^3_{(q_1, q_2, q_3)} \times \mathbb{R}^3_{(p_1, p_2, p_3)}.$$
More generally, a particle in a manifold $M$ has a position $q \in M$ and a velocity vector $v \in T_q M$; for various reasons, it’s more natural to consider the dual, momentum vector $p \in (T_q M)^*$. 

The phase space of the particle is the **cotangent bundle**

$$T^* M = \{(q, p) \mid q \in M, \ p \in (T_q M)^*\}.$$ 

If $\dim_{\mathbb{R}} M = n$, then $\dim_{\mathbb{R}} T^* M = 2n$. 

![Diagram](image-url)
Symplectic manifolds

Cotangent bundles $T^*M$ are examples of symplectic manifolds.

**Definition**

A 2-form $\omega$ on a $2n$-dim’l manifold $W$ is a **symplectic form** if
- $d\omega = 0$ ($\omega$ is closed)
- $\omega^n$ is a nowhere zero $2n$-form ($\omega$ is nondegenerate).

**Definition**

An even-dimensional manifold is a **symplectic manifold** if it has a symplectic form.
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**Definition**

An even-dimensional manifold is a **symplectic manifold** if it has a symplectic form.

The “prototypical” symplectic manifold is $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ with coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ and symplectic form

$$\omega = dq_1 \wedge dp_1 + \cdots + dq_n \wedge dp_n.$$
More generally, on a cotangent bundle $T^*M$ with local coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$, we can define a 2-form $\omega \in \Omega^2(T^*M)$ by

$$\omega = dq_1 \wedge dp_1 + \cdots + dq_n \wedge dp_n.$$ 

**Theorem**

- For any smooth manifold $M$, $\omega$ is independent of coordinates, and $(T^*M, \omega)$ is a symplectic manifold.
- If $M$ and $M'$ are diffeomorphic (equivalent as smooth manifolds), then the symplectic manifolds $T^*M$ and $T^*M'$ are symplectomorphic (equivalent as symplectic manifolds).
The symplectic form on $T^*M$

Coordinate-free definition of $\omega \in \Omega^2(T^*M)$:

There is a canonical 1-form $\lambda_{\text{can}} \in \Omega^1(T^*M)$, the Liouville form: for $v \in T_{(q,p)}(T^*M)$,

$$\lambda_{\text{can}}(v) = \langle \pi(v), d\pi(v) \rangle.$$ 

Then

$$\omega = -d\lambda_{\text{can}}.$$
Arnol’d’s strategy

V. I. Arnol’d: study the smooth topology of $M$ via the symplectic topology of $T^*M$.

Question

If $M, M'$ are closed smooth manifolds such that $T^*M$ and $T^*M'$ are symplectomorphic, are $M$ and $M'$ necessarily diffeomorphic?

Note: recent result of Adam Knapp (2012) shows that this is not necessarily true without the closed condition: exotic $\mathbb{R}^4$’s have symplectomorphic cotangent bundles.
One way to produce invariants of smooth manifolds:

\[ M \quad \text{smooth manifold} \quad \longrightarrow \quad T^* M \quad \text{symplectic manifold} \]

\[ \text{smooth invariant of } M \quad \overset{:=}{\longleftarrow} \quad \text{symplectic invariant of } T^* M \]
Smooth invariants from symplectic geometry

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The symplectic invariants are often given by counts of holomorphic curves.
Gromov, 1980s: one can create interesting invariants of symplectic manifolds \((W, \omega)\) by studying **holomorphic curves** in \(W\): Riemann surfaces in \(W\) satisfying a certain compatibility condition with \(\omega\) (involving an almost complex structure on \(W\) tamed by \(\omega\)).

Gromov’s insight: in many cases, there are only *finitely many* holomorphic curves, and counting them yields symplectic invariants (cf. algebraic geometry).
More generally, the *moduli space* of holomorphic curves is often well-behaved (e.g., a manifold with corners) and studying this moduli space yields symplectic invariants.
Hamiltonian Floer Homology

One invariant of (certain) symplectic manifolds: Hamiltonian Floer homology (based on Floer, 1988).


The Hamiltonian Floer homology of the symplectic manifold $T^*M$ is isomorphic to the singular homology of the free loop space $\mathcal{L}M$:

$$HF_\ast(T^*M) \cong H_\ast(\mathcal{L}M).$$

Thus the symplectic structure on $T^*M$ remembers at least some homotopic data about $M$. 
Recently, Mohammed Abouzaid has shown that the symplectic structure on $T^*M$ can encode more than the homotopic/topological structure of $T^*M$: it can encode smooth information.

**Theorem (Abouzaid, 2008)**

*If $\Sigma$ is an exotic $S^{4k+1}$ that does not bound a parallelizable manifold, then $T^*\Sigma$ is not symplectomorphic to $T^*S^{4k+1}$.***

Kervaire–Milnor: there are 8 different smooth structures on $S^9$; this shows that 6 of them are distinct from the standard smooth structure.

Abouzaid’s argument studies certain moduli spaces of holomorphic curves on $T^*\Sigma$. 
We will focus on a relative of the cotangent construction.

**Definition**

Let $K \subset M$ be a submanifold. The **conormal bundle** to $K$ is

$$L_K := \{(q, p) \mid q \in K \text{ and } \langle p, v \rangle = 0 \text{ for all } v \in T_qK\} \subset T^*M.$$
If \( \dim(M) = n \), then \( \dim(T^*M) = 2n \) and dimension counting shows that \( \dim(L_K) = n \) regardless of the dimension of \( K \).

**Theorem**

*For any submanifold \( K \subset M \),

\[
L_K \subset T^*M
\]

is **Lagrangian**: a maximal-dimensional submanifold of \( T^*M \) on which the symplectic form \( \omega \) is identically 0.*

We will be interested in the case where \( M = \mathbb{R}^3 \) and \( K \subset \mathbb{R}^3 \) is a knot: a smooth embedding of \( S^1 \) in \( \mathbb{R}^3 \). In this case, \( L_K \cong S^1 \times \mathbb{R}^2 \) is a Lagrangian submanifold of \( T^*\mathbb{R}^3 \cong \mathbb{R}^6 \).
Knots in $\mathbb{R}^3$

We consider knots in $\mathbb{R}^3$ up to smooth isotopy: two knots $K_0$ and $K_1$ are smoothly isotopic if there is a 1-parameter family of knots $K_t$ for $0 \leq t \leq 1$.

Smoothly isotopic knots (here, the right-handed trefoil).
If knots $K_0, K_1 \subset \mathbb{R}^3$ are smoothly isotopic, then there is a 1-parameter family of Lagrangian submanifolds $L_{K_t} \subset T^*\mathbb{R}^3$: $L_{K_0}, L_{K_1}$ are Lagrangian isotopic.

**Question**

*How much of the topology of the knot $K \subset \mathbb{R}^3$ is encoded in the symplectic/Lagrangian structure of $L_K \subset T^*\mathbb{R}^3$?*
If knots $K_0, K_1 \subset \mathbb{R}^3$ are smoothly isotopic, then there is a 1-parameter family of Lagrangian submanifolds $L_{K_t} \subset T^*\mathbb{R}^3$: $L_{K_0}, L_{K_1}$ are Lagrangian isotopic.

**Question**

*How much of the topology of the knot $K \subset \mathbb{R}^3$ is encoded in the symplectic/Lagrangian structure of $L_K \subset T^*\mathbb{R}^3$?*

**Conjecture?**

*The Lagrangian submanifold $L_K$ is a complete knot invariant: if $K_0, K_1$ are knots such that $L_{K_0}$ and $L_{K_1}$ are Lagrangian isotopic, then $K_0$ and $K_1$ are smoothly isotopic.

(More precise conjecture involves “Legendrian isotopy” in the contact manifold $ST^*\mathbb{R}^3$ of $\Lambda_K := L_K \cap ST^*\mathbb{R}^3$.)
Theorem (N., 2005)

\[ L_K \text{ detects the unknot } O: \text{ if } K \subset \mathbb{R}^3 \text{ is a knot such that } \Lambda_K \text{ and } \Lambda_O \text{ are Legendrian isotopic, then } K \text{ is} \text{ unknotted: } K = O. \]
Legendrian contact homology

To distinguish between Lagrangians $L_K$ for different knots $K$, need good invariants of Lagrangian submanifolds in symplectic manifolds.

One is given by Legendrian contact homology (LCH) (Eliashberg–Hofer, 1990s; Etnyre–Ekholm–Sullivan, 2005). LCH inputs a Legendrian submanifold $\Lambda$ of a contact manifold $\mathcal{V}$, and outputs a count of holomorphic curves in the symplectization $\mathbb{R} \times \mathcal{V}$ with boundary on $\mathbb{R} \times \Lambda$ and certain asymptotic behavior.
Legendrian contact homology

To distinguish between Lagrangians $L_K$ for different knots $K$, need good invariants of Lagrangian submanifolds in symplectic manifolds.

In our setting, LCH counts certain holomorphic disks in $T^*M$ with boundary on $L_K$. 
Knot contact homology

\[ K \subset \mathbb{R}^3 \text{ knot} \rightarrow L_K \subset T^*\mathbb{R}^3 \text{ Lagrangian} \]

\[ LCH \]

\[ HC_*(L_K), \text{ symplectic invariant} \]
Knot contact homology

Let $K \subset \mathbb{R}^3$ be a knot. The knot contact homology $HC_*(K)$ is the LCH associated to $L_K \subset T^*\mathbb{R}^3$. This is a knot invariant (an invariant of knots up to smooth isotopy).

There is a combinatorially-defined differential graded algebra $(\mathcal{A}, \partial)$ associated to a knot $K$, for which

$$H_*(\mathcal{A}, \partial) = HC_*(K).$$

The algebra $\mathcal{A}$ is a finitely-generated noncommutative algebra over the ring $\mathbb{Z}[^{\pm1}\lambda, ^{\pm1}\mu, ^{\pm1}U]$. 

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The algebra $\mathcal{A}$ is a finitely-generated noncommutative algebra over the ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}, U^{\pm 1}]$.

Conjecture?

Knot contact homology is a complete knot invariant: if knots $K_1, K_2$ satisfy

$$HC_*(K_1) \cong HC_*(K_2)$$

then $K_1 = K_2$. 
Properties of knot contact homology

Theorem (N., 2005)

- Knot contact homology $HC_*(K)$ determines the Alexander polynomial $\Delta_K(t)$.
- Knot contact homology is “relatively strong” as a knot invariant: it can distinguish mirrors, mutants, etc.

Two famous “mutant” knots: the Kinoshita–Terasaka knot and the Conway knot.
A new polynomial knot invariant

Definition

The **augmentation variety** of a knot $K$ (with DGA $(\mathcal{A}, \partial)$) is

$$\{(\lambda, \mu, U) \in (\mathbb{C} \setminus \{0\})^3 | \text{there is an algebra map } \epsilon : \mathcal{A} \to \mathbb{C} \text{ with } \epsilon \circ \partial = 0\} \subset (\mathbb{C} \setminus \{0\})^3.$$
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This appears to be a codimension-1 algebraic set for all knots $K$.

Definition

The augmentation polynomial of a knot $K$

$$\text{Aug}_K(\lambda, \mu, U) \in \mathbb{Z}[\lambda, \mu, U]$$

is the polynomial for which the augmentation variety is

$$\{\text{Aug}_K(\lambda, \mu, U) = 0\}.$$
Computing the augmentation polynomial

In practice, to a knot $K$, knot contact homology associates a finite, combinatorially defined collection of polynomials in some variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{Z}[\lambda, \mu, U]$:

$$K \mapsto \{ p_1(x_1, \ldots, x_n), \ldots, p_m(x_1, \ldots, x_n) \}.$$  

The augmentation variety is the set of $(\lambda, \mu, U)$ for which these polynomials have a common root in $x_1, \ldots, x_n$:

$$p_1(x_1, \ldots, x_n) = 0,$$
$$p_2(x_1, \ldots, x_n) = 0,$$
$$\vdots,$$
$$p_m(x_1, \ldots, x_n) = 0.$$
Augmentation polynomial: unknot

For $K = O$, the unknot: the collection of polynomials in $n = 0$ variables is

$$\{U - \lambda - \mu + \lambda\mu\}.$$ 

Thus

$$\text{Aug}_O(\lambda, \mu, U) = U - \lambda - \mu + \lambda\mu.$$
Augmentation polynomial: trefoil

For $K = T$, the right-handed trefoil: the collection of polynomials in $n = 1$ variable is

$$\{Ux_1^2 - \mu Ux_1 + \lambda \mu^3(1 - \mu), Ux_1^2 + \lambda \mu^2x_1 + \lambda \mu^2(\mu - U)\}.$$ 

Then take the resultant of these two polynomials:

$$\text{Aug}_T(\lambda, \mu, U) = (U^3 - \mu U^2) + (-U^3 + \mu U^2 - 2\mu^2 U + 2\mu^2 U^2 + \mu^3 U - \mu^4 U)\lambda + (-\mu^3 + \mu^4)\lambda^2.$$
Relation to other knot invariants

Theorem (N. 2005)

A specialization of the augmentation polynomial,

$$\text{Aug}_K(\lambda, \mu, 1),$$

contains the A-polynomial $A_K(\lambda, \mu^2)$ as a factor.

Here the A-polynomial is a knot invariant related to $SL_2\mathbb{C}$-representations of the knot complement and hyperbolic structures.

Corollary (N. 2005)

The augmentation polynomial $\text{Aug}_K(\lambda, \mu, U)$, and thus knot contact homology, detects the unknot: if $\text{Aug}_K = \text{Aug}_O$ then $K = O$. 
It appears that knot contact homology in general is intimately related with the topology of the knot complement.

In a different direction, knot contact homology is also related to the HOMFLY-PT polynomial, a two-variable knot polynomial that generalizes the Alexander and Jones polynomials:

**Conjecture**

The augmentation polynomial encodes a specialization of the HOMFLY-PT polynomial, $P_K(a, 1)$.

The motivation for this conjecture comes from physics.
Conifold transition

\[ T^*S^3 \quad \text{cone on } S^2 \times S^3 \quad X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \]

Gopakumar–Vafa (1998), building on work of Witten: starting with \( T^*S^3 \), pass through the “conifold transition” to obtain a 6-manifold \( X \), the total space of the rank 2 complex vector bundle

\[ \mathcal{O}(-1) \oplus \mathcal{O}(-1) \]

\[ \mathbb{C}P^1. \]
Conifold transition

\[ T^* S^3 \quad \text{cone on } S^2 \times S^3 \quad X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \]

**Conjecture (Gopakumar–Vafa)**

*In the large N limit:*

\[
\begin{align*}
SU(N) \text{ Chern–Simons theory on } S^3 \\
\Downarrow \\
\text{closed topological string theory on } X.
\end{align*}
\]
Conifold transition and $L_K$

Ooguri–Vafa (1999): given a knot $K \subset S^3$, follow the Lagrangian $L_K$ through the conifold transition to obtain a Lagrangian $\tilde{L}_K \subset X$. 
Conifold transition and $L_K$

Conjecture (Ooguri–Vafa)

*In the large $N$ limit:*

- $SU(N)$ Chern–Simons theory for $K \subset S^3$
- open topological string theory for $\tilde{L}_K \subset X$. 

Conifold transition and $L_K$

$T^*S^3$ \[ \rightarrow \] conifold \[ \rightarrow \] $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$
Conifold transition and $L_K$

Checked for unknot, some torus knots.

Slightly more mathematical statement:

Chern–Simons knot invariants for $K \subset S^3$
(e.g. Jones polynomial)

open Gromov–Witten invariants for $\tilde{L}_K \subset X$. 
Aganagic–Vafa (2012) propose a “generalized Strominger–Yau–Zaslow conjecture” that uses $\tilde{L}_K \subset X$ to produce a mirror to $X$.

**Conjecture (Aganagic–Vafa)**

The pair $(X, \tilde{L}_K)$ produces a mirror Calabi–Yau 3-fold to $X$,

$$X_K = \{(u, v, x, p) \mid uv = A_K(e^x, e^p, Q)\}$$

$$\subset \mathbb{C}^4.$$

Here $Q$ is a parameter measuring the complexified Kähler class of $\mathbb{CP}^1$ and $A_K$ is a three-variable polynomial.
The mirror and knot invariants

The dashed arrows use string-theoretic arguments of Gukov–Schwarz–Vafa (2004) and others.
Conjecture (Aganagic–Ekholm–N.-–Vafa 2012)

The two polynomials $A_K$ and $\text{Aug}_K$ are equal for all knots $K$.

This would imply that the augmentation polynomial $\text{Aug}_K(\lambda, \mu, U)$ is at least as strong as many other known knot invariants.

Currently: a great deal of circumstantial evidence for this conjecture, but no proof.
Summary of knot invariants

\[ K \subset \mathbb{R}^3 \text{ smooth knot} \]

\[ L_K \subset T^*\mathbb{R}^3 \text{ Lagrangian submanifold} \]

\[ HC_*(K) = H_*(\mathcal{A}, \partial) \text{ knot contact homology} \]

\[ \text{Aug}_K(\lambda, \mu, U) \text{ augmentation polynomial} \]

HOMFLY, knot homologies, ???

unknot detection
For further reading:

- T. Perutz, *The symplectic topology of cotangent bundles*, article in the March 2010 EMS Newsletter
- L. Ng, *Conormal bundles, contact homology, and knot invariants*, math/0412330
- T. Ekholm and J. Etnyre, *Invariants of knots, embeddings and immersions via contact geometry*, math/0412517
- L. Ng, *A topological introduction to knot contact homology*, forthcoming
- Another forthcoming survey paper?