COMBINATORIAL KNOT CONTACT HOMOLOGY AND TRANSVERSE KNOTS

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Abstract. We give a combinatorial treatment of transverse homology, a new invariant of transverse knots that is an extension of knot contact homology. The theory comes in several flavors, including one that is an invariant of topological knots and produces a three-variable knot polynomial related to the A-polynomial. We provide a number of computations of transverse homology that demonstrate its effectiveness in distinguishing transverse knots, including knots that cannot be distinguished by the Heegaard Floer transverse invariants or other previous invariants.

1. Introduction and Results

In this paper, we extend the combinatorial form for knot contact homology introduced in [Ng05a, Ng05b, Ng08] to an invariant of knots in $\mathbb{R}^3$ that are transverse to the standard contact structure $\ker(dz - y\,dx)$. This new invariant, which we call *transverse homology*, comes in several flavors. One version, *infinity transverse homology*, can be seen as an invariant of topological knots in $\mathbb{R}^3$ with no reference to a contact structure; as such, it is an extension and generalization of knot contact homology.

The transverse invariant described here is a combinatorial form of a more general invariant of transverse knots in contact 3-manifolds. This general invariant is introduced in [EENSa], where it is also proven to specialize to the combinatorial complex studied in this paper in the case of knots in standard contact $\mathbb{R}^3$.[1] Here is a brief summary of the geometric construction in [EENSa], building on the work in [EENSb], in the case of interest to us: a knot $K \subset (\mathbb{R}^3, \xi_{\text{std}})$ lifts to a Legendrian torus Λ_K, the conormal bundle to K, in the cosphere bundle $ST^*\mathbb{R}^3$; the contact structure ξ similarly lifts to two submanifolds $\widetilde{\xi}, \widetilde{-\xi} \subset ST^*\mathbb{R}^3$ corresponding to the two possible coorientations of ξ. If K is transverse to ξ, then Λ_K is disjoint from both $\widetilde{\xi}$ and $\widetilde{-\xi}$. Now $\widetilde{\xi}$ and $\widetilde{-\xi}$ lift to 4-dimensional submanifolds in the symplectization of $ST^*\mathbb{R}^3$, and these submanifolds are holomorphic with respect to an appropriate almost complex structure. Counting intersections of holomorphic

[1]The combinatorial invariant defined in [EENSa] uses slightly different sign conventions from the one defined here, but the two invariants are equivalent. See Section 3.4 for further discussion.

disks with these holomorphic 4-manifolds induces filtrations on the Legendrian contact homology complex for Λ_K, resulting in the invariant that we call transverse homology. We keep track of these two filtrations through two parameters U and V, which encode intersection numbers with $\widetilde{\xi}$ and $\widetilde{-\xi}$.

We will henceforth suppress the geometric origins of transverse homology, and treat the invariant purely combinatorially. In particular, we will present combinatorial proofs of the invariance results for transverse homology. Please see [EENSa] for the parallel contact-geometric story.

The formulation of transverse homology described in this paper is defined in terms of a braid whose closure is the topological or transverse knot under consideration. By the classical Markov Theorem, isotopy classes of topological knots (or links) correspond to equivalence classes of braids modulo conjugation and positive/negative stabilization/destabilization. For the purposes of this paper, we can define a transverse knot likewise.

Definition 1.1. A *transverse knot* or link in $\mathbb{R}^3$ is an equivalence class of braids modulo conjugation and positive de/stabilization.

This definition is equivalent to the usual definition, a knot everywhere transverse to the standard contact structure, by the Transverse Markov Theorem [OS03, Wri].

We now present our main results. Throughout this paper, we write $R = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$. To any braid B whose closure has one component[2], we will associate a differential graded algebra over $R[U, V]$, the *transverse complex*

$$B \rightsquigarrow (CT_*^-(B), \partial^-(B)).$$

We remark that unlike in Heegaard Floer homology, U and V have grading 0. Define the *transverse homology* of B to be the graded $R[U, V]$-algebra $HT_*^-(B) = H_*(CT^-(B), \partial^-(B))$.

Theorem 1.2. *Up to stable tame isomorphism, the transverse complex $(CT_*^-(B), \partial^-(B))$ is invariant under braid conjugation and positive braid stabilization, and thus constitutes an invariant of the transverse knot underlying the braid B. In particular, $HT_*^-(B)$ is a transverse invariant. In addition, setting $U = V = 1$ in $(CT_*^-(B), \partial^-(B))$ yields the framed knot DGA from* [Ng08].

As is familiar from Heegaard Floer homology or similar settings, we can derive several auxiliary complexes from the transverse complex.

Definition 1.3. Let B be a braid.

- The *hat transverse complex* $(\widehat{CT}_*(B), \widehat{\partial}(B))$ is the differential graded algebra over R obtained from $(CT_*^-(B), \partial^-(B))$ by setting $U = 0$

[2] All of the invariants we present in this paper have corresponding analogues for multi-component links, where the base ring R is replaced by $\mathbb{Z}[\lambda_1^{\pm 1}, \mu_1^{\pm 1}, \ldots, \lambda_k^{\pm 1}, \mu_k^{\pm 1}]$ for k the number of components of the link. We have specialized to the one-component knot case for simplicity; see [EENSa] for more on the general link invariants.

and $V = 1$; the *hat transverse homology* $\widehat{HT}_*(B)$ is the graded R-algebra $H_*(\widehat{CT}(B), \widehat{\partial}(B))$.

- The *double-hat transverse complex* $(\widehat{\widehat{CT}}_*(B), \widehat{\widehat{\partial}}(B))$ is the differential graded algebra over R obtained from $(CT^-_*(B), \partial^-(B))$ by setting $U = V = 0$; the *double-hat transverse homology* $\widehat{\widehat{HT}}_*(B)$ is the graded R-algebra $H_*(\widehat{\widehat{CT}}(B), \widehat{\widehat{\partial}}(B))$.
- The *infinity transverse complex* $(CT^\infty_*(B), \partial^\infty(B))$ is the differential graded algebra over $R[U^{\pm 1}, V^{\pm 1}]$ obtained from $(CT^-_*(B), \partial^-(B))$ by tensoring with $R[U^{\pm 1}, V^{\pm 1}]$ and replacing λ by $\lambda(U/V)^{-(\mathrm{sl}(B)+1)/2}$, where $\mathrm{sl}(B) = w(B) - n(B)$ is the difference between the writhe (algebraic crossing number) of B and the number of strands in B; the *infinity transverse homology* $HT^\infty_*(B)$ is the graded $R[U^{\pm 1}, V^{\pm 1}]$-algebra $H_*(CT^\infty(B), \partial^\infty(B))$.

Note the substitution $\lambda \mapsto \lambda(U/V)^{-(\mathrm{sl}(B)+1)/2}$ in the infinity theory, which is allowed once U, V are invertible, and which is necessary for the invariance statement below.

Besides the complexes listed in Definition 1.3 above, there are many others that one might consider: e.g., the DGA over R obtained by setting $U = 1, V = 0$. Many of these are equivalent to one of the complexes from Definition 1.3; see the discussion of symmetries in Section 4. There are other variants, e.g., a "+ version", that we will not consider here.

Theorem 1.4. *Let B be a braid.*

(1) *Up to stable tame isomorphism, the DGAs $(\widehat{CT}_*(B), \widehat{\partial}(B))$ and $(\widehat{\widehat{CT}}_*(B), \widehat{\widehat{\partial}}(B))$ are invariants of the transverse knot underlying B. In particular, $\widehat{HT}_*(B)$ and $\widehat{\widehat{HT}}_*(B)$ are transverse invariants.*
(2) *Up to stable tame isomorphism, the DGA $(CT^\infty_*(B), \partial^\infty(B))$ is an invariant of the knot type given by the closure of B. In particular, $HT^\infty_*(B)$ is a topological invariant.*

Statement (1) in Theorem 1.4 is a direct corollary of Theorem 1.2, while the proof of statement (2) requires checking that $(CT^\infty_*(B), \partial^\infty(B))$ is invariant under negative braid stabilization as well as conjugation and positive stabilization.

We summarize the various flavors of transverse homology in the diagram below. We indicate which of the flavors are transverse knot invariants and which are topological knot invariants by using inputs denoted T for a transverse knot and K for a topological knot. The final DGA $CC_*(K)$ in the diagram is the framed knot DGA from [Ng08].

$$CT^-_*(T) \text{ over } R[U,V] \xrightarrow{U=0,\,V=1} \widehat{CT}_*(T) \text{ over } R$$

$$CT^-_*(T) \text{ over } R[U,V] \xrightarrow{U=V=0} \widehat{\widehat{CT}}_*(T) \text{ over } R$$

$$CT^-_*(T) \text{ over } R[U,V] \xrightarrow{\otimes R[U^{\pm 1},V^{\pm 1}];\ \lambda\mapsto\lambda(U/V)^{-(\mathrm{sl}+1)/2}} CT^\infty_*(K) \text{ over } R[U^{\pm 1},V^{\pm 1}]$$

$$CT^\infty_*(K) \text{ over } R[U^{\pm 1},V^{\pm 1}] \xrightarrow{U=V=1} CC_*(K) \text{ over } R$$

We will see that transverse homology constitutes an *effective transverse invariant*; that is, it can be used to distinguish pairs of transverse knots with the same topological type and self-linking number. In particular, we can use transverse homology to produce the following result.

Proposition 1.5. *The knot types* $m(7_2)$*,* $m(7_6)$*,* $m(9_{44})$*,* 9_{48}*,* $m(10_{132})$*,* 10_{136}*,* $m(10_{140})$*,* $m(10_{145})$*,* $m(10_{161})$*, and* $12n_{591}$ *are transversely nonsimple.*

Six of the ten knots in Proposition 1.5 were previously known to be nonsimple, using a Heegaard Floer transverse invariant (either grid-diagram [OST08] or LOSS [LOSS09]). The other four, $m(7_6)$, $m(9_{44})$, 9_{48}, and 10_{136}, cannot be distinguished by any previously known invariant, including Heegaard Floer.

As a crude gauge of the effectiveness of our invariant, there are thirteen total knots with arc index at most 9 that are conjectured to be transversely nonsimple by the "Legendrian knot atlas" [CN]; of these, transverse homology can prove nonsimplicity for at least the ten listed in Proposition 1.5. Transverse homology could in theory be applied to the three remaining knots as well, though probably with some difficulty. See Section 5 for further discussion, including a table of all thirteen knots with transverse representatives in grid-diagram and braid form.

The proof of Proposition 1.5 via transverse homology involves auxiliary calculations in *Mathematica*, using a program `transverse.m` that computes the transverse DGA of any braid and can compute various numerical invariants derived from the transverse DGA. Readers interested in doing computations with transverse homology can download `transverse.m` from the author's web site.

All of the applications to transverse nonsimplicity given in this paper use only a small part of the transverse homology invariant, the degree 0 invariant $\widehat{HT}_0$ in the hat version. A reader familiar with knot contact homology may recall that the degree 0 part of knot contact homology, HC_0, has a simple topological interpretation as the "cord algebra" of the knot [Ng05b, Ng08]. A similar interpretation of degree 0 transverse homology would be interesting, and in particular might lead to transverse nonsimplicity results for general families of knots, but does not presently exist. We hope to return to this issue in the future.

The topological version of transverse homology, HT_*^∞, may be of interest in its own right as a generalization of knot contact homology. On the contact-geometric side, passing from knot contact homology $HC_*(K)$ to infinity transverse homology $HT_*^\infty(K)$ represents an extension of the base ring for the Legendrian contact homology of ΛK in the contact manifold $ST^*\mathbb{R}^3$ from $\mathbb{Z}[H_1(\Lambda K)]$ to $\mathbb{Z}[H_2(ST^*\mathbb{R}^3, \Lambda K)]$. See [EENSa] for further discussion.

We will not discuss topological aspects of infinity transverse homology much in this paper, but one can produce from $HT_0^\infty(K)$ a three-variable knot polynomial $\mathrm{Aug}_K(\lambda, \mu, U)$, the "augmentation polynomial", in an analogous manner to the two-variable augmentation polynomial $\tilde{A}_K(\lambda, \mu)$ produced from $HC_0(K)$ in [Ng08]. It was shown in [Ng08] that $\tilde{A}_K(\lambda, \mu)$ contains the familiar A-polynomial of K as a factor. We suspect that $\mathrm{Aug}_K(\lambda, \mu, 1) = \tilde{A}_K(\lambda, \mu)$ in general, in which case $\mathrm{Aug}_K(\lambda, \mu, U)$ could be viewed as some sort of three-variable generalization of the A-polynomial.

Here is an outline of the rest of the paper. We present algebraic definitions for the various flavors of transverse homology in Section 2 and prove invariance in Section 3. In Section 4, we establish some basic properties and symmetries of transverse homology. Section 5 presents a number of computations of transverse homology, discussing the three-variable augmentation polynomial for topological knots, and demonstrating the effectiveness of transverse homology as an invariant of transverse knots.

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2. The Invariants

Let $B \in B_n$ be an n-strand braid. In this section, we associate a differential graded algebra $(\mathcal{A}, \partial)$ to B, the transverse DGA of B. Algebraically, the transverse DGA is just an easy enhancement of the framed knot DGA from [Ng08]. A main result of this paper is that, up to stable tame isomorphism, $(\mathcal{A}, \partial)$ is invariant under braid conjugation and positive braid stabilization, and is thus an invariant of the underlying transverse knot.

Recall that R represents the ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$. Write $\mathcal{A}_n$ for the tensor algebra over $R[U,V]$ generated by $n(n-1)$ formal variables a_{ij} with $1 \leq i,j \leq n$ and $i \neq j$. As in [Ng08], there is a representation $\phi : B_n \to \operatorname{Aut}\mathcal{A}_n$ defined on the generators σ_k of B_n by

$$\phi_{\sigma_k} : \begin{cases} a_{ki} \mapsto -a_{k+1,i} - a_{k+1,k}a_{ki}, & i \neq k, k+1 \\ a_{ik} \mapsto -a_{i,k+1} - a_{ik}a_{k,k+1}, & i \neq k, k+1 \\ a_{k+1,i} \mapsto a_{ki}, & i \neq k, k+1 \\ a_{i,k+1} \mapsto a_{ik}, & i \neq k, k+1 \\ a_{k,k+1} \mapsto a_{k+1,k} & \\ a_{k+1,k} \mapsto a_{k,k+1} & \\ a_{ij} \mapsto a_{ij}, & i,j \neq k, k+1. \end{cases}$$

We will write ϕ_B for the automorphism of $\mathcal{A}_n$ assigned to a braid B by the map ϕ.

It will be convenient to consider $n \times n$ matrices with entries in $\mathcal{A}_n$; write $\operatorname{Mat}_n(\mathcal{A}_n)$ for the $R[U,V]$-algebra of such matrices. To a braid $B \in B_n$, the homomorphism ϕ associates two matrices $\mathbf{\Phi}^L_B, \mathbf{\Phi}^R_B \in \operatorname{Mat}_n(\mathcal{A}_n)$, as follows. View B as a braid in B_{n+1} by adding an additional strand labeled $n+1$, and write ϕ^{ext}_B for the resulting automorphism of $\mathcal{A}_{n+1}$; then for any $i = 1, \ldots, n$, we can write

$$\phi^{\text{ext}}_B(a_{i,n+1}) = \sum_{\ell=1}^{n} (\mathbf{\Phi}^L_B)_{i\ell} a_{\ell,n+1}$$
$$\phi^{\text{ext}}_B(a_{n+1,i}) = \sum_{\ell=1}^{n} a_{n+1,\ell} (\mathbf{\Phi}^R_B)_{\ell i}$$

for $(\mathbf{\Phi}^L_B)_{i\ell}, (\mathbf{\Phi}^R_B)_{\ell i} \in \mathcal{A}_n$.

Assemble the generators a_{ij} into two matrices $\mathbf{A}_{\geq}, \mathbf{A}_{\leq} \in \operatorname{Mat}_n(\mathcal{A}_n)$, with

$$(\mathbf{A}_{\geq})_{ij} = \begin{cases} a_{ij} & i > j \\ -1 & i = j \\ 0 & i < j \end{cases} \qquad \text{and} \qquad (\mathbf{A}_{\leq})_{ij} = \begin{cases} 0 & i > j \\ -1 & i = j \\ a_{ij} & i < j. \end{cases}$$

Define two more matrices $\hat{\mathbf{A}}$, $\check{\mathbf{A}} \in \operatorname{Mat}_n(\mathcal{A}_n)$ by

$$\hat{\mathbf{A}} = \mathbf{A}_{\geq} + \mu U \mathbf{A}_{\leq}$$
$$\check{\mathbf{A}} = V\mathbf{A}_{\geq} + \mu \mathbf{A}_{\leq};$$

that is,

$$(\hat{\mathbf{A}})_{ij} = \begin{cases} a_{ij} & i > j \\ -1 - \mu U & i = j \\ \mu U a_{ij} & i < j \end{cases} \qquad \text{and} \qquad (\check{\mathbf{A}})_{ij} = \begin{cases} V a_{ij} & i > j \\ -V - \mu & i = j \\ \mu a_{ij} & i < j. \end{cases}$$

Note that when $U = V = 1$, the matrices $\hat{\mathbf{A}}$ and $\check{\mathbf{A}}$ are identical and equal to the matrix labeled A in [Ng08].

Finally, let $\mathbf{\Lambda}_B, \mathbf{\Lambda}'_B$ denote the $n\times n$ diagonal matrices $\mathrm{diag}(\lambda\mu^{-w(B)}, 1, 1, \ldots, 1)$ and $\mathrm{diag}(\lambda\mu^{-w(B)}(U/V)^{-(\mathrm{sl}(B)+1)/2}, 1, 1, \ldots, 1)$, where $w(B), \mathrm{sl}(B)$ are the writhe and self-linking number of B.

Definition 2.1. (1) The *degree*-0 *transverse homology* of B, written $HT_0^-(B)$, is the $R[U,V]$-algebra given by the quotient

$$HT_0^-(B) = \mathcal{A}_n \,/\, (\hat{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \mathbf{\Phi}_B^L \cdot \check{\mathbf{A}}, \check{\mathbf{A}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}_B^R \cdot \mathbf{\Lambda}_B^{-1}),$$

where $(\mathbf{M}_1, \mathbf{M}_2)$ represents the two-sided ideal in $\mathcal{A}_n \otimes R[U,V]$ generated by the entries of the two matrices $\mathbf{M}_1$ and $\mathbf{M}_2$.

(2) The *degree*-0 *hat transverse homology* and *degree*-0 *double hat transverse homology* of B are the R-algebras given by

$$\widehat{HT}_0(B) = \mathcal{A}_n \,/\, (\hat{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \mathbf{\Phi}_B^L \cdot \check{\mathbf{A}}, \check{\mathbf{A}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}_B^R \cdot \mathbf{\Lambda}_B^{-1})|_{U=0,V=1}$$

$$\widehat{\widehat{HT}}_0(B) = \mathcal{A}_n \,/\, (\hat{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \mathbf{\Phi}_B^L \cdot \check{\mathbf{A}}, \check{\mathbf{A}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}_B^R \cdot \mathbf{\Lambda}_B^{-1})|_{U=V=0}.$$

(3) The *degree*-0 *infinity transverse homology* of B is the $R[U^{\pm1}, V^{\pm1}]$-algebra given by

$$HT_0^\infty(B) = (\mathcal{A}_n \otimes R[U^{\pm1}, V^{\pm1}]) \,/\, (\hat{\mathbf{A}} - \mathbf{\Lambda}'_B \cdot \mathbf{\Phi}_B^L \cdot \check{\mathbf{A}}, \check{\mathbf{A}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}_B^R \cdot (\mathbf{\Lambda}'_B)^{-1}).$$

Note that if we set $U = V = 1$ in $HT_0^-(B)$, we recover the degree-0 framed knot contact homology (or "cord algebra") $HC_0(B)$ from [Ng08].

In practice, the degree-0 transverse homologies defined above are easier to work with than the full transverse complex defined below. In particular, the proofs that the degree-0 transverse homologies are transverse invariants (or topological invariants, in the case of HT_0^∞) are easier than the invariance proofs for the full transverse DGA, and the applications in Section 5 rely only on $\widehat{HT}_0$.

We next construct the full invariant, a differential graded algebra $(CT_*^-(B), \partial^-(B))$ whose homology in degree 0 is $HT_0^-(B)$. The algebra $CT_*^-(B)$ is the tensor algebra over $R[U,V]$ freely generated by the following:

- the $n(n-1)$ generators a_{ij}, where $1 \le i,j \le n$ and $i \ne j$, of degree 0
- $n(n-1)$ generators b_{ij}, where $1 \le i,j \le n$ and $i \ne j$, of degree 1
- $2n^2$ generators c_{ij} and d_{ij}, where $1 \le i,j \le n$, of degree 1
- $2n^2$ generators e_{ij} and f_{ij}, where $1 \le i,j \le n$, of degree 2.

Assemble these six families of generators into six $n\times n$ matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}$, where $\mathbf{A} = (a_{ij})$ and so forth, and the diagonal entries in $\mathbf{A}, \mathbf{B}$ are all $-2, 0$ respectively; also define two auxiliary matrices $\hat{\mathbf{B}}$, $\check{\mathbf{B}}$ by

$$(\hat{\mathbf{B}})_{ij} = \begin{cases} b_{ij} & i > j \\ 0 & i = j \\ \mu U b_{ij} & i < j \end{cases} \qquad \text{and} \qquad (\check{\mathbf{B}})_{ij} = \begin{cases} V b_{ij} & i > j \\ 0 & i = j \\ \mu b_{ij} & i < j, \end{cases}$$

cf. the definitions of $\hat{\mathbf{A}}$ and $\check{\mathbf{A}}$ (which reduce to $\mathbf{A}$ if we set $U = V = \mu = 1$).

One final piece of notation: if $\mathbf{M} \in \mathrm{Mat}_n(\mathcal{A}_n)$, then write $\phi_B(\mathbf{M})$ and $\partial^-(\mathbf{M})$ for the matrices whose entries are the images of the entries of M under ϕ_B and ∂^-.

Definition 2.2. The *transverse DGA* of B, written $(CT_*^-(B), \partial^-(B))$, is the differential graded algebra over $R[U,V]$ with $CT_*^-(B)$ defined as above and the differential $\partial^-(B)$ given by

$$\begin{aligned}
\partial^-(\mathbf{A}) &= 0\\
\partial^-(\mathbf{B}) &= \mathbf{A} - \mathbf{\Lambda}_B \cdot \phi_B(\mathbf{A}) \cdot \mathbf{\Lambda}_B^{-1}\\
\partial^-(\mathbf{C}) &= \hat{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \mathbf{\Phi}_B^L \cdot \check{\mathbf{A}}\\
\partial^-(\mathbf{D}) &= \check{\mathbf{A}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}_B^R \cdot \mathbf{\Lambda}_B^{-1}\\
\partial^-(\mathbf{E}) &= \hat{\mathbf{B}} - \mathbf{C} - \mathbf{\Lambda}_B \cdot \mathbf{\Phi}_B^L \cdot \mathbf{D}\\
\partial^-(\mathbf{F}) &= \check{\mathbf{B}} - \mathbf{D} - \mathbf{C} \cdot \mathbf{\Phi}_B^R \cdot \mathbf{\Lambda}_B^{-1}.
\end{aligned}$$

As described in Section 1, we can construct other flavors of transverse complexes by applying various algebraic operations to the transverse DGA:

$$\begin{aligned}
(\widehat{CT}_*(B), \widehat{\partial}(B)) &= (CT_*^-(B), \partial^-(B))|_{U=0,V=1}\\
(\widehat{\widehat{CT}}_*(B), \widehat{\widehat{\partial}}(B)) &= (CT_*^-(B), \partial^-(B))|_{U=0,V=0}\\
(CT_*^\infty(B), \partial^\infty(B)) &= (CT_*^-(B) \otimes R[U^{\pm1}, V^{\pm1}], \partial^-(B))\ (\lambda \mapsto \lambda(U/V)^{-(\mathrm{sl}(B)+1)/2}).
\end{aligned}$$

The first two complexes are DGAs over R, while the last is a DGA over $R[U^{\pm1}, V^{\pm1}]$.

Example 2.3. Let T_U and $S(T_U)$ denote the standard $\mathrm{sl} = -1$ transverse unknot and the $\mathrm{sl} = -3$ transverse unknot, respectively, so that $S(T_U)$ is the transverse stabilization of T_U. Note that T_U and $S(T_U)$ are the closures of the trivial braid in B_1 and $\sigma_1^{-1} \in B_2$, respectively. The transverse DGA for T_U is generated by four generators c, d, e, f, with differential

$$\begin{aligned}
\partial^-(c) &= -1 - \mu U + \lambda V + \lambda\mu\\
\partial^-(d) &= -\lambda^{-1}(-1 - \mu U + \lambda V + \lambda\mu)\\
\partial^-(e) &= -c - \lambda d\\
\partial^-(f) &= -d - \lambda^{-1} c.
\end{aligned}$$

Up to stable tame isomorphism, the generators d, f in this DGA can be eliminated, resulting in the DGA generated by c, e with differential

$$\begin{aligned}
\partial^-(c) &= -1 - \mu U + \lambda V + \lambda\mu\\
\partial^-(e) &= 0;
\end{aligned}$$

see Proposition 3.3 for a more general result. The degree-0 transverse homology of T_U is

$$HT_0^-(U) = R[U,V]/(-1 - \mu U + \lambda V + \lambda\mu),$$

with corresponding results for the other flavors of transverse homology.

For $S(T_U)$, the transverse DGA is more involved to write down explicitly, and we will not do it here. However, it is straightforward to calculate that the degree-0 transverse homology is

$$HT_0^-(S(T_U)) = (R[U,V])[a_{12}]/(1 + \mu U + V a_{12}, V + \mu + U a_{12}/\lambda).$$

On replacing λ by $\lambda U/V$, we deduce that $HT_0^\infty(S(T_U)) \cong R[U^{\pm 1}, V^{\pm 1}]/(-1-\mu U + \lambda V + \lambda\mu) \cong HT_0^\infty(T_U)$.

The main invariance results about the transverse complexes, Theorems 1.2 and 1.4, state that, up to equivalence, $CT_*^-(B)$, $\widehat{CT}_*(B)$, $\widehat{\widehat{CT}}_*(B)$ are invariants of transverse knots, while $CT_*^\infty(B)$ is an invariant of topological knots; we will prove these results in Section 3. The precise notion of equivalence is stable tame isomorphism, originally defined by Chekanov [Che02] for DGAs over $\mathbb{Z}/2$. Since our equivalences are for DGAs over $R[U,V]$ or $R[U^{\pm 1}, V^{\pm 1}]$, we recall here the definition of stable tame isomorphism, stated for DGAs over general rings.

Definition 2.4. Let $\mathcal{R}$ be a commutative ring with unit, and let $(\mathcal{A}, \partial)$, $(\tilde{\mathcal{A}}, \tilde{\partial})$ be DGAs (more precisely, finitely generated semifree noncommutative unital differential graded algebras) over $\mathcal{R}$: $\mathcal{A}$ has a distinguished set of generators $\{a_1, \ldots, a_n\}$ and is generated as an $\mathcal{R}$-module by words in the a_i's, and similarly $\tilde{\mathcal{A}}$ has a distinguished set of generators $\{\tilde{a}_1, \ldots, \tilde{a}_{\tilde{n}}\}$.

- An *elementary automorphism* of $\mathcal{A}$ is an $\mathcal{R}$-algebra automorphism of the form
$$a_i \mapsto \alpha\, a_i + v \qquad \text{for some } i$$
$$a_j \mapsto a_j \qquad \text{for all } j \neq i,$$
where v is an element of $\mathcal{A}$ not involving a_i, and α is an invertible element of $\mathcal{R}$.
- In the case where $n = \tilde{n}$, a *tame isomorphism* ϕ between $(\mathcal{A}, \partial)$ and $(\tilde{\mathcal{A}}, \tilde{\partial})$ is a composition of some number of elementary automorphisms of $\mathcal{A}$ and the isomorphism $\mathcal{A} \to \tilde{\mathcal{A}}$ that sends the generators $\{a_i\}$ to the generators $\{\tilde{a}_i\}$ in some order, such that $\phi \circ \partial = \tilde{\partial} \circ \phi$.
- A *stabilization* of $(\mathcal{A}, \partial)$ is a DGA generated by $a_1, \ldots, a_n$ and two new generators e_1, e_2, with $|e_1| = |e_2| + 1$ and differential induced by the differential on $\mathcal{A}$ along with $\partial(e_1) = e_2$, $\partial(e_2) = 0$.
- Two DGAs are *stable tame isomorphic* if, after some number of stabilizations of each, there is a tame isomorphism between the resulting DGAs.

A key feature of stable tame isomorphism is that it is a special case of quasi-isomorphism.

Proposition 2.5 (cf. [Che02, ENS02])**.** *If $(\mathcal{A}, \partial)$ and $(\tilde{\mathcal{A}}, \tilde{\partial})$ are stable tame isomorphic DGAs over $\mathcal{R}$, then we have an isomorphism of $\mathcal{R}$-modules $H_*(\mathcal{A}, \partial) \cong H_*(\tilde{\mathcal{A}}, \tilde{\partial})$.*

We conclude this section with a couple of results about the consistency of our definitions for the transverse DGA and transverse homology.

Proposition 2.6. *In the transverse DGA* $(CT_*^-(B), \partial^-(B))$*, we have* $(\partial^-)^2 = 0$*.*

Proposition 2.7. *The homology in degree* 0 *of the DGAs* $CT_*^-(B)$*,* $\widehat{CT}_*(B)$*,* $\widehat{\widehat{CT}}_*(B)$*,* $CT_*^\infty(B)$ *agrees with* $HT_0^-(B)$*,* $\widehat{HT}_*(B)$*,* $\widehat{\widehat{HT}}_*(B)$*,* $HT_*^\infty(B)$ *as given in Definition 2.1.*

Both of these results are easy consequences of the following algebraic lemma, which is used throughout this paper.

Lemma 2.8. *(cf.* [Ng05a, Prop. 4.7], [Ng05b, Prop. 2.10]*) Let* $\phi_B(\mathbf{A}_\geq), \phi_B(\mathbf{A}_\leq)$ *denote the images of* $\mathbf{A}_\geq, \mathbf{A}_\leq$ *under* ϕ*; in particular, these are lower and upper triangular matrices, respectively, with* -1 *along the diagonal. Then we have*

$$\phi_B(\mathbf{A}_\geq) = \mathbf{\Phi}_B^L \cdot \mathbf{A}_\geq \cdot \mathbf{\Phi}_B^R$$
$$\phi_B(\mathbf{A}_\leq) = \mathbf{\Phi}_B^L \cdot \mathbf{A}_\leq \cdot \mathbf{\Phi}_B^R.$$

As a consequence, we also have

$$\phi_B(\hat{\mathbf{A}}) = \mathbf{\Phi}_B^L \cdot \hat{\mathbf{A}} \cdot \mathbf{\Phi}_B^R$$
$$\phi_B(\check{\mathbf{A}}) = \mathbf{\Phi}_B^L \cdot \check{\mathbf{A}} \cdot \mathbf{\Phi}_B^R.$$

Proof. This lemma is implicit in the proof of Theorem 2.10 in [Ng08], but can be explicitly proven by induction by exactly following the proof of Proposition 4.7 in [Ng05a]: first verify the identities explicitly for $B = \sigma_k$, then use the chain rule for $\mathbf{\Phi}^L, \mathbf{\Phi}^R$ to deduce the identities for $B = \sigma_k^{-1}$, and then for $B = B_1 B_2$ assuming the identities hold for $B = B_1$ and $B = B_2$. □

Proof of Proposition 2.6. It suffices to show that $(\partial^-)^2(\mathbf{E}) = (\partial^-)^2(\mathbf{F}) = 0$. From the definitions of $\hat{\mathbf{B}}, \check{\mathbf{B}}, \partial^-(\mathbf{B})$, we see that $\partial^-(\hat{\mathbf{B}}) = \hat{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \phi_B(\hat{\mathbf{A}}) \cdot \mathbf{\Lambda}_B^{-1}$ and $\partial^-(\check{\mathbf{B}}) = \check{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \phi_B(\check{\mathbf{A}}) \cdot \mathbf{\Lambda}_B^{-1}$. The desired result is then a consequence of Lemma 2.8. □

Proof of Proposition 2.7. (See Proposition 3.3 for another proof.) Since the transverse DGA is supported in nonnegative degree, it suffices to check that the entries of the matrix $\mathbf{A} - \mathbf{\Lambda}_B \cdot \phi_B(\mathbf{A}) \cdot \mathbf{\Lambda}_B^{-1}$ are in the ideal of $\mathcal{A}_n$ generated by the entries of $\hat{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \mathbf{\Phi}_B^L \cdot \check{\mathbf{A}}$ and $\check{\mathbf{A}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}_B^R \cdot \mathbf{\Lambda}_B^{-1}$. Now by Lemma 2.8, we have

$$\hat{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \phi_B(\hat{\mathbf{A}}) \cdot \mathbf{\Lambda}_B^{-1} = (\hat{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \mathbf{\Phi}_B^L \cdot \check{\mathbf{A}}) + (\mathbf{\Lambda}_B \cdot \mathbf{\Phi}_B^L) \cdot (\check{\mathbf{A}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}_B^R \cdot \mathbf{\Lambda}_B^{-1})$$
$$\check{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \phi_B(\check{\mathbf{A}}) \cdot \mathbf{\Lambda}_B^{-1} = (\check{\mathbf{A}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}_B^R \cdot \mathbf{\Lambda}_B^{-1}) + (\hat{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \mathbf{\Phi}_B^L \cdot \check{\mathbf{A}}) \cdot (\mathbf{\Phi}_B^R \cdot \mathbf{\Lambda}_B^{-1}).$$

But the matrix $\mathbf{A} - \mathbf{\Lambda}_B \cdot \phi_B(\mathbf{A}) \cdot \mathbf{\Lambda}_B^{-1}$ is zero along the diagonal, agrees below the diagonal with the first of these matrices, and agrees above the diagonal with μ times the second of these matrices. □

3. Proofs of Invariance

In this section, we prove the main invariance theorems, Theorems 1.2 and 1.4, along with some other isomorphism results. We first establish some auxiliary results giving alternate forms for the transverse DGA in Section 3.1. In Section 3.2, we use these to prove the invariance of degree-0 transverse homology, which is technically a corollary of the main invariance theorems but has a more streamlined proof, and which is all we need to deduce the applications in Section 5. In Section 3.3, we present the full proofs of invariance, which are somewhat messy and may probably be skipped with impunity by most readers. Finally, in Section 3.4, we describe the version of the transverse DGA presented in [EENSa], which is slightly different from the transverse DGA in this paper, and prove that the two are stable tame isomorphic.

3.1. **Equivalent forms for the transverse DGA.** In this subsection, we prove a few auxiliary results that show that various DGAs related to the transverse DGA are stable tame isomorphic. These results will be used in the invariance proofs in Sections 3.2, 3.3, and 3.4, and also in Section 4.

First, it will sometimes be useful to replace the diagonal matrix $\mathbf{\Lambda}_B$ in the definition of the transverse DGA by some other diagonal matrix. Here we prove that the transverse DGA depends only on the determinant of $\mathbf{\Lambda}_B$ rather than its particular entries. Recall that we only consider the case of knots in this paper; the corresponding result for multi-component links states that the transverse DGA depends only on the determinants of the submatrices of $\mathbf{\Lambda}_B$ corresponding to the various link components.

Proposition 3.1. *Let B be a braid. Replace $\mathbf{\Lambda}_B$ in the definition of the transverse DGA of B, Definition 2.2, by an arbitrary diagonal matrix $\mathbf{\Lambda}$ with invertible determinant. If the closure of B is a single-component knot, then the transverse DGA, up to tame isomorphism, depends only on* $\det \mathbf{\Lambda}$.

Proof. This is a straightforward extension of Proposition 3.1 in [Ng08], whose proof we follow here. For $B \in B_n$, let $s(B) \in S_n$ denote the permutation of $\{1, \ldots, n\}$ corresponding to B under the usual map $B_n \to S_n$. If v is a vector of length n, let $\mathbf{\Delta}(v)$ denote the diagonal $n \times n$ matrix whose diagonal entries are v, and write $s(B)v$ for the vector that results from permuting the entries of v by $s(B)$. We recall [Ng08, Lemma 3.2]: if we define $\tilde{\mathbf{A}} = \mathbf{\Delta}(v) \cdot \mathbf{A} \cdot \mathbf{\Delta}(v)^{-1}$, then

$$\mathbf{\Phi}^L_B(\tilde{\mathbf{A}}) = \mathbf{\Delta}(s(B)v) \cdot \mathbf{\Phi}^L_B(\mathbf{A}) \cdot \mathbf{\Delta}(v)^{-1}$$
$$\mathbf{\Phi}^R_B(\tilde{\mathbf{A}}) = \mathbf{\Delta}(v) \cdot \mathbf{\Phi}^R_B(\mathbf{A}) \cdot \mathbf{\Delta}(s(B)v)^{-1}.$$

Here, as in [Ng05a, Ng08], $\mathbf{\Phi}^L_B(\tilde{\mathbf{A}}), \mathbf{\Phi}^R_B(\tilde{\mathbf{A}})$ denote the result of replacing the generators a_{ij} by $\tilde{\mathbf{A}}_{ij}$ (for all i, j) in the matrices $\mathbf{\Phi}^L_B, \mathbf{\Phi}^R_B$.

Now given any vector v of length n with invertible entries, let (CT^-, ∂^-) and $(\widetilde{CT}^-, \tilde{\partial}^-)$ be the transverse DGAs for B with $\mathbf{\Lambda}_B$ replaced by $\mathbf{\Lambda}$ and

$\tilde{\mathbf{\Lambda}} = \mathbf{\Delta}(v) \cdot \mathbf{\Delta}(s(B)v)^{-1} \cdot \mathbf{\Lambda}$, respectively, for some diagonal matrix $\mathbf{\Lambda}$ with invertible diagonal entries. Then the identification of the algebras CT^- with $\widetilde{CT}^-$ given in matrix form by $\tilde{\mathbf{A}} = \mathbf{\Delta}(v)\mathbf{A}\mathbf{\Delta}(v)^{-1}$, $\tilde{\mathbf{B}} = \mathbf{\Delta}(v)\mathbf{B}\mathbf{\Delta}(v)^{-1}$, $\tilde{\mathbf{C}} = \mathbf{\Delta}(v)\mathbf{C}\mathbf{\Delta}(v)^{-1}$, $\tilde{\mathbf{D}} = \mathbf{\Delta}(v)\mathbf{D}\mathbf{\Delta}(v)^{-1}$, $\tilde{\mathbf{E}} = \mathbf{\Delta}(v)\mathbf{E}\mathbf{\Delta}(v)^{-1}$, $\tilde{\mathbf{F}} = \mathbf{\Delta}(v)\mathbf{F}\mathbf{\Delta}(v)^{-1}$ provides a tame isomorphism between (CT^-, ∂^-) and $(\widetilde{CT}^-, \tilde{\partial}^-)$:

$$\begin{aligned}
\tilde{\partial}^-(\tilde{\mathbf{C}}) &= \hat{\tilde{\mathbf{A}}} - \tilde{\mathbf{\Lambda}} \cdot \mathbf{\Phi}^L_B(\tilde{\mathbf{A}}) \cdot \check{\tilde{\mathbf{A}}} \\
&= \mathbf{\Delta}(v) \cdot \hat{\mathbf{A}} \cdot \mathbf{\Delta}(v)^{-1} - \tilde{\mathbf{\Lambda}} \cdot \mathbf{\Delta}(s(B)v) \cdot \mathbf{\Phi}^L_B(\mathbf{A}) \cdot \check{\mathbf{A}} \cdot \mathbf{\Delta}(v)^{-1} \\
&= \mathbf{\Delta}(v) \cdot (\hat{\mathbf{A}} - \mathbf{\Lambda} \cdot \mathbf{\Phi}^L_B(\mathbf{A}) \cdot \check{\mathbf{A}}) \cdot \mathbf{\Delta}(v)^{-1} \\
&= \partial^-(\tilde{\mathbf{C}}),
\end{aligned}$$

with similar computations for the differentials of $\mathbf{B}, \mathbf{D}, \mathbf{E}, \mathbf{F}$.

Thus, in the definition of the transverse DGA, we can replace $\mathbf{\Lambda}_B$ by any matrix of the form $\mathbf{\Delta}(v) \cdot \mathbf{\Delta}(s(B)v)^{-1} \cdot \mathbf{\Lambda}_B$, up to tame isomorphism. Since the closure of B is a knot, $s(B)$ is an n-cycle; it follows that for any diagonal $\mathbf{\Lambda}$ with the same determinant as $\mathbf{\Lambda}_B$, one can choose a vector v with $\mathbf{\Lambda} = \mathbf{\Delta}(v) \cdot \mathbf{\Delta}(s(B)v)^{-1} \cdot \mathbf{\Lambda}_B$. (We need $\mathbf{\Lambda}_{ii} = v_i v^{-1}_{s(B)(i)} (\mathbf{\Lambda}_B)_{ii}$ for $i = 1, \ldots, n$, and this formula allows us to define v up to an overall multiplicative factor.) The proposition follows. □

Corollary 3.2. *Let $B \in B_n$ be a braid whose closure is a knot, and let $\mathbf{\Lambda}$ be a diagonal $n \times n$ matrix with invertible determinant. Up to isomorphism, the degree-0 transverse homology*

$$HT^-_0(B) = \mathcal{A}_n \,/\, (\hat{\mathbf{A}} - \mathbf{\Lambda} \cdot \mathbf{\Phi}^L_B \cdot \check{\mathbf{A}}, \check{\mathbf{A}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}^R_B \cdot \mathbf{\Lambda}^{-1})$$

depends only on $\det \mathbf{\Lambda}$, with corresponding results for the other flavors of degree-0 transverse homology.

Next, there are several variants of the transverse DGA, akin to the "modified framed DGA" from [Ng08], that are stable tame isomorphic to the transverse DGA but have fewer generators. It will occasionally be convenient to use one of the variants instead of the original transverse DGA.

Proposition 3.3. *Let B be a braid. The following DGAs over $R[U,V]$ are stable tame isomorphic:*

(1) *the transverse DGA from Definition 2.2;*
(2) *the DGA with generators and differential:*
   - $\{a_{ij}\}_{1 \leq i \neq j \leq n}$ *of degree 0 with* $\partial^-(\mathbf{A}) = 0$,
   - $\{c_{ij}\}_{1 \leq i,j \leq n}$ *of degree 1 with* $\partial^-(\mathbf{C}) = \hat{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \mathbf{\Phi}^L_B \cdot \check{\mathbf{A}}$,
   - $\{d_{ij}\}_{1 \leq i,j \leq n}$ *of degree 1 with* $\partial^-(\mathbf{D}) = \check{\mathbf{A}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}^R_B \cdot \mathbf{\Lambda}^{-1}_B$,
   - $\{e_{ij}\}_{1 \leq i \leq j \leq n}$ *of degree 2 with* $\partial^-(e_{ii}) = (\mathbf{C} + \mathbf{\Lambda}_B \cdot \mathbf{\Phi}^L_B \cdot \mathbf{D})_{ii}$ *and*

   $$\partial^-(e_{ij}) = (\mathbf{C} - U\mathbf{D} + \mathbf{\Lambda}_B \cdot \mathbf{\Phi}^L_B \cdot \mathbf{D} - U\mathbf{C} \cdot \mathbf{\Phi}^R_B \cdot \mathbf{\Lambda}^{-1}_B)_{ij}$$

   for $i < j$,

- $\{f_{ij}\}_{1\leq j\leq i\leq n}$ *of degree* 2 *with* $\partial^-(f_{ii}) = (\mathbf{D}+\mathbf{C}\cdot\mathbf{\Phi}^R_B\cdot\mathbf{\Lambda}_B^{-1})_{ii}$ *and*

  $$\partial^-(f_{ij}) = (\mathbf{D} - V\mathbf{C} + \mathbf{C}\cdot\mathbf{\Phi}^R_B\cdot\mathbf{\Lambda}_B^{-1} - V\mathbf{\Lambda}_B\cdot\mathbf{\Phi}^L_B\cdot\mathbf{D})_{ij}$$

  for $j < i$*;*

(3) *the DGA with generators and differential:*
- $\{a_{ij}\}_{1\leq i\neq j\leq n}$ *of degree* 0 *with* $\partial^-(\mathbf{A}) = 0$*,*
- $\{b_{ij}\}_{1\leq i\neq j\leq n}$ *of degree* 1 *with* $\partial^-(\mathbf{B}) = \mathbf{\Lambda}_B^{-1}\cdot\mathbf{A}\cdot\mathbf{\Lambda}_B - \phi_B(\mathbf{A})$*,*
- $\{d_{ij}\}_{1\leq i,j\leq n}$ *of degree* 1 *with* $\partial^-(\mathbf{D}) = \check{\mathbf{A}}\cdot\mathbf{\Lambda}_B - \hat{\mathbf{A}}\cdot\mathbf{\Phi}^R_B$*,*
- $\{f_{ij}\}_{1\leq i,j\leq n}$ *of degree* 2 *with*

$$\partial^-(\mathbf{F}) = \check{\mathbf{B}}\cdot(\mathbf{\Phi}^R_B)^{-1} - \hat{\mathbf{B}}\cdot\mathbf{\Lambda}_B^{-1} + \mathbf{\Phi}^L_B\cdot\mathbf{D}\cdot\mathbf{\Lambda}_B^{-1} - \mathbf{\Lambda}_B^{-1}\cdot\mathbf{D}\cdot(\mathbf{\Phi}^R_B)^{-1}.$$

Proof. To obtain DGA (2) from the transverse DGA (1), apply the tame automorphisms

$$b_{ij} \mapsto b_{ij} + (\mathbf{C} + \mathbf{\Lambda}_B\cdot\mathbf{\Phi}^L_B\cdot\mathbf{D})_{ij}$$

for $i > j$,

$$b_{ij} \mapsto b_{ij} + \mu^{-1}(\mathbf{D} + \mathbf{C}\cdot\mathbf{\Phi}^R_B\cdot\mathbf{\Lambda}_B^{-1})_{ij}$$

for $i < j$, to the transverse DGA. The result has $\partial^-(e_{ij}) = b_{ij}$ for $i > j$, $\partial^-(f_{ij}) = b_{ij}$ for $i < j$, and $\partial^-(\mathbf{B}) = 0$ (since $(\partial^-)^2 = 0$). We can then destabilize by removing b_{ij}, e_{ij} for $i > j$ and b_{ij}, f_{ij} for $i < j$, and the result is DGA (2).

To obtain DGA (3) from the transverse DGA (1), successively apply the tame automorphisms

$$\begin{aligned}
\mathbf{C} &\mapsto \mathbf{C} + \hat{\mathbf{B}} - \mathbf{\Lambda}_B\cdot\mathbf{\Phi}^L_B\cdot\mathbf{D}\\
\mathbf{D} &\mapsto \mathbf{D}\cdot\mathbf{\Lambda}_B^{-1}\\
\mathbf{B} &\mapsto \mathbf{\Lambda}_B\cdot\mathbf{B}\cdot\mathbf{\Lambda}_B^{-1}\\
\mathbf{F} &\mapsto (\mathbf{\Lambda}_B\cdot\mathbf{F} + \mathbf{E})\cdot\mathbf{\Phi}^R_B\cdot\mathbf{\Lambda}_B^{-1}
\end{aligned}$$

to the transverse DGA. The result has $\partial^-(\mathbf{E}) = -\mathbf{C}$, $\partial^-(\mathbf{C}) = 0$ (since $(\partial^-)^2 = 0$), and $\partial^-(\mathbf{B}), \partial^-(\mathbf{D}), \partial^-(\mathbf{F})$ as given in the statement of the proposition. We can then destabilize by removing the c and e generators, and the result is DGA (3). □

DGA (2) from Proposition 3.3 is the analogue of the framed DGA from [Ng08] and is convenient to use in the invariance proofs under braid stabilization presented in Section 3.3 below. DGA (3) is, up to signs and powers of λ and μ, the version of the transverse DGA considered in [EENSa]; see Section 3.4 below for the exact relation.

3.2. **Invariance proofs I: degree-0 transverse homology.** Before embarking on the full invariance proofs for the transverse DGA, we first prove the invariance of degree-0 transverse homology HT_0. This is technically superfluous, since it follows from the invariance of the DGA, but we have included it because it is a simplified (and more readable) version of the full invariance proof. In addition, as mentioned before, the applications of transverse homology that we present in Section 5 rely only on the invariance of HT_0, and the reader interested in applications rather than the theory behind transverse homology can skip the full invariance proofs in favor of the proofs in this section.

Proposition 3.4. *The degree-0 transverse homology $HT_0^-(B)$ is an invariant of transverse knots. More precisely, if braids $B, \tilde{B}$ are related by some sequence of conjugation and positive de/stabilization, then there is an isomorphism of $R[U,V]$-algebras*

$$HT_0^-(B) \cong HT_0^-(\tilde{B}).$$

Proof. Let $B \in B_n$. It suffices to consider $B, \tilde{B}$ related by one of the following: $\tilde{B} = \sigma_k^{-1} B \sigma_k$ for some k; $\tilde{B} = B\sigma_0$, where we view $B \subset B_{n+1}$ with the strands in B_{n+1} numbered $0, \ldots, n$ instead of $1, \ldots, n+1$.

CASE 1: $\tilde{B} = \sigma_k^{-1} B \sigma_k$.

We follow the proofs of Theorem 4.10 from [Ng05a] and Theorem 2.7 from [Ng08]. Define $\mathbf{\Lambda}$ to be $\mathbf{\Lambda}_B = \operatorname{diag}(\lambda\mu^{-w(B)}, 1, \ldots, 1)$ if $k \neq 1$, and $\operatorname{diag}(1, \lambda\mu^{-w(B)}, 1, \ldots, 1)$ if $k = 1$. Write $\mathcal{I}, \tilde{\mathcal{I}}$ for the ideals

$$\mathcal{I} = (\hat{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \mathbf{\Phi}_B^L \cdot \check{\mathbf{A}}, \check{\mathbf{A}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}_B^R \cdot \mathbf{\Lambda}_B^{-1})$$
$$\tilde{\mathcal{I}} = (\hat{\mathbf{A}} - \mathbf{\Lambda} \cdot \mathbf{\Phi}_{\tilde{B}}^L \cdot \check{\mathbf{A}}, \check{\mathbf{A}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}_{\tilde{B}}^R \cdot \mathbf{\Lambda}^{-1})$$

in $\mathcal{A}_n$. By Corollary 3.2, we have $HT_0^-(B) \cong \mathcal{A}_n/\mathcal{I}$ and $HT_0^-(\tilde{B}) \cong \mathcal{A}_n/\tilde{\mathcal{I}}$.

We claim that $\phi_{\sigma_k} : \mathcal{A}_n \to \mathcal{A}_n$ maps $\tilde{\mathcal{I}}$ into $\mathcal{I}$; a similar argument shows that $\phi_{\sigma_k}^{-1}$ maps $\mathcal{I}$ into $\tilde{\mathcal{I}}$, which proves that ϕ_{σ_k} induces an isomorphism $HT_0^-(B) \to HT_0^-(\tilde{B})$. To prove the claim, we will show that the entries of

$$\mathbf{M} = \phi_{\sigma_k}(\hat{\mathbf{A}}) - \mathbf{\Lambda} \cdot \mathbf{\Phi}_{\tilde{B}}^L(\phi_{\sigma_k}(\mathbf{A})) \cdot \phi_{\sigma_k}(\check{\mathbf{A}})$$

are in $\mathcal{I}$. Similarly, the entries of $\phi_{\sigma_k}(\check{\mathbf{A}}) - \phi_{\sigma_k}(\hat{\mathbf{A}}) \cdot \mathbf{\Phi}_{\tilde{B}}^R(\phi_{\sigma_k}(\mathbf{A})) \cdot \mathbf{\Lambda}^{-1}$ are in $\mathcal{I}$, and the claim follows.

We have

$$\begin{aligned}
\mathbf{M} &= \mathbf{\Phi}_{\sigma_k}^L(\mathbf{A})\hat{\mathbf{A}}\mathbf{\Phi}_{\sigma_k}^R(\mathbf{A}) - \mathbf{\Lambda}\left(\mathbf{\Phi}_{\sigma_k}^L(\phi_B(\mathbf{A}))\mathbf{\Phi}_B^L(\mathbf{A})(\mathbf{\Phi}_{\sigma_k}^L(\mathbf{A}))^{-1}\right)\left(\mathbf{\Phi}_{\sigma_k}^L(\mathbf{A})\check{\mathbf{A}}\mathbf{\Phi}_{\sigma_k}^R(\mathbf{A})\right) \\
&= \mathbf{\Phi}_{\sigma_k}^L(\mathbf{A})\left\{\hat{\mathbf{A}} - \mathbf{\Lambda}_B\mathbf{\Phi}_B^L(\mathbf{A})\check{\mathbf{A}}\right\}\mathbf{\Phi}_{\sigma_k}^R(\mathbf{A}) \\
&\quad - \left\{\mathbf{\Lambda}\mathbf{\Phi}_{\sigma_k}^L(\phi_B(\mathbf{A})) - \mathbf{\Phi}_{\sigma_k}^L(\mathbf{A})\mathbf{\Lambda}_B\right\}\mathbf{\Phi}_B^L(\mathbf{A})\check{\mathbf{A}}\mathbf{\Phi}_{\sigma_k}^R(\mathbf{A}).
\end{aligned}$$

But both terms in braces in this last expression have entries in $\mathcal{I}$. This is clear for the first term in braces; by the expression for $\mathbf{\Phi}_{\sigma_k}^L$ from [Ng05a, Lemma 4.6], the second term in braces is the 0 matrix except in the kk entry, where it is $a_{k+1,k} - \phi_B(a_{k+1,k})$ (or $\lambda\mu^{-w(B)}a_{21} - \phi_B(a_{21})$ if $k = 1$), which is a scalar multiple of the $k+1, k$ entry of the matrix $\mathbf{A} - \mathbf{\Lambda}_B \cdot \phi_B(\mathbf{A}) \cdot \mathbf{\Lambda}_B^{-1}$ and hence in $\mathcal{I}$ by the proof of Proposition 2.7.

CASE 2: $\tilde{B} = B\sigma_0$. Write $\tilde{\mathcal{A}}$ for the algebra over $R[U,V]$ obtained from $\mathcal{A}_n$ by adding generators a_{i0}, a_{0i} for $1 \leq i \leq n$. Also write ∂^-, $\tilde{\partial}^-$ for the differentials on the transverse DGAs of B and $\tilde{B}$, respectively. Then we can write $HT_0^-(B) = \mathcal{A}_n/\mathcal{I}$ and $HT_0^-(\tilde{B}) = \tilde{\mathcal{A}}/\tilde{\mathcal{I}}$, where $\mathcal{I}$ is generated by $\partial^-(c_{ij}), \partial^-(d_{ij})$ for $1 \leq i \neq j \leq n$, and $\tilde{\mathcal{I}}$ is generated by $\tilde{\partial}^-(c_{ij}), \tilde{\partial}^-(d_{ij})$ for $0 \leq i \neq j \leq n$. Using the expressions for $\mathbf{\Phi}^L_{B\sigma_0}, \mathbf{\Phi}^R_{B\sigma_0}$ from the proof of Theorem 4.10 in [Ng05a], we calculate, for $1 \leq i \leq n$ and $2 \leq j \leq n$:

$$\begin{aligned}
\tilde{\partial}^-(c_{00}) &= -1 - \mu U - \lambda\mu^{-w(B)-1}(\phi(a_{10})(V+\mu) - V\mathbf{\Phi}^L_{1\ell}a_{\ell 0}) \\
&= -1 - \mu U - \lambda\mu^{-w(B)}\mathbf{\Phi}^L_{1\ell}a_{\ell 0} \\
\tilde{\partial}^-(c_{0i}) &= \mu U a_{0i} - \lambda\mu^{-w(B)-1}(-\mu\phi(a_{10})a_{0i} - \mathbf{\Phi}^L_{1\ell}\check{a}_{\ell i}) \\
\tilde{\partial}^-(c_{10}) &= a_{10} + V + \mu \\
\tilde{\partial}^-(c_{1i}) &= \hat{a}_{1i} - \mu a_{0i} \\
\tilde{\partial}^-(c_{j0}) &= a_{j0} - V\mathbf{\Phi}^L_{j\ell}a_{\ell 0} \\
\tilde{\partial}^-(c_{ji}) &= \hat{a}_{ji} - \mathbf{\Phi}^L_{j\ell}\check{a}_{\ell i} \\
\tilde{\partial}^-(d_{00}) &= -V - \mu - \lambda^{-1}\mu^{w(B)+1}((1+\mu U)\phi(a_{01}) - \mu U a_{0\ell}\mathbf{\Phi}^R_{\ell 1}) \\
&= -V - \mu - \lambda^{-1}\mu^{w(B)+1}a_{0\ell}\mathbf{\Phi}^R_{\ell 1} \\
\tilde{\partial}^-(d_{i0}) &= Va_{i0} - \lambda^{-1}\mu^{w(B)+1}(-a_{i0}\phi(a_{01}) - \hat{a}_{i\ell}\mathbf{\Phi}^R_{\ell 1}) \\
\tilde{\partial}^-(d_{01}) &= \mu a_{01} + 1 + \mu U \\
\tilde{\partial}^-(d_{i1}) &= \check{a}_{i1} - a_{i0} \\
\tilde{\partial}^-(d_{0j}) &= \mu a_{0j} - \mu U a_{0\ell}\mathbf{\Phi}^R_{\ell j} \\
\tilde{\partial}^-(d_{ij}) &= \check{a}_{ij} - \hat{a}_{i\ell}\mathbf{\Phi}^R_{\ell j}.
\end{aligned}$$

Here $\hat{a}_{ij}, \check{a}_{ij}$ are the ij entries of $\hat{\mathbf{A}}, \check{\mathbf{A}}$ respectively, $\phi, \mathbf{\Phi}^L, \mathbf{\Phi}^R$ are shorthand for $\phi_B, \mathbf{\Phi}^L_B, \mathbf{\Phi}^R_B$, and all appearances of ℓ are understood to represent sums over $\ell = 1, \dots, n$.

From $\tilde{\partial}^-(c_{1i}), \tilde{\partial}^-(d_{i1})$, we see that in $\tilde{\mathcal{A}}/\tilde{\mathcal{I}}$, we have $a_{0i} = \hat{a}_{1i}/\mu$ and $a_{i0} = \check{a}_{i1}$ for $1 \leq i \leq n$. Using these relations to replace all appearances of a_{0i}, a_{i0}, we find that the remaining relations in $\tilde{\mathcal{A}}/\tilde{\mathcal{I}}$ become precisely the generators of $\mathcal{I}$. For instance,

$$\tilde{\partial}^-(c_{00}) = -1 - \mu U - \lambda\mu^{-w(B)}\mathbf{\Phi}^L_{1\ell}\check{a}_{\ell 1} = \partial^-(c_{11})$$

$$\tilde{\partial}^-(c_{0i}) = U\hat{a}_{1i} + \lambda\mu^{-w(B)-1}\left(\mathbf{\Phi}^L_{1\ell}\check{a}_{\ell 1}\hat{a}_{1i} + \mathbf{\Phi}^L_{1\ell}\check{a}_{\ell i}\right) = -\mu^{-1}\left((\partial^-(c_{11}))\hat{a}_{1i} + \partial^-(c_{1i})\right)$$

and so forth. It follows that $\tilde{\mathcal{A}}/\tilde{\mathcal{I}} \cong \mathcal{A}_n/\mathcal{I}$, as desired. □

We next establish the invariance result for degree-0 infinity transverse homology.

Proposition 3.5. *The degree-0 infinity transverse homology $HT_0^\infty(B)$ is an invariant of topological knots. More precisely, if braids $B, \tilde{B}$ have the same* *knot closure, then there is an isomorphism of* $R[U^{\pm1}, V^{\pm1}]$*-algebras*

$$HT_0^\infty(B) \cong HT_0^\infty(\tilde{B}).$$

Proof. Because of Proposition 3.4, it suffices to establish the result when $\tilde{B}$ is a negative stabilization of B: $\tilde{B} = B\sigma_0^{-1}$. We use the same notation as the proof of Proposition 3.4 in the case $\tilde{B} = B\sigma_0$. Using the expressions for $\mathbf{\Phi}^L_{B\sigma_0^{-1}}, \mathbf{\Phi}^R_{B\sigma_0^{-1}}$ from the proof of Theorem 4.10 in [Ng05a], we calculate, for $1 \le i \le n$ and $2 \le j \le n$:

$$\begin{aligned}
\tilde{\partial}^\infty(c_{00}) &= -1 - \mu U - \lambda\mu^{-w(B)+1} V (U/V)^{-(\mathrm{sl}(B)-1)/2} \mathbf{\Phi}^L_{1\ell} a_{\ell 0} \\
\tilde{\partial}^\infty(c_{0i}) &= \mu U a_{0i} - \lambda\mu^{-w(B)+1} (U/V)^{-(\mathrm{sl}(B)-1)/2} \mathbf{\Phi}^L_{1\ell} \check{a}_{\ell i} \\
\tilde{\partial}^\infty(c_{10}) &= a_{10} - (V + \mu - V\phi(a_{01}) \mathbf{\Phi}^L_{1\ell} a_{\ell 0}) \\
\tilde{\partial}^\infty(c_{1i}) &= \hat{a}_{1i} - (-\mu a_{0i} - \phi(a_{01}) \mathbf{\Phi}^L_{1\ell} \check{a}_{\ell i}) \\
\tilde{\partial}^\infty(c_{j0}) &= a_{j0} - V \mathbf{\Phi}^L_{j\ell} a_{\ell 0} \\
\tilde{\partial}^\infty(c_{ji}) &= \hat{a}_{ji} - \mathbf{\Phi}^L_{j\ell} \check{a}_{\ell i} \\
\tilde{\partial}^\infty(d_{00}) &= -V - \mu - \lambda^{-1}\mu^{w(B)} U (U/V)^{(\mathrm{sl}(B)-1)/2} a_{0\ell} \mathbf{\Phi}^R_{\ell 1} \\
\tilde{\partial}^\infty(d_{i0}) &= V a_{i0} - \lambda^{-1}\mu^{w(B)-1} (U/V)^{(\mathrm{sl}(B)-1)/2} \hat{a}_{i\ell} \mathbf{\Phi}^R_{\ell 1} \\
\tilde{\partial}^\infty(d_{01}) &= \mu a_{01} - (1 + \mu U - \mu U a_{0\ell} \mathbf{\Phi}^R_{\ell 1} \phi(a_{10})) \\
\tilde{\partial}^\infty(d_{i1}) &= \check{a}_{i1} - (-a_{i0} - \hat{a}_{i\ell} \mathbf{\Phi}^R_{\ell 1} \phi(a_{10})) \\
\tilde{\partial}^\infty(d_{0j}) &= \mu a_{0j} - \mu U a_{0\ell} \mathbf{\Phi}^R_{\ell j} \\
\tilde{\partial}^\infty(d_{ij}) &= \check{a}_{ij} - \hat{a}_{i\ell} \mathbf{\Phi}^R_{\ell j}.
\end{aligned}$$

From $\tilde{\partial}^\infty(c_{0i}), \tilde{\partial}^\infty(d_{i0})$, we see that in $\tilde{\mathcal{A}}/\tilde{\mathcal{I}}$, we have

$$\begin{aligned}
a_{0i} &= \lambda\mu^{-w(B)} V^{-1} (U/V)^{-(\mathrm{sl}(B)+1)/2} \mathbf{\Phi}^L_{1\ell} \check{a}_{\ell i} \\
a_{i0} &= \lambda^{-1}\mu^{w(B)-1} U^{-1} (U/V)^{(\mathrm{sl}(B)+1)/2} \hat{a}_{i\ell} \mathbf{\Phi}^R_{\ell 1}
\end{aligned}$$

for $1 \le i \le n$. Using these relations to replace all appearances of a_{0i}, a_{i0}, we find that the relations $\tilde{\partial}^\infty(c_{00}) = 0$ and $\tilde{\partial}^\infty(d_{00}) = 0$ become $0 = 0$, and the remaining relations in $\tilde{\mathcal{A}}/\tilde{\mathcal{I}}$ become precisely the generators of $\mathcal{I}$. It follows that $\tilde{\mathcal{A}}/\tilde{\mathcal{I}} \cong \mathcal{A}_n/\mathcal{I}$, as desired. □

3.3. **Invariance proofs II: the full transverse DGA.** Here we extend the arguments from Section 3.2 to prove invariance for the full transverse DGA, Theorems 1.2 and 1.4. We provide detailed outlines for the invariance proofs, which follow the proof of Theorem 2.7 in [Ng08], and leave some amount of easy but tedious checking to the reader. We remark that the proofs given here reduce, upon setting $U = V = 1$, to a proof of invariance of the knot DGA that is very similar to, but slightly different from, the original proof from [Ng08, Thm. 2.7]. The few changes are changes of convenience for our current setup.

Outline of proof of Theorem 1.2. As in the proof of Proposition 3.4, it suffices to prove invariance for the transverse DGA under stable tame isomorphism for braids $B, \tilde{B}$ related by either conjugation or positive stabilization.

CASE 1: $\tilde{B} = \sigma_k^{-1} B \sigma_k$.

Let $(\mathcal{A} = CT^-, \partial^-)$ denote the transverse DGA of B, and let $(\tilde{\mathcal{A}} = \widetilde{CT}^-, \tilde{\partial}^-)$ denote the transverse DGA of $\tilde{B}$ but with $\mathbf{\Lambda}_{\tilde{B}}$ replaced by $\tilde{\mathbf{\Lambda}}$, defined to be $\operatorname{diag}(\lambda\mu^{-w(B)}, 1, \dots, 1)$ if $k \neq 1$ and $\operatorname{diag}(1, \lambda\mu^{-w(B)}, 1, \dots, 1)$ if $k = 1$; this replacement is allowed by Proposition 3.1. We claim that $(\mathcal{A}, \partial^-)$ and $(\tilde{\mathcal{A}}, \tilde{\partial}^-)$ are tamely isomorphic.

The tame isomorphism, which is nearly identical to the corresponding map considered in the proof of [Ng08, Thm. 2.7][3], is given by the identification:

$$
\begin{aligned}
\tilde{\mathbf{A}} &= \phi_{\sigma_k}(\mathbf{A}) \\
\tilde{\mathbf{B}} &= \mathbf{\Phi}^L_{\sigma_k}(\epsilon_k^{-1}\phi_B(\mathbf{A})) \cdot \mathbf{B} \cdot \mathbf{\Phi}^R_{\sigma_k}(\epsilon_k\phi_B(\mathbf{A})) + \mathbf{\Phi}^L_{\sigma_k}(\mathbf{A}) \cdot \mathbf{A} \cdot \mathbf{\Theta}^R_k \\
&\qquad + \mathbf{\Theta}^L_k \cdot \mathbf{A} \cdot \mathbf{\Phi}^R_{\sigma_k}(\epsilon_k\phi_B(\mathbf{A})) + \partial^-(\mathbf{\Theta}^L_k \cdot \mathbf{\Theta}^R_k) \\
\tilde{\mathbf{C}} &= \mathbf{\Phi}^L_{\sigma_k}(\epsilon_k^{-1}\phi_B(\mathbf{A})) \cdot \mathbf{C} \cdot \mathbf{\Phi}^R_{\sigma_k}(\mathbf{A}) + \mathbf{\Theta}^L_k \cdot \hat{\mathbf{A}} \cdot \mathbf{\Phi}^R_{\sigma_k}(\mathbf{A}) \\
\tilde{\mathbf{D}} &= \mathbf{\Phi}^L_{\sigma_k}(\mathbf{A}) \cdot \mathbf{D} \cdot \mathbf{\Phi}^R_{\sigma_k}(\epsilon_k\phi_B(\mathbf{A})) + \mathbf{\Phi}^L_{\sigma_k}(\mathbf{A}) \cdot \check{\mathbf{A}} \cdot \mathbf{\Theta}^R_k \\
\tilde{\mathbf{E}} &= \mathbf{\Phi}^L_{\sigma_k}(\epsilon_k^{-1}\phi_B(\mathbf{A})) \cdot \mathbf{E} \cdot \mathbf{\Phi}^R_{\sigma_k}(\epsilon_k\phi_B(\mathbf{A})) + \mathbf{\Phi}^L_{\sigma_k}(\epsilon_k^{-1}\phi_B(\mathbf{A})) \cdot C \cdot \mathbf{\Theta}^R_k + \mathbf{\Theta}^L_k \cdot (\hat{\mathbf{A}} + \mu U) \cdot \mathbf{\Theta}^R_k \\
\tilde{\mathbf{F}} &= \mathbf{\Phi}^L_{\sigma_k}(\epsilon_k^{-1}\phi_B(\mathbf{A})) \cdot \mathbf{F} \cdot \mathbf{\Phi}^R_{\sigma_k}(\epsilon_k\phi_B(\mathbf{A})) - \mathbf{\Theta}^L_k \cdot \mathbf{D} \cdot \mathbf{\Phi}^R_{\sigma_k}(\epsilon_k\phi_B(\mathbf{A})) + \mu\mathbf{\Theta}^L_k \cdot \mathbf{\Theta}^R_k.
\end{aligned}
$$

Here, as in [Ng08], ϵ_k is $\lambda\mu^{-w(B)}$ if $k = 1$ and 1 otherwise; $\mathbf{\Phi}^L_{\sigma_k}(\epsilon_k^{-1}\phi_B(\mathbf{A})), \mathbf{\Phi}^R_{\sigma_k}(\epsilon_k\phi_B(\mathbf{A}))$ are the matrices $\mathbf{\Phi}^L_{\sigma_k}(\mathbf{A}), \mathbf{\Phi}^R_{\sigma_k}(\mathbf{A})$ with $a_{k+1,k}, a_{k,k+1}$ replaced by $\epsilon_k^{-1}\phi_B(a_{k+1,k}), \epsilon_k\phi_B(a_{k,k+1})$; and $\mathbf{\Theta}^L_k, \mathbf{\Theta}^R_k$ are the matrices that are identically zero except in the (k,k) entry, where they are $-b_{k+1,k}, -b_{k,k+1}$. (Note however that $\mathbf{A}$ here, which has -2 entries along the diagonal, is the result of setting $\mu = -1$ in the matrix A from [Ng08].)

We leave the verification that this identification intertwines ∂^- and $\tilde{\partial}^-$ to the reader. In addition to the identities provided in the proof of [Ng08, Thm. 2.7], the following two identities are useful in this regard:

$$
\begin{aligned}
\hat{\tilde{\mathbf{B}}} &= \mathbf{\Phi}^L_{\sigma_k}(\epsilon_k^{-1}\phi_B(\mathbf{A})) \cdot \hat{\mathbf{B}} \cdot \mathbf{\Phi}^R_{\sigma_k}(\epsilon_k\phi_B(\mathbf{A})) + \mathbf{\Phi}^L_{\sigma_k}(\mathbf{A}) \cdot \hat{\mathbf{A}} \cdot \mathbf{\Theta}^R_k \\
&\qquad + \mathbf{\Theta}^L_k \cdot \hat{\mathbf{A}} \cdot \mathbf{\Phi}^R_{\sigma_k}(\epsilon_k\phi_B(\mathbf{A})) + \mu U\,\partial^-(\mathbf{\Theta}^L_k \cdot \mathbf{\Theta}^R_k) \\
\check{\tilde{\mathbf{B}}} &= \mathbf{\Phi}^L_{\sigma_k}(\epsilon_k^{-1}\phi_B(\mathbf{A})) \cdot \check{\mathbf{B}} \cdot \mathbf{\Phi}^R_{\sigma_k}(\epsilon_k\phi_B(\mathbf{A})) + \mathbf{\Phi}^L_{\sigma_k}(\mathbf{A}) \cdot \check{\mathbf{A}} \cdot \mathbf{\Theta}^R_k \\
&\qquad + \mathbf{\Theta}^L_k \cdot \check{\mathbf{A}} \cdot \mathbf{\Phi}^R_{\sigma_k}(\epsilon_k\phi_B(\mathbf{A})) + \mu\,\partial^-(\mathbf{\Theta}^L_k \cdot \mathbf{\Theta}^R_k).
\end{aligned}
$$

CASE 2: $\tilde{B} = B\sigma_0$.

[3]Note that our e and f generators are respectively the f and e generators in [Ng08].

Let $(\mathcal{A} = CT^-, \partial^-)$, $(\tilde{\mathcal{A}} = \widetilde{CT}^-, \tilde{\partial}^-)$ denote the modified versions of the transverse DGAs of B, $\tilde{B}$, respectively, given by (2) in Proposition 3.3. We wish to show that $(\mathcal{A}, \partial^-)$ and $(\tilde{\mathcal{A}}, \tilde{\partial}^-)$ are stable tame isomorphic.

The generators of $\mathcal{A}$ are: a_{ij} for $1 \leq i \neq j \leq n$; c_{ij}, d_{ij} for $1 \leq i, j \leq n$; and e_{ij}, f_{ji} for $1 \leq i \leq j \leq n$. In order to give $\mathcal{A}$ as many generators as $\tilde{\mathcal{A}}$ has, add to $\mathcal{A}$ the generators

$$a_{0i}, a_{i0}, c_{00}, c_{0i}, c_{i0}, d_{00}, d_{0i}, d_{i0}, e_{00}, e_{0i}, f_{00}, f_{i0}$$

for $1 \leq i \leq n$, and extend the differential on $\mathcal{A}$ to these generators by: $\partial^- c_{0i} = -\mu a_{0i} + \hat{a}_{1i}$, $\partial^- d_{i0} = -a_{i0} + \check{a}_{i1}$ for $1 \leq i \leq n$; $\partial^- e_{00} = c_{00}$, $\partial^- f_{00} = d_{00}$, $\partial^- e_{01} = c_{10}$, $\partial^- f_{10} = d_{01}$; $\partial^- e_{0j} = d_{0j}$, $\partial^- f_{j0} = c_{j0}$ for $2 \leq j \leq n$; and the differential on the other new generators is 0. The resulting DGA, which we also write as $(\mathcal{A}, \partial^-)$, is stable tame isomorphic to the original DGA for B.

We will present a tame isomorphism between the new $(\mathcal{A}, \partial^-)$ and $(\tilde{\mathcal{A}}, \tilde{\partial}^-)$. For clarity, add tildes to the generators of $\tilde{\mathcal{A}}$: $\tilde{a}_{ij}$ for $0 \leq i \neq j \leq n$; $\tilde{c}_{ij}, \tilde{d}_{ij}$ for $0 \leq i, j \leq n$; $\tilde{e}_{ij}, \tilde{f}_{ji}$ for $0 \leq i \leq j \leq n$. The tame isomorphism is the identification between $\mathcal{A}$ and $\tilde{\mathcal{A}}$ given as follows: $a_{i_1 i_2} = \tilde{a}_{i_1 i_2}$ for $0 \leq i_1 \neq i_2 \leq n$ and

$$
\begin{aligned}
c_{00} &= \tilde{c}_{00} - \lambda \mu^{-w(B)} (\mathbf{\Phi}^L_B)_{1\ell} \tilde{d}_{\ell 1} - c_{11} & d_{00} &= \tilde{d}_{00} - \lambda^{-1} \mu^{w(B)} \tilde{c}_{1\ell} (\mathbf{\Phi}^R_B)_{\ell 1} - d_{11} \\
c_{0i} &= \tilde{c}_{1i} & d_{i0} &= \tilde{d}_{i1} \\
c_{10} &= \tilde{d}_{01} + \tilde{c}_{11} & d_{01} &= \tilde{c}_{10} + \tilde{d}_{11} \\
c_{1i} &= -\mu \tilde{c}_{0i} + \tilde{c}_{1i} - \mu \tilde{c}_{00} a_{0i} & d_{i1} &= -\mu^{-1} \tilde{d}_{i0} + \tilde{d}_{i1} - \mu^{-1} a_{i0} \tilde{d}_{00} \\
c_{j0} &= \tilde{c}_{j0} + \tilde{d}_{j1} - V \tilde{c}_{j1} - V (\mathbf{\Phi}^L_B)_{j\ell} \tilde{d}_{\ell 1} & d_{0j} &= \tilde{d}_{0j} + \tilde{c}_{1j} - U \tilde{d}_{1j} - U \tilde{c}_{1\ell} (\mathbf{\Phi}^R_B)_{\ell j} \\
c_{ji} &= \tilde{c}_{ji} & d_{ij} &= \tilde{d}_{ij} \\
e_{00} &= \mu \tilde{e}_{01} - \tilde{e}_{11} - \tilde{c}_{00} \tilde{d}_{01} & f_{00} &= \mu^{-1} \tilde{f}_{10} - \tilde{f}_{11} + \mu^{-1} \tilde{c}_{10} \tilde{d}_{00} \\
e_{0i} &= \tilde{e}_{1i} & f_{i0} &= \tilde{f}_{i1} \\
e_{11} &= \tilde{e}_{00} + \tilde{e}_{11} - \mu \tilde{e}_{01} + \tilde{c}_{00} \tilde{d}_{01} & f_{11} &= \tilde{f}_{00} + \tilde{f}_{11} - \mu^{-1} \tilde{f}_{10} - \mu^{-1} \tilde{c}_{10} \tilde{d}_{00} \\
e_{1j} &= -\mu \tilde{e}_{0j} - \tilde{e}_{1j} + \tilde{c}_{00} \tilde{d}_{0j} & f_{j1} &= -\mu^{-1} \tilde{f}_{j0} + \tilde{f}_{j1} - \mu^{-1} \tilde{c}_{j0} \tilde{d}_{00} \\
e_{j_1 j_2} &= \tilde{e}_{j_1 j_2} & f_{j_2 j_1} &= \tilde{f}_{j_2 j_1}
\end{aligned}
$$

for $1 \leq i \leq n$, $2 \leq j \leq n$, and $2 \leq j_1 \leq j_2 \leq n$. Here expressions involving ℓ are summations over $1 \leq \ell \leq n$.

The above identification intertwines the differentials ∂^- and $\tilde{\partial}^-$, a fact that we leave to the reader to verify. To complete the proof of the theorem, we need to check that the identification represents a tame isomorphism between $\mathcal{A}$ and $\tilde{\mathcal{A}}$. To this end, we can express the identification as a composition of maps beginning at $\tilde{\mathcal{A}}$ and ending at $\mathcal{A}$, where each map replaces some of the generators of $\tilde{\mathcal{A}}$ by generators of $\mathcal{A}$. Beginning with $\tilde{\mathcal{A}}$, we successively replace generators as follows:

- the $\tilde{a}$'s by the a's;
- then $\tilde{e}_{00}$, $\tilde{f}_{00}$, $\tilde{e}_{0j}$, $\tilde{f}_{j0}$ by e_{11}, f_{11}, e_{1j}, f_{j1} for $2 \leq j \leq n$;
- then $\tilde{e}_{01}$, $\tilde{f}_{10}$ by e_{00}, f_{00};
- then the remaining $\tilde{e}$'s, $\tilde{f}$'s by the remaining e's, f's;
- then $\tilde{c}_{10}$, $\tilde{d}_{01}$, $\tilde{c}_{0i}$, $\tilde{d}_{i0}$, $\tilde{c}_{j0}$, $\tilde{d}_{0j}$ by d_{01}, c_{10}, c_{1i}, d_{i1}, c_{j0}, d_{0j} for $1 \leq i \leq n$ and $2 \leq j \leq n$;
- then $\tilde{c}_{00}$, $\tilde{d}_{00}$ by c_{00}, d_{00};
- then the remaining $\tilde{c}$'s, $\tilde{d}$'s by the remaining c's, d's.

An inspection of the above identification shows that each of these maps is a tame isomorphism (in particular, each map involves only generators present at that moment), and so their composition is as well. □

Outline of proof of Theorem 1.4. Because of Theorem 1.2, it suffices to show that the infinity transverse DGA over $R[U^{\pm 1}, V^{\pm 1}]$ is invariant under negative braid stabilization.

Let B be an n-strand braid and $\tilde{B} = B\sigma_0^{-1}$ be its negative stabilization, and let $(\mathcal{A} = CT^\infty, \partial^\infty)$ and $(\tilde{\mathcal{A}} = \widetilde{CT}^\infty, \partial^\infty)$ denote the infinity flavor of the version of the transverse DGAs for B and $\tilde{B}$, respectively, given as (2) in Proposition 3.3. Write $\lambda' = \lambda\mu^{-w(B)}(U/V)^{-(\mathrm{sl}(B)+1)/2}$.

Add to the DGA $(\mathcal{A}, \partial^\infty)$ the generators

$$a_{0i}, a_{i0}, c_{00}, c_{0i}, c_{i0}, d_{00}, d_{0i}, d_{i0}, e_{00}, e_{0i}, f_{00}, f_{i0},$$

and extend the differential on $\mathcal{A}$ to these generators by setting $\partial^\infty e_{00} = c_{00}$, $\partial^\infty f_{00} = d_{00}$, $\partial^\infty e_{0i} = d_{0i}$, $\partial^\infty f_{i0} = c_{i0}$,

$$\partial^\infty c_{0i} = \mu U a_{0i} - \tfrac{\lambda' \mu U}{V} \mathbf{\Phi}^L_{1\ell} \hat{a}_{\ell i}, \qquad \partial^\infty d_{i0} = V a_{i0} - \tfrac{V}{\lambda' \mu U} \hat{a}_{i\ell} \mathbf{\Phi}^R_{\ell 1}$$

for $1 \leq i \leq n$, and $\partial^\infty = 0$ for the other new generators. The result is a DGA that we now call $(\mathcal{A}, \partial^\infty)$ and is stable tame isomorphic to the original infinity DGA for B.

Now the following identification between $(\mathcal{A}, \partial^\infty)$ and $(\tilde{\mathcal{A}}, \tilde{\partial}^\infty)$ is an isomorphism of DGAs (as usual, we leave this as an exercise):

$$
\begin{aligned}
c_{00} &= \tilde{c}_{00} + \tfrac{\lambda'\mu U}{V} \mathbf{\Phi}^L_{1\ell} \tilde{d}_{\ell 0} & d_{00} &= \tilde{d}_{00} + \tfrac{V}{\lambda'\mu U} \tilde{c}_{0\ell} \mathbf{\Phi}^R_{\ell 1} \\
c_{0i} &= \tilde{c}_{0i} & d_{i0} &= \tilde{d}_{i0} \\
c_{10} &= -\tilde{c}_{10} + \tilde{d}_{11} - \tilde{c}_{1\ell} \mathbf{\Phi}^R_{\ell 1} \phi(a_{10}) & d_{01} &= -\tilde{d}_{01} + \tilde{c}_{11} - \phi(a_{01}) \mathbf{\Phi}^L_{1\ell} \tilde{d}_{\ell 1} \\
c_{1i} &= \tilde{c}_{1i} - \tfrac{1}{U} \tilde{c}_{0i} + \tfrac{\lambda'}{V} \tilde{d}_{00} \mathbf{\Phi}^L_{1\ell} \check{a}_{\ell i} & d_{i1} &= \tilde{d}_{i1} - \tfrac{1}{V} \tilde{d}_{i0} + \tfrac{1}{\lambda'\mu U} \hat{a}_{i\ell} \mathbf{\Phi}^R_{\ell 1} \tilde{c}_{00} \\
c_{j0} &= -\tilde{c}_{j0} + \tilde{d}_{j1} - V \tilde{c}_{j1} & d_{0j} &= -\tilde{d}_{0j} + \tilde{c}_{1j} - U \tilde{d}_{1j} \\
&\quad - \tilde{c}_{j\ell} \mathbf{\Phi}^R_{\ell 1} \phi(a_{10}) - V \mathbf{\Phi}^L_{j\ell} \tilde{d}_{\ell 1} & &\quad - \phi(a_{01}) \mathbf{\Phi}^L_{1\ell} \tilde{d}_{\ell j} - U \tilde{c}_{1\ell} \mathbf{\Phi}^R_{\ell j} \\
c_{ji} &= \tilde{c}_{ji} & d_{ij} &= \tilde{d}_{ij} \\
e_{00} &= \tilde{e}_{00} & f_{00} &= \tilde{f}_{00} \\
e_{0i} &= \tilde{e}_{1i} & f_{i0} &= \tilde{f}_{i1} \\
e_{11} &= -\tfrac{1}{U} \tilde{e}_{01} + \tilde{e}_{11} - \tfrac{1}{\mu U} \tilde{e}_{00} & f_{11} &= -\tfrac{1}{V} \tilde{f}_{10} + \tilde{f}_{11} - \tfrac{\mu}{V} \tilde{f}_{00} \\
&\quad - \tfrac{\lambda'}{V} \tilde{d}_{00} \mathbf{\Phi}^L_{1\ell} \tilde{d}_{\ell 1} + \tfrac{\lambda'\mu U}{V} \tilde{f}_{00} \phi(a_{10}) & &\quad + \tfrac{1}{\lambda'\mu U} \tilde{c}_{1\ell} \mathbf{\Phi}^R_{\ell 1} \tilde{c}_{00} + \tfrac{UV}{\lambda'\mu} \phi(a_{01}) \tilde{e}_{00} \\
e_{1j} &= -\tfrac{1}{U} \tilde{e}_{0j} + \tilde{e}_{1j} - \tfrac{\lambda'}{V} \tilde{d}_{00} \mathbf{\Phi}^L_{1\ell} \tilde{d}_{\ell j} & f_{j1} &= -\tfrac{1}{V} \tilde{f}_{j0} + \tilde{f}_{j1} + \tfrac{1}{\lambda'\mu U} \tilde{c}_{j\ell} \mathbf{\Phi}^R_{\ell 1} \tilde{c}_{00} \\
e_{j_1 j_2} &= \tilde{e}_{j_1 j_2} & f_{j_2 j_1} &= \tilde{f}_{j_2 j_1}.
\end{aligned}
$$

Here $1 \leq i \leq n$, $2 \leq j \leq n$, and $2 \leq j_1 \leq j_2 \leq n$; also, $\phi = \phi_B$, $\mathbf{\Phi}^L = \mathbf{\Phi}^L_B$, $\mathbf{\Phi}^R = \mathbf{\Phi}^R_B$, and all expressions involving ℓ are summations over $1 \leq \ell \leq n$. To see that the above identification represents a tame isomorphism, we note, as in the proof of Theorem 1.2, that it can be expressed as a composition of generator replacements beginning at $\tilde{\mathcal{A}}$ and ending at $\mathcal{A}$:

- replace the $\tilde{a}$'s by the a's;
- then $\tilde{e}_{0i}$, $\tilde{f}_{i0}$ by e_{1i}, f_{i1} for $1 \leq i \leq n$;
- then the remaining $\tilde{e}$'s, $\tilde{f}$'s by the remaining e's, f's;
- then $\tilde{c}_{i0}$, $\tilde{d}_{0i}$ by c_{i0}, d_{0i} for $1 \leq i \leq n$;
- then $\tilde{c}_{1i}$, $\tilde{d}_{i1}$ by c_{1i}, d_{i1} for $1 \leq i \leq n$;
- then $\tilde{c}_{00}$, $\tilde{d}_{00}$ by c_{00}, d_{00};
- then the remaining $\tilde{c}$'s, $\tilde{d}$'s by the remaining c's, d's.

An inspection of the above identification shows that each of these maps is a tame isomorphism, and so their composition is as well. □

3.4. **Comparison of transverse DGA conventions.** In [EENSa], a version of the transverse DGA is presented that differs slightly from the version given in this paper. The two transverse DGAs agree up to signs and powers of λ and μ, and are in fact stable tame isomorphic except for a shift of λ to $-\lambda$ and μ to μ^{-1}. Here we compare the conventions and demonstrate this isomorphism. Although the transverse DGA in [EENSa], which is written there as $(KC\mathcal{A}^-, \partial^-)$, is defined for general transverse links, we will consider only the single-component knot case for simplicity.

We recall the definition of the transverse DGA from [EENSa], which we decorate with tildes where appropriate for clarity of comparison. Thus for instance let $\tilde{\mathcal{A}}_n$ be the tensor algebra $R[U,V]$ generated by $n(n-1)$ formal variables $\tilde{a}_{ij}$, $1 \leq i \neq j \leq n$. Let $\tilde{\phi} : B_n \to \operatorname{Aut} \tilde{\mathcal{A}}_n$ be the homomorphism defined by

$$\tilde{\phi}_{\sigma_k} : \begin{cases} \tilde{a}_{ki} \mapsto \tilde{a}_{k+1,i} - \tilde{a}_{k+1,k}\tilde{a}_{ki}, & i \neq k, k+1 \\ \tilde{a}_{ik} \mapsto \tilde{a}_{i,k+1} - \tilde{a}_{ik}\tilde{a}_{k,k+1}, & i \neq k, k+1 \\ \tilde{a}_{k+1,i} \mapsto \tilde{a}_{ki}, & i \neq k, k+1 \\ \tilde{a}_{i,k+1} \mapsto \tilde{a}_{ik}, & i \neq k, k+1 \\ \tilde{a}_{k,k+1} \mapsto -\tilde{a}_{k+1,k} & \\ \tilde{a}_{k+1,k} \mapsto -\tilde{a}_{k,k+1} & \\ \tilde{a}_{ij} \mapsto \tilde{a}_{ij}, & i,j \neq k, k+1; \end{cases}$$

note that this agrees mod 2 with the homomorphism ϕ introduced in Section 2. For $B \in B_n$, let $\tilde{\phi}_B$ be the image of B under this map, let $\tilde{\phi}_B^{\text{ext}}$ be the corresponding automorphism of $\tilde{\mathcal{A}}_{n+1}$ obtained by including B_n into B_{n+1}, and define matrices $\tilde{\mathbf{\Phi}}_B^L(\tilde{\mathbf{A}}), \tilde{\mathbf{\Phi}}_B^R(\tilde{\mathbf{A}})$ by

$$\tilde{\phi}_B^{\text{ext}}(\tilde{a}_{i,n+1}) = \sum_{\ell=1}^{n} (\tilde{\mathbf{\Phi}}_B^L(\tilde{\mathbf{A}}))_{i\ell}\tilde{a}_{\ell,n+1}$$

$$\tilde{\phi}_B^{\text{ext}}(\tilde{a}_{n+1,i}) = \sum_{\ell=1}^{n} \tilde{a}_{n+1,\ell}(\tilde{\mathbf{\Phi}}_B^R(\tilde{\mathbf{A}}))_{\ell i}.$$

Define four more $n \times n$ matrices $\tilde{\mathbf{A}}^U, \tilde{\mathbf{A}}^V, \tilde{\mathbf{B}}^U, \tilde{\mathbf{B}}^V$ by

$$(\tilde{\mathbf{A}}^U)_{ij} = \begin{cases} \mu\tilde{a}_{ij} & i > j \\ \mu + U & i = j \\ U\tilde{a}_{ij} & i < j \end{cases} \qquad (\tilde{\mathbf{A}}^V)_{ij} = \begin{cases} \mu V\tilde{a}_{ij} & i > j \\ 1 + \mu V & i = j \\ \tilde{a}_{ij} & i < j \end{cases}$$

$$(\tilde{\mathbf{B}}^U)_{ij} = \begin{cases} \mu\tilde{b}_{ij} & i > j \\ 0 & i = j \\ U\tilde{b}_{ij} & i < j \end{cases} \qquad (\tilde{\mathbf{B}}^V)_{ij} = \begin{cases} \mu V\tilde{b}_{ij} & i > j \\ 0 & i = j \\ \tilde{b}_{ij} & i < j. \end{cases}$$

(Here $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$ have $2, 0$ along the diagonal, respectively.) The transverse DGA $(KC\mathcal{A}^-, \tilde{\partial}^-)$ defined in [EENSa] has generators $\{\tilde{a}_{ij}\}_{1\leq i\neq j\leq n}$ of degree 0, $\{\tilde{b}_{ij}\}_{1\leq i\neq j\leq n}$ and $\{\tilde{d}_{ij}\}_{1\leq i,j\leq n}$ of degree 1, and $\{\tilde{f}_{ij}\}_{1\leq i,j\leq n}$ of degree 2, with differential:

$$\tilde{\partial}^-(\tilde{\mathbf{A}}) = 0,$$
$$\tilde{\partial}^-(\tilde{\mathbf{B}}) = -\mathbf{\Lambda}_B^{-1} \cdot \tilde{\mathbf{A}} \cdot \mathbf{\Lambda}_B + \tilde{\phi}_B(\tilde{\mathbf{A}}),$$
$$\tilde{\partial}^-(\tilde{\mathbf{D}}) = \tilde{\mathbf{A}}^V \cdot \mathbf{\Lambda}_B + \tilde{\mathbf{A}}^U \cdot \tilde{\mathbf{\Phi}}_B^R(\tilde{\mathbf{A}}),$$
$$\tilde{\partial}^-(\tilde{\mathbf{F}}) = \tilde{\mathbf{B}}^V \cdot (\tilde{\mathbf{\Phi}}_B^R(\tilde{\mathbf{A}}))^{-1} + \tilde{\mathbf{B}}^U \cdot \mathbf{\Lambda}_B^{-1} - \tilde{\mathbf{\Phi}}_B^L(\tilde{\mathbf{A}}) \cdot \tilde{\mathbf{D}} \cdot \mathbf{\Lambda}_B^{-1} + \mathbf{\Lambda}_B^{-1} \cdot \tilde{\mathbf{D}} \cdot (\tilde{\mathbf{\Phi}}_B^R(\tilde{\mathbf{A}}))^{-1}.$$

Proposition 3.6. *The version of the transverse DGA from* [EENSa] *is stable tame isomorphic to the version from this paper, once we replace λ by $-\lambda$ and μ by μ^{-1} in the latter.*

We first present a lemma relating the two homomorphisms $\phi, \tilde{\phi}$ and the matrices $\mathbf{\Phi}^L, \mathbf{\Phi}^R, \tilde{\mathbf{\Phi}}^L, \tilde{\mathbf{\Phi}}^R$. Let $R : B_n \to GL_n(\mathbb{Z})$ be the representation given by

$$R(\sigma_k)(e_i) = \begin{cases} e_{k+1} & i = k \\ -e_k & i = k+1 \\ e_i & i \neq k, k+1 \end{cases}$$

where e_i is the usual "basis" vector in $\mathbb{Z}^n$ with 1 in the i-th coordinate and 0 elsewhere.

Lemma 3.7. *Set $\tilde{\mathbf{A}} = -\mathbf{A}$; that is, $\tilde{a}_{ij} = -a_{ij}$ for all $i \neq j$. Then*

$$\begin{aligned} \tilde{\mathbf{\Phi}}^L_B(\tilde{\mathbf{A}}) &= \mathbf{\Delta}(v_B) \cdot \mathbf{\Phi}^L_B(\mathbf{A}), \\ \tilde{\mathbf{\Phi}}^R_B(\tilde{\mathbf{A}}) &= \mathbf{\Phi}^R_B(\mathbf{A}) \cdot \mathbf{\Delta}(v_B)^{-1}, \\ \tilde{\phi}_B(\tilde{\mathbf{A}}) &= -\mathbf{\Delta}(v_B) \cdot \phi_B(\mathbf{A}) \cdot \mathbf{\Delta}(v_B)^{-1}, \end{aligned}$$

where $v_B = R(B^{-1})(e_1 + \cdots + e_n)$ and $\mathbf{\Delta}(v_B)$ is the diagonal $n \times n$ matrix with diagonal entries given by v_B.

Proof. The proofs of the expressions for $\tilde{\mathbf{\Phi}}^L_B(\tilde{\mathbf{A}})$ and $\tilde{\mathbf{\Phi}}^R_B(\tilde{\mathbf{A}})$ are straightforward inductions on the length of the braid word B. For $\tilde{\mathbf{\Phi}}^L_B(\tilde{\mathbf{A}})$ (with a similar proof for $\tilde{\mathbf{\Phi}}^R_B(\tilde{\mathbf{A}})$), use the chain rules

$$\mathbf{\Phi}^L_{B\sigma_k}(\mathbf{A}) = \mathbf{\Phi}^L_{\sigma_k}(\phi_B(\mathbf{A})) \cdot \mathbf{\Phi}^L_B(\mathbf{A}), \qquad \tilde{\mathbf{\Phi}}^L_{B\sigma_k}(\tilde{\mathbf{A}}) = \tilde{\mathbf{\Phi}}^L_{\sigma_k}(\tilde{\phi}_B(\tilde{\mathbf{A}})) \cdot \tilde{\mathbf{\Phi}}^L_B(\tilde{\mathbf{A}}),$$

cf. [Ng05a, Prop. 4.4], and the fact that $\mathbf{\Phi}^L_{\sigma_k}(\phi_B(\mathbf{A}))$ and $\tilde{\mathbf{\Phi}}^L_{\sigma_k}(\tilde{\phi}_B(\tilde{\mathbf{A}}))$ are the identity matrix except for the 2×2 submatrices formed by rows $k, k+1$ and columns $k, k+1$, where they are $\begin{pmatrix} -\phi_B(a_{k+1,k}) & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -\tilde{\phi}_B(\tilde{a}_{k+1,k}) & 1 \\ 1 & 0 \end{pmatrix}$ respectively, cf. [Ng05a, Lemma 4.6]. The expression for $\tilde{\phi}_B(\tilde{\mathbf{A}})$ follows from the expressions for $\tilde{\mathbf{\Phi}}^L_B(\tilde{\mathbf{A}})$ and $\tilde{\mathbf{\Phi}}^R_B(\tilde{\mathbf{A}})$, along with the identity $\tilde{\phi}_B(\tilde{\mathbf{A}}) = \tilde{\mathbf{\Phi}}^L_B(\tilde{\mathbf{A}}) \cdot \tilde{\mathbf{A}} \cdot \tilde{\mathbf{\Phi}}^R_B(\tilde{\mathbf{A}})$, cf. Lemma 2.8. □

Proof of Proposition 3.6. Let B be a braid whose closure is a knot, and let $\mathbf{\Delta} = \mathbf{\Delta}(v_B)$ be the diagonal matrix defined in Lemma 3.7. Let $(\tilde{\mathcal{A}}, \tilde{\partial}^-)$ be the transverse DGA from [EENSa] as defined above; let $(\mathcal{A}, \partial^-)$ be the version of our transverse DGA presented as (3) in Proposition 3.3 above, but with μ replaced by μ^{-1} and $\mathbf{\Lambda}_B$ replaced by $-\mathbf{\Lambda}_B \cdot \mathbf{\Delta}$:

$$\begin{aligned}
\partial^-(\mathbf{A}) &= 0 \\
\partial^-(\mathbf{B}) &= \mathbf{\Delta}^{-1} \cdot \mathbf{\Lambda}_B^{-1} \cdot \mathbf{A} \cdot \mathbf{\Lambda}_B \cdot \mathbf{\Delta} - \phi_B(\mathbf{A}) \\
\partial^-(\mathbf{D}) &= -(\check{\mathbf{A}}|_{\mu \mapsto \mu^{-1}}) \cdot \mathbf{\Lambda}_B \cdot \mathbf{\Delta} - (\hat{\mathbf{A}}|_{\mu \mapsto \mu^{-1}}) \cdot \mathbf{\Phi}_B^R(\mathbf{A}) \\
\partial^-(\mathbf{F}) &= (\check{\mathbf{B}}|_{\mu \mapsto \mu^{-1}}) \cdot (\mathbf{\Phi}_B^R(\mathbf{A}))^{-1} + (\hat{\mathbf{B}}|_{\mu \mapsto \mu^{-1}}) \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{\Lambda}_B^{-1} \\
&\qquad - \mathbf{\Phi}_B^L(\mathbf{A}) \cdot \mathbf{D} \cdot \mathbf{\Delta}^{-1} \cdot \mathbf{\Lambda}_B^{-1} + \mathbf{\Delta}^{-1} \cdot \mathbf{\Lambda}_B^{-1} \cdot \mathbf{D} \cdot (\mathbf{\Phi}_B^R)^{-1}.
\end{aligned}$$

Note that if B has index n and writhe w, then $-\mathbf{\Delta}(v_B)$ has determinant $(-1)^{w+n} = -1$, and so replacing $\mathbf{\Lambda}_B$ by $-\mathbf{\Lambda}_B \cdot \mathbf{\Delta}$ is equivalent to replacing λ by $-\lambda$, by Proposition 3.1.

It thus suffices to show that $(\mathcal{A}, \partial^-)$ and $(\tilde{\mathcal{A}}, \tilde{\partial}^-)$ are tamely isomorphic. But under the tame isomorphism between $\mathcal{A}$ and $\tilde{\mathcal{A}}$ given by $\tilde{\mathbf{A}} = -\mathbf{A}$, $\tilde{\mathbf{B}} = \mathbf{\Delta} \cdot \mathbf{B} \cdot \mathbf{\Delta}^{-1}$, $\tilde{\mathbf{D}} = \mu \mathbf{D} \cdot \mathbf{\Delta}^{-1}$, $\tilde{\mathbf{F}} = \mu \mathbf{\Delta} \cdot \mathbf{F}$, we have

$$\begin{aligned}
\tilde{\mathbf{A}}^U &= -\mu(\hat{\mathbf{A}}|_{\mu \mapsto \mu^{-1}}) & \tilde{\mathbf{A}}^V &= -\mu(\check{\mathbf{A}}|_{\mu \mapsto \mu^{-1}}) \\
\tilde{\mathbf{B}}^U &= \mu \mathbf{\Delta} \cdot (\hat{\mathbf{B}}|_{\mu \mapsto \mu^{-1}}) \cdot \mathbf{\Delta}^{-1} & \tilde{\mathbf{B}}^V &= \mu \mathbf{\Delta} \cdot (\check{\mathbf{B}}|_{\mu \mapsto \mu^{-1}}) \cdot \mathbf{\Delta}^{-1};
\end{aligned}$$

then Lemma 3.7, along with the definitions of ∂^- and $\tilde{\partial}^-$, gives $\tilde{\partial}^-(\tilde{\mathbf{A}}) = \partial^-(\tilde{\mathbf{A}})$, $\tilde{\partial}^-(\tilde{\mathbf{B}}) = \partial^-(\tilde{\mathbf{B}})$, $\tilde{\partial}^-(\tilde{\mathbf{D}}) = \partial^-(\tilde{\mathbf{D}})$, and $\tilde{\partial}^-(\tilde{\mathbf{F}}) = \partial^-(\tilde{\mathbf{F}})$. □

4. Properties

In this section, we present some basic properties of transverse homology.

Proposition 4.1. *If T is a destabilizable transverse knot (i.e., it is the stabilization of another transverse knot), then the double-hat transverse DGA of T is trivial. In particular,*

$$\widehat{\widehat{HT}}_*(T) = 0.$$

Proof. As in the proof of Proposition 3.5, we calculate that if $B \in B_n$ and $\tilde{B} = B\sigma_0^{-1}$ is a transverse stabilization, then we have

$$\partial^-(c_{00}) = -1 - \mu U - \lambda \mu^{-w(B)+1} V (\mathbf{\Phi}_B^L)_{1\ell} a_{\ell 0}.$$

Setting $U = V = 0$ gives $\hat{\hat{\partial}}(c_{00}) = -1$. □

Note the formal similarity of Proposition 4.1 to the result of Chekanov [Che02] that the Legendrian DGA for a stabilized Legendrian knot in $(\mathbb{R}^3, \xi_{\mathrm{std}})$ is trivial.

We next examine various symmetries of the transverse DGA.

Proposition 4.2. *If α is a unit, then we have a chain isomorphism of $R[U, V]$-algebras*

$$(CT_*^-(B; \lambda, \mu, U, V), \partial^-) \cong (CT_*^-(B; \lambda\alpha^{-\mathrm{sl}(B)}, \mu/\alpha, U\alpha, V/\alpha), \partial^-)$$

with corresponding isomorphisms for $\widehat{CT}$ and $\widehat{\widehat{CT}}$, and a chain isomorphism of $R[U^{\pm 1}, V^{\pm 1}]$-algebras

$$(CT^\infty_*(B; \lambda, \mu, U, V), \partial^\infty) \cong (CT^\infty_*(B; \lambda\alpha, \mu/\alpha, U\alpha, V/\alpha), \partial^\infty).$$

Proof. Replacing (λ, μ, U, V) by $(\lambda\alpha^{-\operatorname{sl}(B)}, \mu/\alpha, U\alpha, V/\alpha)$ keeps $\hat{\mathbf{A}}$ unchanged, multiplies $\check{\mathbf{A}}$ by α^{-1}, and replaces $\mathbf{\Lambda}_B$ by $\operatorname{diag}(\lambda\mu^{-w(B)}\alpha^{n(B)}, 1, \ldots, 1)$. In the differential for the transverse DGA, this has the same effect as keeping $\hat{\mathbf{A}}$ and $\check{\mathbf{A}}$ unchanged, replacing $\mathbf{D}$ by $\alpha\mathbf{D}$ (a tame isomorphism), and replacing $\mathbf{\Lambda}_B$ by $\operatorname{diag}(\lambda\mu^{-w(B)}\alpha^{n(B)-1}, \alpha^{-1}, \ldots, \alpha^{-1})$. Since this matrix has the same determinant, $\lambda\mu^{-w(B)}$, as $\mathbf{\Lambda}_B$, the result follows from Proposition 3.1. □

It follows from Proposition 4.2 that localizing at V (allowing V to be invertible) in the transverse DGA is equivalent to setting $V = 1$. In particular, one can set $V = 1$ in infinity transverse homology without losing any information.

For our next symmetry, if (CT^-_*, ∂^-) is a transverse DGA, then we can construct another DGA $(CT^-_*, \partial^-_{\mathrm{op}})$ where ∂^-_{op} is defined on generators of CT^-_* as ∂^- but with every word reversed and appropriate signs introduced. More precisely, on any graded tensor algebra over $R[U, V]$, we can define an $R[U, V]$-module involution, op, by

$$\operatorname{op}(x_1 x_2 \cdots x_n) = (-1)^{\sum_{i<j} |x_i||x_j|}(x_n \cdots x_2 x_1),$$

where $x_1, x_2, \ldots, x_n$ are generators of the tensor algebra. Now define ∂^-_{op} on CT^-_* by

$$\partial^-_{\mathrm{op}} = \operatorname{op} \circ \partial^- \circ \operatorname{op}.$$

Since $\operatorname{op}(ab) = (-1)^{|a||b|}(\operatorname{op} b)(\operatorname{op} a)$ for any a, b, it is easy to check that ∂^-_{op} satisfies the Leibniz rule and $(\partial^-_{\mathrm{op}})^2 = 0$.

For B a braid, the transverse complex $CT^-_*(B)$ involves parameters λ, μ, U, V; we make this explicit, where relevant, by writing $CT^-_*(B; \lambda, \mu, U, V)$ for $CT^-_*(B)$.

Proposition 4.3. *We have a chain isomorphism of $R[U, V]$-algebras*

$$(CT^-_*(B; \lambda^{-1}, \mu^{-1}, V, U), \partial^-(B; \lambda^{-1}, \mu^{-1}, V, U)) \cong (CT^-_*(B; \lambda, \mu, U, V), \partial^-_{\mathrm{op}}(B; \lambda, \mu, U, V)).$$

In other words, one can switch the roles of U and V in the transverse DGA, at the price of reversing word order and inverting λ and μ.

We first establish a lemma. Define an algebra isomorphism $\psi : CT^-_* \to CT^-_*$ by

$$\psi(\mathbf{A}) = \mathbf{A}^T, \quad \psi(\mathbf{B}) = \mathbf{B}^T, \quad \psi(\mathbf{C}) = \mu\mathbf{D}^T,$$
$$\psi(\mathbf{D}) = \mu\mathbf{C}^T, \quad \psi(\mathbf{E}) = \mu\mathbf{F}^T, \quad \psi(\mathbf{F}) = \mu\mathbf{E}^T,$$

where T represents transpose; that is, $\psi(a_{ij}) = a_{ji}$ and so forth. Note that $\psi \circ \operatorname{op} = \operatorname{op} \circ \psi$. Extend ψ and op to matrices as usual: $(\psi(\mathbf{M}))_{ij} = \psi(\mathbf{M}_{ij})$, and similarly for op.

Lemma 4.4. (1) *If all of the entries in the matrix* $\mathbf{M}_1$ *(or all of the entries in* $\mathbf{M}_2$*) have degree* 0*, then* $(\mathbf{M}_1\mathbf{M}_2)^T = \operatorname{op}((\operatorname{op}\mathbf{M}_2)^T(\operatorname{op}\mathbf{M}_1)^T)$*.*

(2) $\hat{\mathbf{A}}|_{\mu\mapsto\mu^{-1},U\mapsto V,V\mapsto U} = \mu^{-1}\psi(\check{\mathbf{A}}^T)$ *and* $\check{\mathbf{A}}|_{\mu\mapsto\mu^{-1},U\mapsto V,V\mapsto U} = \mu^{-1}\psi(\hat{\mathbf{A}}^T)$*.*

(3) $\psi\operatorname{op}\mathbf{\Phi}^L_B = (\mathbf{\Phi}^R_B)^T$ *and* $\psi\operatorname{op}\mathbf{\Phi}^R_B = (\mathbf{\Phi}^L_B)^T$*.*

Proof. (1) is immediate from the definition of op, and (2) is immediate from the definition of $\hat{\mathbf{A}}$ and $\check{\mathbf{A}}$. (3) can be seen from the fact that the homomorphism ϕ satisfies $\psi\phi(a_{ij}) = \operatorname{op}\phi(a_{ji})$ for all i,j by the construction of ϕ. □

Proof of Proposition 4.3. For clarity, denote the differential ∂^- on CT^-_* with (λ,μ,U,V) replaced by $(\lambda^{-1},\mu^{-1},V,U)$ by $\tilde{\partial}^-$. We claim that $\psi\circ\partial^-_{\mathrm{op}} = \tilde{\partial}^-\circ\psi$.

This is a fairly routine claim to verify, given Lemma 4.4; we will check that $\psi\circ\partial^-_{\mathrm{op}}(\mathbf{C}) = \tilde{\partial}^-\circ\psi(\mathbf{C})$, and leave verification for the other generators of CT^-_* to the reader:

$$\begin{aligned}
\tilde{\partial}^-(\psi(\mathbf{C})) &= \mu\tilde{\partial}^-(\mathbf{D}^T)\\
&= \mu(\check{\mathbf{A}}^T|_{\mu\mapsto\mu^{-1},U\mapsto V,V\mapsto U}) - \mu((\hat{\mathbf{A}}|_{\mu\mapsto\mu^{-1},U\mapsto V,V\mapsto U})\cdot\mathbf{\Phi}^R_B\cdot\mathbf{\Lambda}_B)^T\\
&= \psi(\hat{\mathbf{A}}) - \operatorname{op}(\mathbf{\Lambda}_B\cdot\operatorname{op}(\mathbf{\Phi}^R_B)^T\cdot\operatorname{op}\psi(\check{\mathbf{A}}))\\
&= \psi(\hat{\mathbf{A}} - \operatorname{op}(\mathbf{\Lambda}_B\cdot\mathbf{\Phi}^L_B\cdot\check{\mathbf{A}}))\\
&= \psi(\partial^-_{\mathrm{op}}(\mathbf{C})).
\end{aligned}$$

The proposition follows. □

We can use Proposition 4.3 to understand the effect on the transverse DGA of an operation on transverse knots known as mirroring.

Definition 4.5. Let T be a transverse knot represented by the closure of a braid B. The *transverse mirror* of T, written $\operatorname{mir}(T)$, is the transverse knot represented by the closure of the braid obtained by reversing the order of the letters in B.

Note that in the topological category, transverse mirroring takes a knot to its orientation reverse. In [NT09], transverse mirrors are defined in terms of Legendrian approximations: the transverse mirror of a Legendrian approximation Λ to a transverse knot is the transverse pushoff of $-\mu(\Lambda)$, the orientation reverse of the Legendrian mirror of Λ. It is an easy exercise (cf. [KN10]) to check that this definition agrees with ours.

One example of transverse mirrors is the family of pairs of 3-braids considered by Birman and Menasco in [BM08]: $\sigma_1^u\sigma_2^v\sigma_1^w\sigma_2^{-1}$ and $\sigma_1^w\sigma_2^v\sigma_1^u\sigma_2^{-1}$, which are related by a negative flype and thus topologically isotopic. The transverse mirror of the first of these is $\sigma_2^{-1}\sigma_1^w\sigma_2^v\sigma_1^u$, which is conjugate and thus transversely isotopic to the second.

For the next result, we borrow a definition from [Ng05a]: for $B \in B_n$, define B^* to be the image of B under the group homomorphism on B_n sending σ_k to σ_{n-k}^{-1} for all k; then the braids B and $(B^*)^{-1}$ represent transverse mirrors.

Proposition 4.6. *We have a chain isomorphism of $R[U,V]$-algebras*

$$(CT_*^-((B^*)^{-1};\lambda,\mu,U,V),\partial^-((B^*)^{-1};\lambda,\mu,U,V) \cong (CT_*^-(B;\lambda,\mu,U,V),\partial_{\mathrm{op}}^-(B;\lambda,\mu,U,V)).$$

In other words, if T is a transverse knot, then the transverse DGA of the transverse mirror of T is the opposite of the transverse DGA of T.

Proof. We follow similar proofs from [Ng05a, Ng08]. Let B be an n-strand braid and let CT_*^- denote the usual algebra generated by a, b, c, d, e, f generators. By Proposition 4.3, it suffices to give an isomorphism between $(CT_*^-, \partial^-((B^*)^{-1};\lambda^{-1},\mu^{-1},V,U)$ and $(CT_*^-,\partial^-(B;\lambda,\mu,U,V))$. For clarity, write $\tilde\partial^-$ for $\partial^-((B^*)^{-1};\lambda^{-1},\mu^{-1},V,U)$ as in Definition 2.2, and write ∂^- for $\partial^-(B;\lambda,\mu,U,V)$ as in Definition 2.2 but with $\mathbf{\Lambda}_B$ replaced by $\mathrm{diag}(1,\dots,1,\lambda\mu^{-w(B)})$. By Proposition 3.1, it suffices to show that (CT_*^-,∂^-) and $(CT_*^-,\tilde\partial^-)$ are chain isomorphic.

Let Ξ be the operation on $n\times n$ matrices defined by $\Xi(\mathbf{M})_{ij} = \mathbf{M}_{n+1-i,n+1-j}$, i.e., Ξ conjugates by the $n \times n$ matrix with 1's on the NE-SW diagonal and 0's everywhere else. Define an algebra isomorphism $\xi : CT^- \to CT^-$ by

$$\xi(\mathbf{A}) = \Xi(\phi_B(\mathbf{A})), \qquad \xi(\mathbf{B}) = -\mathbf{\Lambda}_B^{-1}\cdot\Xi(\mathbf{B})\cdot\mathbf{\Lambda}_B,$$
$$\xi(\mathbf{C}) = -\mu^{-1}\mathbf{\Lambda}_B^{-1}\cdot\Xi(\mathbf{C}\cdot\mathbf{\Phi}_B^R(\mathbf{A})), \qquad \xi(\mathbf{D}) = -\mu^{-1}\Xi(\mathbf{\Phi}_B^L(\mathbf{A})\cdot\mathbf{D})\cdot\mathbf{\Lambda}_B,$$
$$\xi(\mathbf{E}) = -\mu^{-1}\mathbf{\Lambda}_B^{-1}\cdot\Xi(\mathbf{F})\cdot\mathbf{\Lambda}_B, \qquad \xi(\mathbf{F}) = -\mu^{-1}\mathbf{\Lambda}_B^{-1}\cdot\Xi(\mathbf{E})\cdot\mathbf{\Lambda}_B.$$

We claim that $\xi\circ\tilde\partial^- = \partial^-\circ\xi$.

From [Ng05a, Cor. 4.5] and the proof of [Ng05a, Prop. 6.9], we have

$$\xi(\mathbf{\Phi}^L_{(B^*)^{-1}}(\mathbf{A})) = \Xi(\mathbf{\Phi}^L_{B^{-1}}(\phi_B(\mathbf{A}))) = \Xi(\mathbf{\Phi}^L_B(\mathbf{A}))^{-1},$$

and similarly $\xi(\mathbf{\Phi}^R_{(B^*)^{-1}}(\mathbf{A})) = \Xi(\mathbf{\Phi}^R_B(\mathbf{A}))^{-1}$. In addition, one readily checks from the definitions that

$$\xi(\hat{\mathbf{A}}|_{\mu\mapsto\mu^{-1},U\mapsto V,V\mapsto U}) = \mu^{-1}\Xi(\phi_B(\check{\mathbf{A}}))$$
$$\xi(\check{\mathbf{A}}|_{\mu\mapsto\mu^{-1},U\mapsto V,V\mapsto U}) = \mu^{-1}\Xi(\phi_B(\hat{\mathbf{A}})).$$

With these identities in hand, it is straightforward to prove the claim; we check that $\xi\circ\tilde\partial^-(\mathbf{C}) = \partial^-\circ\xi(\mathbf{C})$, and leave verification for the other generators of CT_*^- to the reader:

$$\begin{aligned}\xi(\tilde\partial^-(\mathbf{C})) &= \xi\left((\hat{\mathbf{A}}|_{\mu\mapsto\mu^{-1},U\mapsto V,V\mapsto U}) - \mathbf{\Lambda}_B^{-1}\cdot\mathbf{\Phi}^L_{(B^*)^{-1}}(\mathbf{A})\cdot(\check{\mathbf{A}}|_{\mu\mapsto\mu^{-1},U\mapsto V,V\mapsto U})\right)\\ &= \mu^{-1}\Xi(\phi_B(\check{\mathbf{A}})) - \mu^{-1}\mathbf{\Lambda}_B^{-1}\cdot\Xi(\mathbf{\Phi}^L_B(\mathbf{A}))^{-1}\cdot\Xi(\phi_B(\hat{\mathbf{A}}))\\ &= -\mu^{-1}\mathbf{\Lambda}_B^{-1}\Xi\left((\hat{\mathbf{A}} - \Xi(\mathbf{\Lambda}_B)\cdot\mathbf{\Phi}^L_B(\mathbf{A})\cdot\check{\mathbf{A}})\cdot\mathbf{\Phi}^R_B(\mathbf{A})\right)\\ &= \partial^-(\xi(\mathbf{C})).\end{aligned}$$

The proposition follows. □

Like Proposition 4.1, Proposition 4.6 may be compared to an analogous result in Legendrian contact homology, namely that the Chekanov–Eliashberg DGA for the Legendrian mirror of a Legendrian knot Λ is the opposite of the DGA for Λ. See [Ng03].

5. Computations and Applications

In this section, we perform some calculations with transverse homology and demonstrate that it constitutes an effective invariant of transverse knots.

5.1. **The augmentation polynomial of a topological knot.** The infinity transverse homology of a topological knot, a graded algebra over $R[U^{\pm 1}, V^{\pm 1}] = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}, U^{\pm 1}, V^{\pm 1}]$, is a somewhat unwieldy object to compute. There are various ways to extract information out of HT^∞; here we highlight one, a three-variable polynomial that is in some sense a generalization of the two-variable A-polynomial.

Definition 5.1. Let K be a topological knot, and let $\lambda_0, \mu_0, U_0, V_0 \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We say that the DGA $(CT_*^\infty(K), \partial^\infty)$ *has an augmentation at* $(\lambda_0, \mu_0, U_0, V_0)$ if there is a $\mathbb{C}$-algebra map

$$HT_0^\infty(K) \otimes_{R[U^{\pm 1}, V^{\pm 1}]} \mathbb{C} \to \mathbb{C},$$

where $\mathbb{C}$ is viewed as an $R[U^{\pm 1}, V^{\pm 1}]$-algebra with λ, μ, U, V acting as multiplication by the complex scalars $\lambda_0, \mu_0, U_0, V_0$, respectively.

The *augmentation variety* of K is the set

$$\{(\lambda_0, \mu_0, U_0) : (CT_*^\infty(K), \partial^\infty) \text{ has an augmentation at } (\lambda_0, \mu_0, U_0, 1)\} \subset (\mathbb{C}^*)^3.$$

If the augmentation variety is not complex 3-dimensional, then the union of its 2-dimensional components is the zero set of the *augmentation polynomial* $\mathrm{Aug}_K(\lambda, \mu, U) \in \mathbb{C}[\lambda, \mu, U]$, well-defined up to constant multiplication if we specify that it contains no repeated factors and is not divisible by λ, μ, or U.

Note that the V coordinate has been dropped from the definition of the augmentation variety and polynomial; this is because of Proposition 4.2, which implies that the V information is superfluous in the infinity theory.

Example 5.2. For the unknot U, the computation from Example 2.3 shows that $\mathrm{Aug}_U(\lambda, \mu, U) = -1 - \mu U + \lambda + \lambda\mu$.

Example 5.3. Consider the right-handed trefoil T given as the closure of $\sigma_1^3 \in B_2$. The infinity transverse DGA has $\partial^\infty(b_{21}) = a_{21} - \frac{\mu^3 U}{\lambda V} a_{12}$, and so $a_{21} = \frac{\mu^3 U}{\lambda V} a_{12}$ in $HT_0^\infty(T)$. It follows that $HT_0^\infty(T)$ is a quotient of the polynomial ring $(R[U^{\pm 1}, V^{\pm 1}])[x]$ with $x = a_{12}$. A computation of $\partial^\infty(\mathbf{C}), \partial^\infty(\mathbf{D})$ for the trefoil yields

$$HT_0^\infty(T) \cong (\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}, U^{\pm 1}, V^{\pm 1}])[x] \,/ \\ (\mu^4 U x^2 + \mu^3 U x - \lambda\mu V - \lambda V^2, \lambda\mu^2 V x - \mu^3 U x + \lambda V^2 - \lambda\mu^2 U V).$$

Up to multiplication by an overall unit and after setting $V = 1$, the resultant of the two polynomials above in x, and thus the augmentation polynomial of T, is

$$\mathrm{Aug}_T(\lambda, \mu, U) = (\lambda\mu^4 - \mu^4)U^3 + (\lambda\mu^3 - \mu^3 - 2\lambda\mu^2)U^2 + (2\lambda\mu^2 + \lambda\mu + \lambda)U + (-\lambda^2\mu - \lambda^2).$$

For any knot, since transverse homology descends to the knot contact homology from [Ng08] when we set $U = V = 1$, the intersection of the augmentation variety in $(\mathbb{C}^*)^3$ with the plane $U = 1$ yields the augmentation variety in $(\mathbb{C}^*)^2$ considered in [Ng08]. One might reasonably guess that the specialization $\mathrm{Aug}_K(\lambda, \mu, 1)$ should often equal the augmentation polynomial $\tilde{A}_K(\lambda, \mu)$ from [Ng08], whose vanishing set is the augmentation variety in $(\mathbb{C}^*)^2$. This is true for the unknot and the trefoil; we do not know if it is true in general.

From [Ng08], $\tilde{A}_K(\lambda, \mu)$ contains the A-polynomial of K as a factor. One might then consider $\mathrm{Aug}_K(\lambda, \mu, U)$ to be some sort of three-variable generalization of the A-polynomial. A geometric interpretation for the three-variable augmentation polynomial (in terms of representations of $\pi_1(\mathbb{R}^3 \setminus K)$) is currently not known to the author.

5.2. **Transverse computations.** Here we provide evidence that transverse homology is quite effective as an invariant of transverse knots. More precisely, we will see that $\widehat{HT}_0$, considered as an algebra over R, can be used to distinguish pairs of transverse knots with the same classical invariants. (For unknown reasons, it appears in examples that $\widehat{HT}_0$ is more effective at distinguishing transverse knots than $\widehat{\widehat{HT}}_0$.) As an easily implemented computational tool, we use augmentation numbers in a similar manner to [Ng05a, Ng08].

Definition 5.4. Let $\mathbf{k}$ be a finite field and let λ_0, μ_0 be nonzero elements of $\mathbf{k}$. For a finitely generated R-algebra A, the *augmentation number* $\mathrm{Aug}(A, \mathbf{k}, \lambda_0, \mu_0)$ is the number of $\mathbf{k}$-algebra maps $A \otimes_{\mathbb{Z}} \mathbf{k} \to \mathbf{k}$ sending $\lambda \in R$ to λ_0 and $\mu \in R$ to μ_0.

We have the following corollary of Theorem 1.4.

Proposition 5.5. *If T_1, T_2 are transverse knots and there exist $\mathbf{k}, \lambda_0, \mu_0$ for which $\mathrm{Aug}(\widehat{HT}_0(T_1), \mathbf{k}, \lambda_0, \mu_0) \neq \mathrm{Aug}(\widehat{HT}_0(T_2), \mathbf{k}, \lambda_0, \mu_0)$, then T_1, T_2 are not transversely isotopic.*

In practice, augmentation numbers for finitely generated, finitely presented R-algebras are straightforward to calculate by computer. Our computations rely on `transverse.m`, and executable *Mathematica* notebooks producing these computations are available at the author's web site.

On the next two pages, we present a table of the 13 knot types of arc index 9 or fewer that are suggested by the work of [CN] to be transversely nonsimple; this could be conjectured to be a complete list of such knot types (transversely nonsimple with arc index at most 9). For each type, the table includes the guess from [CN] for all of the distinct transverse knots of that type with the relevant self-linking number. Of these 13 topological knots, 6 have been previously shown to be transversely nonsimple, all via the transverse invariant in Heegaard Floer homology; see [CN, NOT08, OS10].

By contrast, at least 10 of the 13 knots can be shown to be transversely nonsimple by transverse homology and Proposition 5.5, including all 6 that can be distinguished by Heegaard Floer homology. Indeed, as the following table shows, it suffices to compute augmentation numbers for as small a field as $\mathbf{k} = \mathbb{Z}/3$.[4]

One particular knot of interest is the twist knot $m(7_2)$. Here Ozsváth and Stipsicz [OS10] have shown transverse nonsimplicity using the Heegaard Floer LOSS invariant, along with a naturality argument. (By contrast, the other 5 knots that can be distinguished by Heegaard Floer can all be treated by a computer program.) Transverse homology, on the other hand, can be used to distinguish the $m(7_2)$ knots by computer, without any geometric input.

The remaining 3 knots of arc index ≤ 9 that are conjectured to be transversely nonsimple are all of the following form: the pair of transverse knots that are conjecturally transversely nonisotopic are related by the transverse mirror operation. Since augmentation numbers depend only on the abelianization of the DGA, and the transverse DGAs for transverse mirrors have the same abelianization by Proposition 4.6, augmentation numbers will never be able to distinguish mirrors. It is not inconceivable that the full transverse DGA might sometimes distinguish mirrors, but this seems to be a very delicate point. We remark that the (grid-diagram) Heegaard Floer transverse invariant cannot currently distinguish transverse mirrors, at least via the

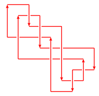
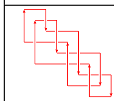
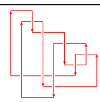
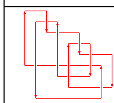
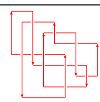
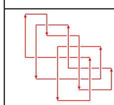
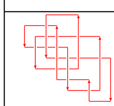
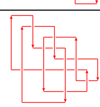
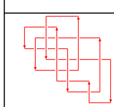
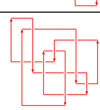
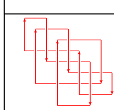
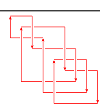
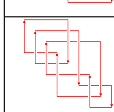
[4]The computer program has difficulty computing augmentation numbers for braids with more than 4 strands. To compute the augmentation numbers for the non-stabilized 5-strand transverse representatives of $m(10_{145})$ and $12n_{591}$ in the table, we used a slightly different version of degree 0 transverse homology than Definition 2.1: it is easy to prove that if $B \in B_n$ is a product of two braids $B = B_1 B_2$, then

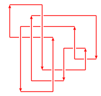
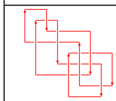
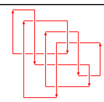
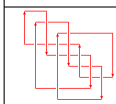
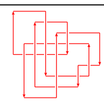
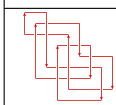
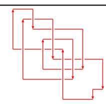
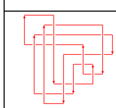
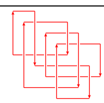
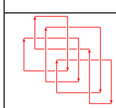
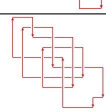
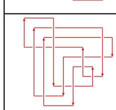
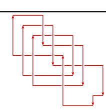
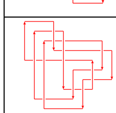
$$HT_0(B) \cong \mathcal{A}_n \,/\, (\mathbf{\Phi}^L_{B_1^{-1}} \cdot \hat{\mathbf{A}} - \mathbf{\Lambda}_B \cdot \mathbf{\Phi}^L_{B_2} \cdot \check{\mathbf{A}},\ \check{\mathbf{A}} \cdot \mathbf{\Phi}^R_{B_1^{-1}} - \hat{\mathbf{A}} \cdot \mathbf{\Phi}^R_{B_2} \cdot \mathbf{\Lambda}_B^{-1}).$$

One can use this formulation of HT_0 to eliminate enough generators of $\mathcal{A}_n$ to allow for the computation of augmentation numbers for $m(10_{145})$ and $12n_{591}$. For the stabilized representatives of $m(10_{145})$ and $12n_{591}$, the computation is much easier: an argument along the lines of Proposition 4.1 shows that any augmentation number $\mathrm{Aug}(\widehat{HT}_0(T), \mathbf{k}, \lambda_0, \mu_0)$ for a stabilized transverse knot must be 0 unless $\mu_0 = -1$.

techniques of [NOT08]; see [OST08, Prop. 1.2]. (It is probable that this will change in the future with the advent of naturality results.)

In the table, for each knot type, transverse representatives that are (conjecturally) distinct are depicted in two ways: grid diagrams corresponding to Legendrian approximations (as taken from [CN]) and braid closures. (For the algorithm to get from one to the other, see, e.g., [KN10]. For braids, numbers represent braid generators and bars represent inverses; e.g., $3\,3\,\overline{2}\,3\,2\,1\,1\,2\,\overline{1}$ is the 4-braid $\sigma_3^2\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_1^{-1}$.) The table then indicates whether the two transverse invariants, the Heegaard Floer invariant and transverse homology, can distinguish the transverse representatives. In the case of a positive answer for transverse homology, the relevant augmentation-number computation is given. Of the 7 knot types with a non-positive answer for the transverse Heegaard Floer invariant, 5 have $\widehat{HFK} = 0$ in the relevant bidegree, while the other 2, $m(9_{45})$ and 10_{128}, are transverse mirrors. (The same result holds for Khovanov homology in the relevant bidegree and Khovanov–Rozansky homology in the relevant triple degree, indicating that the transverse invariants of Plamenevskaya [Pla06] and Wu [Wu08], like the Heegaard Floer invariants, do not distinguish these transverse knots.)

Knot	Grid	Braid	*HFK*?	*HT*?	*HT* computation
$m(7_2)$		$3\,3\,\bar{2}\,3\,2\,1\,1\,2\,\bar{1}$	✓ [OS10]	✓	$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 0$
		$3\,3\,\bar{2}\,3\,2\,\bar{1}\,2\,1\,1$			$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 5$
$m(7_6)$		$1\,\bar{2}\,1\,\bar{2}\,\bar{3}\,2\,3\,3\,3$	✗	✓	$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 5$
		$1\,\bar{2}\,1\,\bar{2}\,3\,3\,3\,2\,\bar{3}$			$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 0$
9_{44}		$\bar{3}\,1\,2\,\bar{3}\,\bar{2}\,3\,1\,\bar{2}\,\bar{3}$	✗	✓/?	$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 5$
		$\bar{2}\,\bar{3}\,2\,1\,2\,\bar{3}\,\bar{2}\,1\,\bar{2}$			$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 0$
		(mirror of previous braid)			$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 0$
$m(9_{45})$		$2\,\bar{3}\,2\,1\,3\,\bar{2}\,3\,1\,2$	?	?	—
		(mirror of previous braid)			—
9_{48}		$\bar{2}\,3\,3\,2\,\bar{1}\,2\,\bar{3}\,2\,1\,1\,\bar{2}$	✗	✓	$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 4$
		$2\,3\,3\,2\,\bar{1}\,\bar{2}\,\bar{2}\,\bar{3}\,2\,1\,1$			$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 0$
10_{128}		$1\,2\,1\,2\,1\,2\,1\,3\,3\,2\,\bar{3}$	?	?	—
		(mirror of previous braid)			—

Knot	Grid	Braid	*HFK*?	*HT*?	*HT* computation
$m(10_{132})$		$3\bar{2}\bar{2}332\bar{3}\bar{1}211$	✓ [NOT08]	✓	$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 1, 1) = 0$
		$3\bar{2}\bar{2}332\bar{3}112\bar{1}$			$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 1, 1) = 1$
10_{136}		$\bar{1}2\bar{1}233\bar{2}1\bar{2}\bar{3}2$	✗	✓	$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 5$
		$\bar{2}3\bar{2}\bar{1}\bar{2}3\bar{2}1113$			$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 0$
$m(10_{140})$		$11\bar{2}12\bar{1}\bar{1}\bar{3}233$	✓ [NOT08]	✓	$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 1$
		$11\bar{2}12\bar{1}\bar{1}332\bar{3}$			$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 2, 1) = 2$
$m(10_{145})$		$\bar{2}3321\bar{1}213221\bar{4}$	✓ [CN]	✓	$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 1, 1) = 0$
		$321\bar{3}\bar{4}\bar{2}\bar{3}1221344$			$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 1, 1) = 1$
10_{160}		$\bar{2}3\bar{2}\bar{1}323231 1$	✗	?	—
		(mirror of previous braid)			—
$m(10_{161})$		$\bar{1}21112211 2\bar{3}$	✓ [CN]	✓	$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 1, 1) = 0$
		$2\bar{1}22133222\bar{1}2\bar{3}$			$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 1, 1) = 1$
$12n_{591}$		$3232\bar{1}321321 21\bar{4}$	✓ [CN]	✓	$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 1, 1) = 0$
		$\bar{2}\bar{3}\bar{1}\bar{2}434321212143 43$			$\mathrm{Aug}(\widehat{HT}_0, \mathbb{Z}/3, 1, 1) = 1$

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