TORSION IN LINEARIZED CONTACT HOMOLOGY FOR LEGENDRIAN KNOTS

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Abstract. We present examples of Legendrian knots in $\mathbb{R}^3$ that have linearized Legendrian contact homology over $\mathbb{Z}$ containing torsion. As a consequence, we show that there exist augmentations of Legendrian knots over $\mathbb{Z}$ that are not induced by exact Lagrangian fillings, even though their mod 2 reductions are.

1. Introduction

Holomorphic-curve invariants are powerful tools for studying Legendrian submanifolds of contact manifolds. This paper concerns the basic setting of Legendrian knots in $\mathbb{R}^3$ equipped with the standard contact structure $\ker(dz - y\,dx)$, and the invariant known as Legendrian contact homology or the Chekanov–Eliashberg differential graded algebra.

In [5], Chekanov provided the first formulation of this invariant as a DGA over $\mathbb{Z}/2$ associated to a Legendrian knot Λ in $\mathbb{R}^3$. Under the appropriate equivalence relation (stable tame isomorphism), this DGA is invariant under Legendrian isotopy, but it is difficult to tell when two DGAs are equivalent. To extract more tractable invariants, Chekanov used augmentations (homomorphisms to $\mathbb{Z}/2$) to construct finite-dimensional chain complexes from the DGA. The homologies of these complexes are called linearized (Legendrian) contact homology and denoted $\mathrm{LCH}^{\epsilon}_*(\Lambda;\mathbb{Z}/2)$; Chekanov showed that the collection of linearized contact homologies over all augmentations ϵ is a Legendrian-isotopy invariant. Using this, he gave the first example of a pair of Legendrian knots, of topological type $m(5_2)$, which have the same classical invariants (knot type, Thurston–Bennequin number, and rotation number) but are not Legendrian isotopic.

Since Chekanov's work, the study of Legendrian contact homology has developed into a rich subject with connections to areas including microlocal sheaf theory, cluster algebras, homological mirror symmetry, and topological string theory. We refer the reader to the survey [14] for a discussion of Legendrian contact homology in $\mathbb{R}^3$, but we briefly mention a couple of developments that are relevant here.

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First, although Chekanov introduced augmentations and linearized contact homology as purely algebraic objects associated to the Chekanov–Eliashberg DGA, we now understand certain augmentations as having geometric origins. Specifically, in [11], Ekholm–Honda–Kálmán showed that any exact Lagrangian filling L of a Legendrian knot Λ induces an augmentation of the DGA of Λ. Furthermore, the linearized contact homology associated to this augmentation is precisely the usual homology of the filling L; this result is colloquially called the Seidel isomorphism.

Second, the coefficient ring of the Chekanov–Eliashberg DGA can be lifted from $\mathbb{Z}/2$ to $\mathbb{Z}$ by assigning coherent orientations to the Floer-type moduli spaces underlying Legendrian contact homology. This was done first for Legendrian knots in $\mathbb{R}^3$ in [15] and then for more general Legendrian submanifolds in arbitrary dimension in [9]. One can subsequently tensor with an arbitrary field $\Bbbk$ and study Legendrian contact homology over $\Bbbk$, as a number of papers have done, or study Legendrian contact homology over $\mathbb{Z}$ directly.

With $\mathbb{Z}$ coefficients, linearized contact homology is lifted from a graded $\mathbb{Z}/2$-vector space to a graded $\mathbb{Z}$-module. This raises the possibility that it might contain torsion. For Legendrians in high dimension (where the contact manifold has dimension ≥ 5), it is well-known that linearized contact homology can indeed contain torsion. The earliest examples of torsion in high dimension were provided in [9], where torsion is used to distinguish between Legendrian submanifolds that share the same classical invariants and Legendrian contact homology over $\mathbb{Z}/2$. As another example, torsion for knot conormal tori has been shown to encode the determinant of the underlying smooth knot (see [21]). More recently, in [17] Golovko showed that any finitely generated abelian group can appear as the linearized contact homology of some high-dimensional Legendrian.

In this paper, we show that torsion in linearized contact homology also appears for Legendrian knots in $\mathbb{R}^3$. To our knowledge, it was previously an open question whether such torsion could exist. Throughout this paper, we use $\mathrm{LCH}_*^\epsilon(\Lambda)$ to denote the linearized contact homology of Λ over $\mathbb{Z}$ associated to a $\mathbb{Z}$-valued augmentation ϵ.

Proposition 1.1. *There are Legendrian knots Λ in $\mathbb{R}^3$ such that for any $n \geq 2$, there is a $\mathbb{Z}$-valued augmentation $\epsilon_n \colon \mathcal{A}_\Lambda \to \mathbb{Z}$ of the Chekanov–Eliashberg DGA of Λ for which the linearized contact homology $\mathrm{LCH}_*^{\epsilon_n}(\Lambda)$ contains a $\mathbb{Z}/n$ summand.*

To prove Proposition 1.1, we give a particular Legendrian knot, of topological type $m(8_{21})$, for which torsion exists for specific augmentations (Section 2.2.1). Afterwards, we generalize this example to larger families of knots which have the same property.

By the universal coefficient theorem, Proposition 1.1 implies that the linearized contact cohomology for these augmentations (which is more directly connected to sheaf theory; cf. [25, 22]) also has torsion.

As a consequence of the proof of Proposition 1.1, we also establish an analogue of Golovko's result for Legendrians in $\mathbb{R}^3$.

Proposition 1.2. *For any finitely generated abelian group G and any $k \in \mathbb{Z}$ with $k \neq 0, 1$, there is a Legendrian knot Λ in $\mathbb{R}^3$ and a $\mathbb{Z}$-valued augmentation ϵ of the Chekanov–Eliashberg DGA of Λ such that $\mathrm{LCH}_k^\epsilon(\Lambda) \cong G$.*

The exclusion $k \neq 0, 1$ is necessary because Sabloff duality imposes constraints on $\mathrm{LCH}_k^\epsilon(\Lambda)$ for $k = 0, 1$. Proposition 1.2 is proven in Section 2.3.1.

We also use our torsion examples to explore the question of when augmentations are geometric, in the sense that they are induced by an exact Lagrangian filling.

Proposition 1.3. *For the family of Legendrian knots Λ_k defined in Section 2.3, there are augmentations to $\mathbb{Z}$ that are not geometric, even though their mod 2 reductions to $\mathbb{Z}/2$ are geometric.*

We conclude the paper by discussing which knots in the Legendrian knot atlas [6] do and do not have torsion, and some speculation based on these observations.

Remark 1.4. In this paper, we consider contact homology linearized with respect to a single augmentation. One can also linearize with respect to two different augmentations. In terms of the augmentation category of [1, 4, 22], these bilinearized contact homologies are the morphism spaces between different objects, while the linearized contact homologies considered in this paper are the endomorphism spaces of a single object. While writing this paper, we learned that Frédéric Bourgeois and Salammbo Connolly have completely solved the geography problem for bilinearized contact homologies in dimension 3, for pairs of non DGA homotopic augmentations. They also obtain some partial results in the linearized case, so that there is some overlap between Bourgeois–Connolly's results and ours.

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2. Results

2.1. Legendrian contact homology, augmentations, and linearized homology.

We start with a brief review of Legendrian contact homology, mostly to fix notation and conventions. For more details, see e.g. [14].

Throughout this paper, we consider only Legendrian knots in $\big(\mathbb{R}^3, \xi_{std} = \ker(dz - y\,dx)\big)$. All knots in this paper will have rotation number 0. We will usually represent a Legendrian knot Λ by its *front projection* $\Pi_{xz}(\Lambda)$ in $\mathbb{R}^2_{xz}$. Assuming that Λ is generic, the front projection $\Pi_{xz}(\Lambda)$ has only two types of singularities: transverse double points (*crossings*) and semicubical cusps (*left cusps* if the cusp singularity is on the left, and *right cusps* if the cusp singularity is on the right).

The Chekanov–Eliashberg DGA is most naturally defined in terms of the *Lagrangian projection* Π_{xy} of a Legendrian knot Λ. Crossings of the knot diagram $\Pi_{xy}(\Lambda)$ are the *Reeb chords* of Λ (flows of the Reeb vector field $\partial/\partial z$ with endpoints on Λ) and generate the DGA. In order to instead produce a DGA associated to a front, we use a procedure called *resolution* from [23] that takes the front projection of Λ and turns it into the Lagrangian projection of a knot that is Legendrian isotopic to Λ. Resolution smooths out left cusps, replaces double points of the front by crossings, and replaces right cusps by loops with a single crossing. There is a one-to-one correspondence between crossings and right cusps of the front of Λ and Reeb chords of the resolved diagram, and we will sometimes accordingly abuse language and use "Reeb chords" to mean the crossings and right cusps of $\Pi_{xz}(\Lambda)$.

Label the Reeb chords of the front $\Pi_{xz}(\Lambda)$ by $a_1, \dots, a_n$, and place a base point at one of the right cusps; we introduce one more indeterminate t corresponding to this base point. We then construct the free unital $\mathbb{Z}$-algebra

$$\mathcal{A}_\Lambda = \mathbb{Z}\langle a_1, \dots, a_n, t^{\pm 1}\rangle,$$

generated by $a_1, \dots, a_n, t^{\pm 1}$, with no relations except for $t \cdot t^{-1} = t^{-1} \cdot t = 1$. The algebra $\mathcal{A}_\Lambda$ is graded by setting $|t| = |t^{-1}| = 0$ and assigning a particular grading $|a_i| \in \mathbb{Z}$ to each a_i: if a_i is a right cusp then $|a_i| = 1$, while if a_i is a crossing then $|a_i|$ is the difference between the number of up cusps and down cusps traversed by a path along $\Pi_{xz}(\Lambda)$ from the undercrossing of a_i (the strand of the crossing with higher slope) to the overcrossing of a_i (the strand with lower slope). The fact that this is well-defined uses our assumption that the rotation number vanishes. For the definition of the differential, see e.g. [23] or [14]. Since signs are important for our computations, we note that our sign conventions follow those references: the sign of a term in ∂_Λ is (-1) raised to the number of corners of the disk that occur at a crossing of even degree and occupy the downward-facing quadrant at that crossing. The differential of t vanishes, and the only place where t appears in ∂a_i for any i is in the differential of the cusp where we have placed the base point, which contains a single term t: $\partial a_i = t + \cdots$. (The sign of t here chooses an orientation for Λ, but our results will be unchanged if we reverse the orientation instead.)

An R-valued augmentation of $\mathcal{A}_\Lambda$ is a DGA map

$$\epsilon \colon \mathcal{A}_\Lambda \to R$$

where R is a unital commutative ring lying in degree 0, with trivial differential. (In this paper, R will usually be $\mathbb{Z}$ or a field $\Bbbk$.) Specifically, this means that $\epsilon(1) = 1$, $\epsilon(a) = 0$ for any $a \in \mathcal{A}_\Lambda$ of nonzero degree, and $\epsilon \circ \partial = 0$.

A fundamental result of Leverson states that for any augmentation ϵ valued in a field $\Bbbk$, $\epsilon(t) = -1$; see [19]. (This uses the fact that Λ is a knot, not a more general link.) It follows that the same holds for $\mathbb{Z}$-valued augmentations, as well.

There is a standard procedure that starts from $(\mathcal{A}_\Lambda, \partial)$ equipped with an augmentation and produces a finitely generated complex of free R-modules, whose homology is defined to be the linearized contact homology $\mathrm{LCH}^\epsilon_*(\Lambda; R)$ of Λ with respect to the augmentation ϵ; see e.g. [14]. Here we present a slight variant that may be useful for computations. Define

$$\mathcal{A}^R = (R[s])\langle a_1, \dots, a_n\rangle = ((\mathcal{A}_\Lambda \otimes \mathbb{Z}[s]) \otimes R)/(t = \epsilon(t))$$

where s should be viewed as a parameter. There is an R-algebra map $\phi^\epsilon\colon \mathcal{A}^R \to \mathcal{A}^R$ defined by $\phi^\epsilon(1) = 1$, $\phi^\epsilon(s) = s$, and

$$\phi^\epsilon(a_i) = s a_i + \epsilon(a_i)$$

for all i. It follows from the fact that $\epsilon \circ \partial = 0$ that $(\phi^\epsilon(\partial a_i))|_{s=0} = 0$ for all i. Define

$$\partial^\epsilon a_i = \left.\frac{d}{ds}\right|_{s=0} \phi^\epsilon(\partial a_i).$$

By construction, $\partial^\epsilon a_i$ is linear in $a_1, \dots, a_n$ for each i. Thus we have

$$\partial^\epsilon \colon V \to V,$$

where V is the free graded R-module generated by $a_1, \dots, a_n$, and $(\partial^\epsilon)^2 = 0$ since $\partial^2 = 0$. We now define the linearized contact homology of Λ with respect to ϵ to be

$$\mathrm{LCH}^\epsilon_*(\Lambda; R) = H_*(V, \partial^\epsilon).$$

When $R = \mathbb{Z}$, we will suppress it from this notation.

A fundamental property of linearized contact homology is *Sabloff duality*, a relationship between linearized contact homology and its dual akin to Poincaré duality for ordinary homology. Specifically, given a Legendrian Λ in $\mathbb{R}^3$ and a $\Bbbk$-valued augmentation ϵ for Λ for some field $\Bbbk$, Sabloff duality states that there are non-canonical isomorphisms

$$\mathrm{LCH}^\epsilon_i(\Lambda; \Bbbk) \cong \mathrm{LCH}^\epsilon_{-i}(\Lambda; \Bbbk) \qquad i \neq 1 \tag{1}$$

$$\mathrm{LCH}^\epsilon_1(\Lambda; \Bbbk) \cong \mathrm{LCH}^\epsilon_{-1}(\Lambda; \Bbbk) \oplus \Bbbk. \tag{2}$$

(A canonical version of the first line identifies LCH^ϵ_i with the dual of $\mathrm{LCH}^\epsilon_{-i}$, and the second becomes a short exact sequence.) This was first proved in the case $\Bbbk = \mathbb{Z}/2$ by Sabloff, in [24], and then extended to higher dimensions and more general rings by Ekholm–Etnyre–Sabloff, in [10]. The form we have used here requires an understanding of what they call the *manifold classes*, which follows either from Sabloff's original argument or from Remark 5.6 of [10].

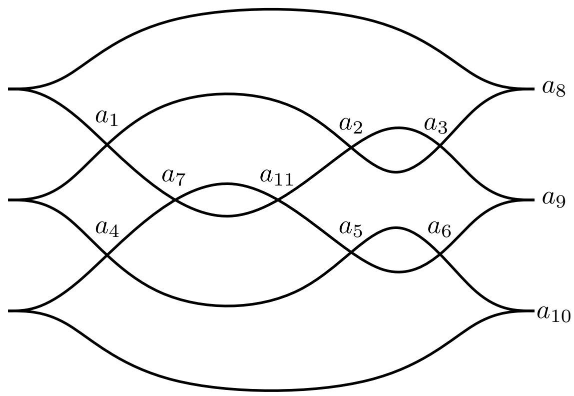

FIGURE 1. **The knot Λ_0.** This knot appears as $m(8_{21})$ in the Legendrian knot atlas, and has this topological knot type.

2.2. **Torsion for $m(8_{21})$.** In this section we consider the Legendrian knot Λ_0 shown in Figure 1. This knot is of topological type $m(8_{21})$ and was studied by Melvin and Shrestha, who showed in [20] that it has two different linearized contact homologies corresponding to different augmentations over $\mathbb{Z}/2$. (The depiction of Λ_0 in [20, Figure 2] differs from ours by reflection in the z axis, but the two are Legendrian isotopic: the contactomorphism $(x, y, z) \mapsto (-x, -y, z)$, which in the front projection is reflection in the z axis, is isotopic to the identity map through contactomorphisms.)

2.2.1. *Linearized contact homology for $m(8_{21})$.* The crossings and right cusps of Λ_0 are labeled in Figure 1, and are graded as follows:

$$
\begin{aligned}
|a_7| = |a_8| = |a_9| = |a_{10}| &= 1 \\
|a_1| = |a_2| = |a_3| = |a_4| = |a_5| = |a_6| &= 0 \\
|a_{11}| &= -1.
\end{aligned}
$$

We place a base point at the cusp a_8.

The differential ∂ on the Chekanov–Eliashberg DGA $\mathcal{A}_{\Lambda_0}$ is:

$$
\begin{aligned}
\partial a_2 &= a_4 a_{11} & \partial a_8 &= t + a_1 + a_3 + a_1 a_2 a_3 + a_7 a_{11} a_3 \\
\partial a_5 &= -a_{11} a_1 & \partial a_9 &= 1 - (1 + a_3 a_2) a_1 a_6 - a_3(a_4 + a_6 + a_4 a_5 a_6) \\
\partial a_7 &= -a_1 a_4 & \partial a_{10} &= 1 - a_4 - a_6 - a_6 a_5 a_4 - a_6 a_{11} a_7
\end{aligned}
$$
$$
\partial a_1 = \partial a_3 = \partial a_4 = \partial a_6 = \partial a_{11} = \partial t = \partial t^{-1} = 0.
$$

Let R be a (unital) integral domain; of interest to us are the cases where R is $\mathbb{Z}$ or a field. A (graded) augmentation $\epsilon\colon \mathcal{A}_{\Lambda_0} \to R$ is determined by the 7-tuple $(\epsilon(a_1), \ldots, \epsilon(a_6), \epsilon(t)) \in R^7$. By solving the system of equations that arise from setting $\epsilon \circ \partial = 0$ ($\epsilon(a_1)\epsilon(a_4) = -\epsilon(\partial a_7) = 0$, etc.), we find that ϵ is an augmentation if and only if one of the following two lines holds:

$$\epsilon(t) = -1 \quad \epsilon(a_4) = 0 \qquad \epsilon(a_6) = 1 \qquad \epsilon(a_1) + \epsilon(a_3) + \epsilon(a_1)\epsilon(a_2)\epsilon(a_3) = 1$$
$$\epsilon(t) = -1 \quad \epsilon(a_1) = 0 \qquad \epsilon(a_3) = 1 \qquad \epsilon(a_4) + \epsilon(a_6) + \epsilon(a_4)\epsilon(a_5)\epsilon(a_6) = 1.$$

In particular, for any $n \in \mathbb{Z}$, there is an augmentation $\epsilon_n \colon \mathcal{A}_{\Lambda_0} \to \mathbb{Z}$ defined by

$$\epsilon_n(a_1, a_2, a_3, a_4, a_5, a_6, t) = (n, -1, 1, 0, 0, 1, -1).$$

If we write V for the graded $\mathbb{Z}$-module generated by $a_1, \ldots, a_{11}$, then ∂ and ϵ_n induce a linear differential $\partial^{\epsilon_n} \colon V \to V$ as described in Section 2.1, and the homology of $(V, \partial^{\epsilon_n})$ is $\mathrm{LCH}_*^{\epsilon_n}(\Lambda_0)$. Specifically, we have

$$\partial^{\epsilon_n} a_5 = -na_{11} \qquad \partial^{\epsilon_n} a_7 = -na_4 \qquad \partial^{\epsilon_n} a_8 = na_2 - (n-1)a_3$$
$$\partial^{\epsilon_n} a_9 = -a_4 - a_6 - na_2 + (n-1)a_3 \qquad \partial^{\epsilon_n} a_{10} = -a_4 - a_6$$
$$\partial^{\epsilon_n} a_1 = \partial^{\epsilon_n} a_2 = \partial^{\epsilon_n} a_3 = \partial^{\epsilon_n} a_4 = \partial^{\epsilon_n} a_6 = \partial^{\epsilon_n} a_{11} = 0$$

and thus

$$\mathrm{LCH}_*^{\epsilon_n}(\Lambda_0) \cong \begin{cases} \mathbb{Z} & * = 1 \\ \mathbb{Z}^2 \oplus \mathbb{Z}/n & * = 0 \\ \mathbb{Z}/n & * = -1 \\ 0 & \text{otherwise.} \end{cases}$$

This proves Proposition 1.1.

2.2.2. *Geometric motivation.* Here we discuss a geometric reason why the existence of torsion for $m(8_{21})$ should not have been unexpected.

For a general Legendrian knot Λ and any field $\Bbbk$, the set of augmentations from the DGA $\mathcal{A}_\Lambda$ to $\Bbbk$ forms a variety over $\Bbbk$, the *augmentation variety* $\mathrm{Aug}(\Lambda, \Bbbk)$. As mentioned above, by [19], any augmentation ϵ must satisfy $\epsilon(t) = -1$. Thus if $a_1, \ldots, a_\ell$ are the degree-0 Reeb chords of Λ, an augmentation is uniquely determined by $(\epsilon(a_1), \ldots, \epsilon(a_\ell))$, and we can view

$$\mathrm{Aug}(\Lambda, \Bbbk) \subset \Bbbk^\ell.$$

This is a variety because it is the vanishing set of a collection of polynomials in $a_1, \ldots, a_\ell$ given by ∂a_i where a_i ranges over all degree-1 generators of $\mathcal{A}_\Lambda$.

Given an augmentation ϵ viewed as a point in $\mathrm{Aug}(\Lambda, \Bbbk)$, we can consider the Zariski tangent space to the augmentation variety at the point ϵ, $T_\epsilon \mathrm{Aug}(\Lambda, \Bbbk)$. It is an exercise in algebra to check that

$$T_\epsilon \mathrm{Aug}(\Lambda, \Bbbk) \cong \ker((\partial^\epsilon)^* \colon A_0^* \to A_1^*)$$

where A_0 and A_1 are the $\Bbbk$-vector spaces generated by the degree 0 and degree 1 Reeb chords of Λ, respectively. In the special case when Λ has no degree -1 Reeb chords, $A_{-1} = 0$ and so $T_\epsilon \mathrm{Aug}(\Lambda, \Bbbk)$ is isomorphic to $\mathrm{LCH}^0_\epsilon(\Lambda; \Bbbk)$, the degree-0 linearized contact cohomology of Λ with respect to ϵ with coefficients in $\Bbbk$. Even when Λ does have degree -1 Reeb chords, there is a surjection $T_\epsilon \mathrm{Aug}(\Lambda, \Bbbk) \twoheadrightarrow \mathrm{LCH}^0_\epsilon(\Lambda; \Bbbk)$. We summarize this by the heuristic "the larger the Zariski tangent space at ϵ, the larger the linearized contact (co)homology": this slogan can be made more precise but will suffice for our purposes. Note that the homology $\mathrm{LCH}_0^{\epsilon}(\Lambda;\Bbbk)$ and cohomology $\mathrm{LCH}^0_{\epsilon}(\Lambda;\Bbbk)$ are isomorphic by the universal coefficient theorem.

In the case that $\Lambda = \Lambda_0$, if we use the obvious coordinates $a_1,\dots,a_6$ on $\Bbbk^6$, then

$$\mathrm{Aug}(\Lambda_0,\Bbbk) = V_1 \cup V_2 \subset \Bbbk^6$$

where

$$\begin{aligned} V_1 &= \{a_4 = 0,\ a_6 = 1,\ a_1 + a_3 + a_1a_2a_3 = 1\} \\ V_2 &= \{a_1 = 0,\ a_3 = 1,\ a_4 + a_6 + a_4a_5a_6 = 1\}. \end{aligned}$$

Note that V_1 and V_2 are smooth 3-dimensional subvarieties of $\mathrm{Aug}(\Lambda_0,\Bbbk)$ that intersect in the 2-dimensional subvariety $V_1 \cap V_2 = \{a_1 = 0,\ a_3 = 1,\ a_4 = 0,\ a_6 = 1\}$. The Zariski tangent space to a point ϵ in $\mathrm{Aug}(\Lambda_0,\Bbbk)$ is larger for $\epsilon \in V_1 \cap V_2$ (where it has dimension 4) than for $\epsilon \in V_1 \setminus V_2$ or $\epsilon \in V_2 \setminus V_1$ (where it has dimension 3). Consistent with the heuristic above, this is borne out in linearized contact homology: one can calculate that

$$\mathrm{LCH}_0^{\epsilon}(\Lambda_0;\Bbbk) \cong \begin{cases} \Bbbk^4 & \epsilon \in V_1 \cap V_2 \\ \Bbbk^2 & \epsilon \in (V_1 \setminus V_2) \cup (V_2 \setminus V_1). \end{cases}$$

Consider the augmentation $\epsilon_n \colon \mathcal{A}_{\Lambda_0} \to \mathbb{Z}$ described in Section 2.2.1. For any prime p, we can compose ϵ_n with the projection $\mathbb{Z} \to \mathbb{Z}/p$ to get the mod p reduction $\epsilon_{n;p} \colon \mathcal{A}_{\Lambda_0} \to \mathbb{Z}/p$. Then $\epsilon_{n;p}$ is in the component V_1 of $\mathrm{Aug}(\Lambda_0,\mathbb{Z}/p)$ for all p, and is also in V_2 if and only if $p \mid n$. Thus,

$$\mathrm{LCH}_0^{\epsilon_{n;p}}(\Lambda_0;\mathbb{Z}/p) \cong \begin{cases} (\mathbb{Z}/p)^4 & p \mid n \\ (\mathbb{Z}/p)^2 & p \nmid n. \end{cases} \tag{3}$$

However, the complex whose homology yields $\mathrm{LCH}_*^{\epsilon_{n;p}}(\Lambda_0;\mathbb{Z}/p)$ is precisely the tensor product of the complex whose homology yields $\mathrm{LCH}_*^{\epsilon_n}(\Lambda_0)$ with $\mathbb{Z}/p$. By the universal coefficient theorem, Formula (3) forces either $\mathrm{LCH}_0^{\epsilon_n}(\Lambda_0)$ or $\mathrm{LCH}_{-1}^{\epsilon_n}(\Lambda_0)$ to have p-torsion if $p \mid n$; and indeed both of these groups have p-torsion.

2.3. **Torsion for the family Λ_k.**

The knot Λ_0 is the smallest of a family of Legendrian knots with augmentations whose linearized contact homology contains torsion. Here we describe the rest of the family.

For $k \geq 1$, let Λ_k denote the Legendrian knot shown in Figure 2. The case $k = 0$ is exactly the $m(8_{21})$ knot considered in Section 2.2; however, restricting to $k \geq 1$ actually makes the computation of linearized homology slightly simpler. (The knot Λ_1 is the second $m(9_{45})$ knot in the Legendrian knot atlas [6].)

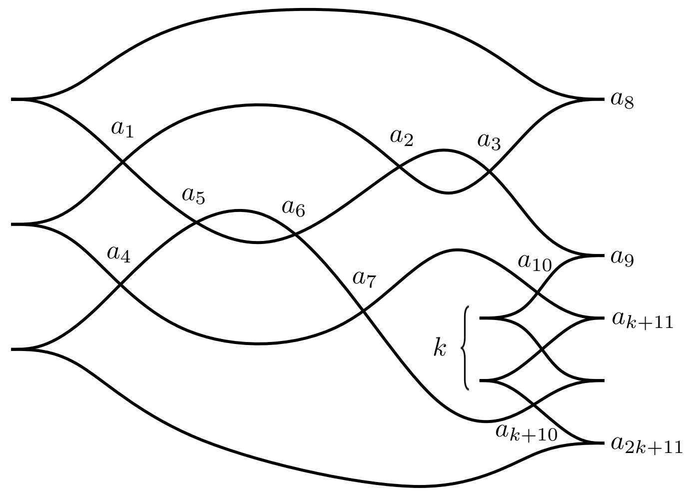

FIGURE 2. **The knot Λ_k for $k \geq 1$.** The depicted knot is Λ_2. The Reeb chords $a_{10}, \ldots, a_{k+10}$ correspond to the crossings on the vertical segment from a_{10} to a_{k+10}, and the chords $a_{k+11}, \ldots, a_{2k+11}$ corresponding to the right cusps on the vertical segment from a_{k+11} to a_{2k+11}.

2.3.1. *Linearized homology for Λ_k.* The Chekanov–Eliashberg DGA of Λ_k is $(\mathcal{A}_{\Lambda_k}, \partial)$, where $\mathcal{A}_{\Lambda_k} = \mathbb{Z}\langle a_1, \ldots, a_{2k+11}, t^{\pm 1}\rangle$ and the grading on $\mathcal{A}_{\Lambda_k}$ is given by

$$\begin{aligned}
|a_5| &= k+1 \\
|a_4| &= k \\
|a_8| = |a_9| = |a_{k+11}| = \cdots = |a_{2k+11}| &= 1 \\
|a_1| = |a_2| = |a_3| = |a_{10}| = \cdots = |a_{k+10}| = |t^{\pm 1}| &= 0 \\
|a_7| &= -k \\
|a_6| &= -k-1.
\end{aligned}$$

The differential is nonzero on the following generators:

$$\begin{aligned}
\partial a_5 &= -a_1 a_4 \\
\partial a_7 &= (-1)^{k+1} a_6 a_1 \\
\partial a_8 &= t + a_1 + a_3 + a_1 a_2 a_3 + a_5 a_6 a_3 \\
\partial a_9 &= 1 - (a_1 + a_3 + a_3 a_2 a_1 + a_3 a_4 a_7) a_{10} \\
\partial a_{k+11+i} &= 1 - a_{10+i} a_{11+i} \qquad \text{for } 0 \leq i \leq k-1 \\
\partial a_{2k+11} &= 1 - a_{k+10}(1 + a_6 a_5 + a_7 a_4),
\end{aligned}$$

where we place the base point at the cusp a_8.

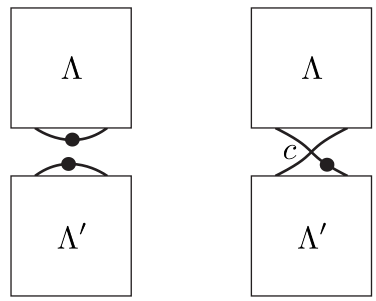

FIGURE 3. **The connected sum.** Left: two front diagrams. Right: their connected sum. (Compare [13, Figure 2].) The dots indicate our choice of basepoints; the new crossing is called c.

From this, it is easy to check that a graded algebra map $\epsilon\colon \mathcal{A}_{\Lambda_k} \to R$ is an augmentation if and only if $\epsilon(t) = -1$, $\epsilon(a_{10}) = \cdots = \epsilon(a_{k+10}) = 1$, and

$$\epsilon(a_1) + \epsilon(a_3) + \epsilon(a_1)\epsilon(a_2)\epsilon(a_3) = 1.$$

In particular, we can define an augmentation ϵ_n for any $n \in \mathbb{Z}$ by $\epsilon_n(a_1) = n$, $\epsilon_n(a_2) = -1$, $\epsilon_n(a_3) = 1$. Then the linearized differentials $\partial^{\epsilon_n} a_5 = -na_4$ and $\partial^{\epsilon_n} a_7 = (-1)^{k+1} na_6$ produce n-torsion in linearized contact homology. To be precise:

$$\mathrm{LCH}_*^{\epsilon_n}(\Lambda_1) \cong \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/n) & * = 1 \\ \mathbb{Z}^2 & * = 0 \\ \mathbb{Z}/n & * = -2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{LCH}_*^{\epsilon_n}(\Lambda_k) \cong \begin{cases} \mathbb{Z} & * = 1 \\ \mathbb{Z}^2 & * = 0 \\ \mathbb{Z}/n & * = k \text{ or } * = -k-1 \\ 0 & \text{otherwise} \end{cases} \qquad k > 1.$$

Proposition 1.2, that any finitely-generated abelian group can be obtained as a linearized contact homology group in any grading $\neq 0, 1$, now follows easily by using connected sums as in Melvin–Shrestha's paper [20]. Given Legendrian knots Λ and Λ', we can form their connected sum $\Lambda \# \Lambda'$ as in Figure 3. Most Reeb chords for $\Lambda \# \Lambda'$ are either Reeb chords for Λ or Λ'; there is one additional chord c with $|c| = 0$. With basepoints as indicated, the differential on $\Lambda \# \Lambda'$ is induced from the differentials on Λ and Λ' by replacing t by c in the differential on $\mathcal{A}_\Lambda$, replacing t' by $-tc$ in the differential on $\mathcal{A}_{\Lambda'}$, and observing that $\partial(c) = 0$. Hence, augmentations for $\Lambda \# \Lambda'$ (sending t to -1) correspond to pairs of augmentations for Λ and

Λ' (sending t, t', c to -1), and with respect to this correspondence,

$$\mathrm{LCH}_i^{\epsilon\#\epsilon'}(\Lambda\#\Lambda') \cong \mathrm{LCH}_i^{\epsilon}(\Lambda) \oplus \mathrm{LCH}_i^{\epsilon'}(\Lambda')$$

for all $i \neq 0, 1$. (One can also analyze the behavior in gradings 0 and 1 using Sabloff duality, as in [20].) Thus, to obtain a group $G = \mathbb{Z}^m \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$ in grading $i > 1$, we simply take the connect sum of Λ_i with itself $m + k$ times, with augmentation

$$\overbrace{\epsilon_0\#\cdots\#\epsilon_0}^{m}\#\epsilon_{n_1}\#\cdots\#\epsilon_{n_k}.$$

2.3.2. *Geometric motivation.* As in Section 2.2.2 for $m(8_{21})$, one can interpret torsion for Λ_k in terms of the augmentation variety, but the interpretation is slightly different for $k \geq 1$ than for $m(8_{21})$. Over a field $\Bbbk$, the augmentation variety of Λ_k is

$$\mathrm{Aug}(\Lambda_k, \Bbbk) = \{a_1 + a_3 + a_1a_2a_3 = 1\} \subset \Bbbk^3.$$

Unlike for Λ_0, this variety is smooth. However, the linearized homology at an augmentation $\epsilon \in \mathrm{Aug}(\Lambda_k, \Bbbk)$ still depends on the point ϵ:

$$\mathrm{LCH}_*^{\epsilon}(\Lambda_k; \Bbbk) \cong \begin{cases} \Bbbk_{k+1} \oplus \Bbbk_k \oplus \Bbbk_1 \oplus \Bbbk_0^2 \oplus \Bbbk_{-k} \oplus \Bbbk_{-k-1} & \epsilon(a_1) = 0 \\ \Bbbk_1 \oplus \Bbbk_0^2 & \epsilon(a_1) \neq 0, \end{cases}$$

where subscripts denote grading. Thus given a $\mathbb{Z}$-valued augmentation ϵ with $\epsilon(a_1) = n$ and a prime p, the linearized homology over $\mathbb{Z}/p$ at the mod p reduction of ϵ is larger if $p \,|\, n$ than if $p \nmid n$. As before, it follows from the universal coefficient theorem that $\mathrm{LCH}_*^{\epsilon}(\Lambda_k)$ contains p-torsion for any prime p dividing n, in line with our calculation in Section 2.3.1.

2.3.3. *A more general family with torsion.* The family Λ_k is itself part of a larger family of Legendrian knots with augmentations whose linearized homology contains torsion. Here we sketch this family.

Suppose that we have two Legendrian knots Λ' and Λ'' with fronts as shown on the left of Figure 4, and x and y are the indicated crossings. Further suppose that the DGAs $\mathcal{A}_{\Lambda'}$ and $\mathcal{A}_{\Lambda''}$ have $\mathbb{Z}$-valued augmentations ϵ' and ϵ'' satisfying the following conditions:

(1) $\epsilon'(x) = n$ for some $n \notin \{-1, 0, 1\}$ (so in particular, $|x| = 0$);
(2) $\epsilon''(y) = 0$; and
(3) y does not appear as a term in the linearized differential $\partial(c)$ for any Reeb chord c of Λ''.

Construct the Legendrian knot Λ shown on the right of Figure 4; this is the connected sum of Λ' and Λ'', but with an additional clasp. For example, if Λ' and Λ'' are the trefoil and twist knot shown in Figure 5, then Λ is the knot Λ_k considered earlier.

Let $(\mathcal{A}_\Lambda, \partial)$ denote the DGA of Λ. We can construct an algebra map $\epsilon\colon \mathcal{A}_\Lambda \to \mathbb{Z}$ by combining ϵ' and ϵ'', as follows. Each Reeb chord a of Λ corresponds to a Reeb chord of either Λ' or Λ'', with the exception of the

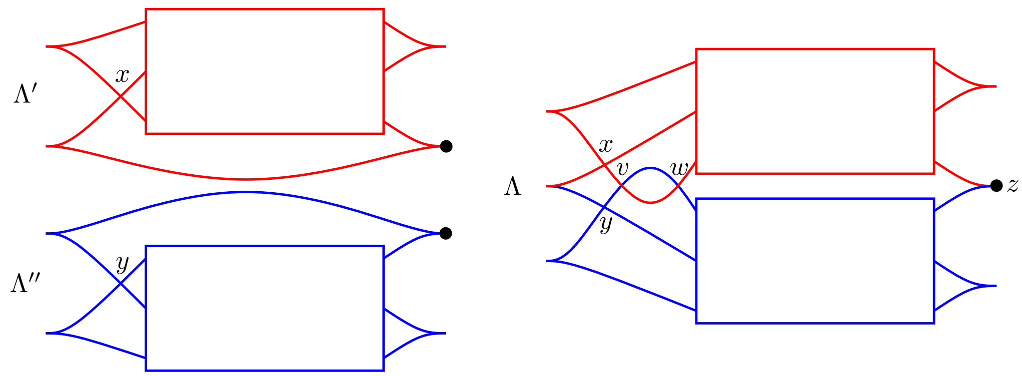

FIGURE 4. **The knots Λ', Λ'', and Λ.** Inside the rectangles, the knots Λ', Λ'' are arbitrary, subject to the condition that they have augmentations ϵ', ϵ'' with the specified properties. Base points are placed as shown.

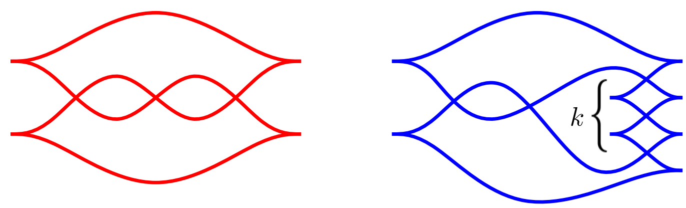

FIGURE 5. **The knots inducing the family $\Lambda = \Lambda_k$.** The knot Λ' is the trefoil (left) while Λ'' is the twist knot (right).

chords labeled v, w, z in Figure 4. Define $\epsilon(t) = -1$, $\epsilon(v) = \epsilon(w) = \epsilon(z) = 0$, and for all other Reeb chords a of Λ, define $\epsilon(a)$ to be either $\epsilon'(a)$ or $\epsilon''(a)$, depending on whether a is a Reeb chord of Λ' or Λ''. It can readily be checked that ϵ is an augmentation of $(\mathcal{A}_\Lambda, \partial)$. Indeed, for chords of Λ coming from Λ' or Λ'', the differential agrees with the differential on $\mathcal{A}_{\Lambda'}$ or $\mathcal{A}_{\Lambda''}$, except for z. For z, since $\epsilon(v) = \epsilon(w) = \epsilon(y) = 0$, only terms in ∂z with no v, w, or y factors contribute; and these terms correspond to the product of the terms (other than t) in the differentials of the bottom right-cusp of Λ' and the top right-cusp of Λ'' (with no y factor).

By the given conditions on ϵ' and ϵ'', we find that $\partial^\epsilon(v) = -ny$, and further that this is the only place where y appears in $\partial^\epsilon(a)$ for any Reeb chord a of Λ. It follows that $\mathrm{LCH}^\epsilon_*(\Lambda)$ contains a $\mathbb{Z}/n$ summand generated by y.

2.4. **Geometric augmentations over $\mathbb{Z}$ and $\mathbb{Z}/2$.** When a Legendrian knot has an exact, embedded Lagrangian filling, the filling induces an augmentation (more precisely, a family of augmentations) of the Legendrian knot. Here we investigate whether particular augmentations of the Legendrian knot Λ_k introduced in Section 2.3 come from a filling, and prove Proposition 1.3.

Recall that an *exact Lagrangian filling* of a Legendrian knot Λ is a Lagrangian surface L in the symplectization $\big(\mathbb{R} \times \mathbb{R}^3, d(e^t(dz - y\,dx))\big)$ such that, for some sufficiently large T, $L \cap \big((T,\infty) \times \mathbb{R}^3\big) = (T,\infty) \times \Lambda$, equipped with a function $f \colon L \to \mathbb{R}$ with $e^t(dz - y\,dx)|_L = df$. In this paper, all of our fillings will be embedded and orientable; henceforth we use the word "filling" to denote an exact, embedded, orientable Lagrangian filling.

Contact homology behaves functorially with respect to cobordisms. In the setting of fillings, this result can be stated as follows.

Proposition 2.1 ([11, 18])**.** *A filling L of a Legendrian knot Λ induces a DGA map $\epsilon_L \colon \mathcal{A}_\Lambda \to \mathbb{Z}[H_1(L)]$ (where the right side has trivial differential).*

Remark 2.2. For a general (embedded, orientable) filling L, the DGA map ϵ_L from Proposition 2.1 may only preserve the $\mathbb{Z}/2$ grading on $\mathcal{A}_\Lambda$ induced from the $\mathbb{Z}$ grading; it is only guaranteed to preserve the full $\mathbb{Z}$ grading if L has Maslov number 0. See e.g. [14] for further discussion.

If we tensor by a field $\Bbbk$, we obtain a DGA map $\mathcal{A}_\Lambda \to \Bbbk[H_1(L)]$. We can obtain a $\Bbbk$-valued augmentation of Λ by composing with a homomorphism $\Bbbk[H_1(L)] \to \Bbbk$; this is equivalent to choosing a rank 1 local system on L, i.e., a group homomorphism $H_1(L) \to \Bbbk^\times$. The same construction works over $\mathbb{Z}$, where by a local system we simply mean a homomorphism $H_1(L) \to \{\pm 1\}$. Note that when $\Bbbk = \mathbb{Z}/2$, there is a unique rank 1 local system on L and the filling L produces a unique augmentation of Λ.

Definition 2.3. A $\mathbb{Z}$-valued augmentation $\epsilon \colon \mathcal{A}_\Lambda \to \mathbb{Z}$ of Λ is *geometric* if there is a filling L of Λ and a group homomorphism $H_1(L) \to \{\pm 1\}$ such that ϵ is equal to the composition

$$\mathcal{A}_\Lambda \xrightarrow{\epsilon_L} \mathbb{Z}[H_1(L)] \longrightarrow \mathbb{Z}.$$

Similarly, if $\Bbbk$ is a field, then a $\Bbbk$-valued augmentation is *geometric* if there is a group homomorphism $H_1(L) \to \Bbbk^\times$ such that ϵ is equal to the composition

$$\mathcal{A}_\Lambda \xrightarrow{\epsilon_L} \mathbb{Z}[H_1(L)] \longrightarrow \Bbbk.$$

It is well-known that not all $\Bbbk$-valued augmentations of Legendrian knots are geometric; see e.g. [3, 11, 12, 16]. However, less is known for $\mathbb{Z}$-valued augmentations. The rest of this section is devoted to proving:

Proposition 2.4. *For any $k \geq 1$ and $n \in \mathbb{Z}$, let Λ_k and ϵ_n be the Legendrian knot and $\mathbb{Z}$-valued augmentation defined in Section 2.3. If n is odd and $n \neq \pm 1$, then $\epsilon_n : \mathcal{A}_{\Lambda_k} \to \mathbb{Z}$ is not geometric, but the mod 2 reduction of ϵ_n is geometric.*

To prove Proposition 2.4, we first show that ϵ_n is not geometric, and then that its mod 2 reduction is geometric. For the former, we use the following result, called the *Seidel isomorphism*, versions of which are due to many authors (e.g., [8, 7, 18, 16]). The variant that we will use is due to Gao and Rutherford [16].

Proposition 2.5 ([16, Proposition 3.4]). *Let Λ be a Legendrian knot and let $\epsilon\colon \mathcal{A}_\Lambda \to \Bbbk$ be a geometric augmentation induced by a filling L of Λ equipped with a rank 1 local system. Then there is a $\mathbb{Z}/2$-graded isomorphism*

$$LCH_*^{\epsilon}(\Lambda;\Bbbk) \cong H_{*+1}(L,\Lambda;\Bbbk).$$

Remark 2.6. Proposition 2.5 is actually a special case of [16, Proposition 3.4]. More precisely, that result states that if ϵ_L is a $\Bbbk$-valued augmentation induced by a filling L, then

$$H^*\operatorname{Hom}_+(\epsilon_L,\epsilon_L) \cong H^*(L;\Bbbk).$$

See [22] for the definition of Hom_+, but to deduce Proposition 2.5, it suffices to note that $H^*\operatorname{Hom}_+(\epsilon_L,\epsilon_L) \cong LCH^{\epsilon}_{1-*}(\Lambda;\Bbbk)$ by Sabloff duality (see [22, §5]), while $H^*(L;\Bbbk) \cong H_{2-*}(L,\Lambda;\Bbbk)$ by Poincaré duality. Also note that the fact that the isomorphism is only $\mathbb{Z}/2$-graded in general is because the Maslov number of L might not be 0; cf. Remark 2.2.

Now consider the augmentation $\epsilon_n\colon \mathcal{A}_{\Lambda_k} \to \mathbb{Z}$ for $n \neq 0, \pm 1$. If this were geometric, then its mod p reduction $\epsilon_{n;p}\colon \mathcal{A}_{\Lambda_k} \to \mathbb{Z}/p$ would also be geometric for any prime p. But it follows from the calculation of $\mathrm{LCH}_*^{\epsilon_n}(\Lambda_k)$ from Section 2.3 that if p is a prime dividing n, then $\mathrm{LCH}_*^{\epsilon_{n;p}}(\Lambda_k;\mathbb{Z}/p)$ has dimension 7 as a vector space over $\mathbb{Z}/p$. On the other hand, by [3], any filling L of Λ_k must satisfy $2g(L)-1 = \mathrm{tb}(\Lambda_k) = 1$, whence L is a punctured torus and $H_*(L,\Lambda;\Bbbk)$ has total dimension 3. This contradicts Proposition 2.5, and we conclude that ϵ_n is not geometric.

To complete the proof of Proposition 2.4, we will show that the mod 2 reduction $\epsilon_{n;2}\colon \mathcal{A}_{\Lambda_k} \to \mathbb{Z}/2$ is geometric when n is odd. To do this, we construct an explicit filling L of Λ_k inducing the augmentation $\epsilon_{n;2}$.

Via the resolution procedure, we can assume that the Lagrangian projection $\Pi_{xy}(\Lambda_k)$ is as shown on the left of Figure 6. In the language of [11], our filling L is decomposable and is constructed as follows. We concatenate two saddle cobordisms that replace the crossings a_1 and a_3 in $\Pi_{xy}(\Lambda_k)$ consecutively by their oriented resolution (in the standard knot theory sense). The result is the Legendrian knot Λ' shown on the right of Figure 6, and the saddle cobordisms yield a cobordism L_1 from Λ' at the bottom to Λ_k at the top. Now by inspection, Λ' is a standard Legendrian unknot and thus has a filling L_2. The concatenation $L_1 \cup L_2$ is the desired filling L of Λ_k.

In constructing the filling L, there is one important point to check: in order for the saddle cobordism L_1 to be realized as an exact Lagrangian, we need to verify that the Reeb chords a_1 and a_3 are contractible in the sense of [11]. That is, we must find a Legendrian isotopy starting at Λ_k such that the Lagrangian projections remain planar isotopic to the knot diagram $\Pi_{xy}(\Lambda_k)$ throughout the isotopy and such that the height of the

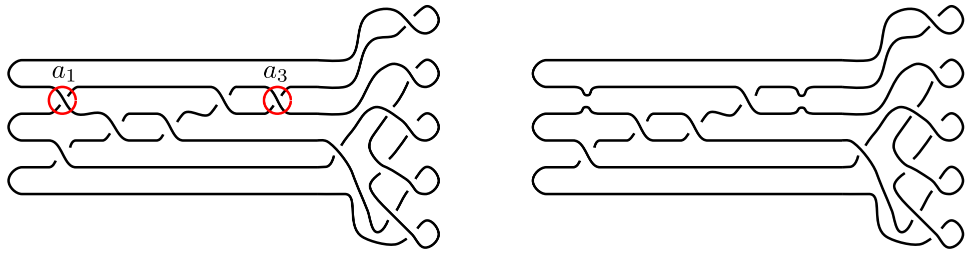

FIGURE 6. **The cobordism L_1 built from two saddle moves.** Resolving the circled crossings on the left via two saddle moves produces the cobordism L_1 from a standard Legendrian unknot Λ' (right) to Λ_k. The case $k = 2$ is shown.

crossings at a_1 and a_3 both approach 0 at the end of the isotopy. To do this, modify $\Pi_{xy}(\Lambda_k)$ by a planar isotopy to obtain the diagram at the top of Figure 7. This diagram is the Lagrangian projection of a Legendrian knot whose front projection is given by the bottom diagram in Figure 7: recall that we translate from xz to xy projection by setting $y = dz/dx$. In the front projection, we have not drawn the entire front, but rather just the key portions drawn in color; the remainder of the front, corresponding to the black portion of the Lagrangian projection, is completed by the usual resolution procedure. It is now apparent that the front can be perturbed by translating the red portion in the negative z direction, in such a way that the two segments labeled a_1 and a_3 both shrink to a point. This does not change any portion of the Lagrangian projection (up to planar isotopy) apart from a_1 and a_3 and verifies that a_1 and a_3 are indeed contractible.

Now let $\epsilon\colon \mathcal{A}_{\Lambda_k} \to \mathbb{Z}/2$ be the augmentation to $\mathbb{Z}/2$ induced by the filling L. Since the cobordism L_1 from Λ' to Λ_k consists of a concatenation of two saddle cobordisms at contractible Reeb chords, there is a combinatorial formula for the resulting DGA map $\Phi_{L_1}\colon (\mathcal{A}_{\Lambda_k}, \partial) \to (\mathcal{A}_{\Lambda'}, \partial)$ given by [11, Proposition 6.18]. For our purposes, it suffices to note that $\Phi_{L_1}(a_1) = \Phi_{L_1}(a_3) = 1$ since a_1 and a_3 are the Reeb chords being removed. Since ϵ is the composition of Φ_{L_1} with a map $\mathcal{A}_{\Lambda'} \to \mathbb{Z}/2$ corresponding to the filling L_2, we conclude that $\epsilon(a_1) = \epsilon(a_3) = 1$.

The conditions that $\epsilon(a_1) = \epsilon(a_3) = 1$ uniquely determine the augmentation ϵ: from the computation in Section 2.3.1, any augmentation ϵ of Λ_k satisfies $\epsilon(a_1) + \epsilon(a_3) + \epsilon(a_1)\epsilon(a_2)\epsilon(a_3) = 1$, whence $\epsilon(a_2) = 1$ in our case, and furthermore ϵ is uniquely determined by $\epsilon(a_i)$ for $i = 1, 2, 3$. But $\epsilon_{n;2}$ also satisfies $\epsilon_{n;2}(a_1) = \epsilon_{n;2}(a_3) = 1$ when n is odd. Hence, $\epsilon = \epsilon_{n;2}$ and thus that $\epsilon_{n;2}$ is geometric. This completes the proof of Proposition 2.4.

Remark 2.7. A similar but slightly more complicated calculation shows that for the knot Λ_0 from Section 2.2, the augmentation ϵ_n is geometric over $\mathbb{Z}/2$ but not over $\mathbb{Z}$ when n is odd and $n \neq \pm 1$.

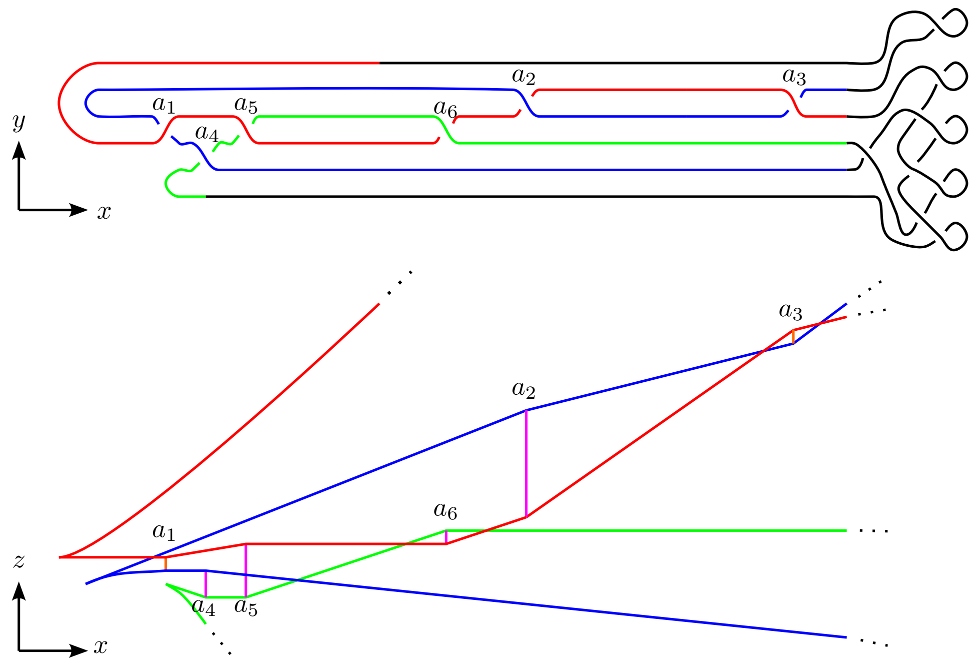

FIGURE 7. The Reeb chords a_1 and a_3 in the Lagrangian projection of Λ_k are simultaneously contractible. Top, the Lagrangian projection for Λ_k; bottom, a corresponding front diagram for Λ_k. The omitted portions of the front follow the standard resolution procedure.

2.5. **Further comments.** We conclude with some observations about when torsion appears for knots in the Legendrian knot atlas [6].

The knots $m(8_{21})$ and $m(9_{45})B$ are Λ_0 and Λ_1 discussed above. (Here we write $m(9_{45})B$ for the second Legendrian representative of $m(9_{45})$ listed in the atlas, and similarly for other knots below.) Direct computation shows that three other knots have torsion: $m(9_{45})A$, 11^n_{95}, and 11^n_{118}. Like the examples discussed above, one can find an augmentation ϵ_n for each of these knots so that the Legendrian contact homology has a $\mathbb{Z}/n$ summand.

There are also many knots which can be shown not to have torsion:

Definition 2.8. A Legendrian knot is *positive* if it has rotation number 0 and is Legendrian isotopic to a knot for which all of the Reeb chords have nonnegative grading.

Proposition 2.9. *Let* Λ *be a positive Legendrian knot with Thurston–Bennequin number* $\operatorname{tb}(\Lambda)$. *Then the linearized contact homology for any*

$\mathbb{Z}$-valued augmentation ϵ of Λ is given by:

$$\mathrm{LCH}^{\epsilon}_{*}(\Lambda) \cong \begin{cases} \mathbb{Z} & * = 1 \\ \mathbb{Z}^{\mathrm{tb}(\Lambda)+1} & * = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, no linearized contact homology for a positive Legendrian knot can contain torsion.

Proof. This follows from Sabloff duality. Specifically, by the universal coefficient theorem, it suffices to prove that for any field $\Bbbk$,

$$\mathrm{LCH}^{\epsilon}_{*}(\Lambda; \Bbbk) \cong \begin{cases} \Bbbk & * = 1 \\ \Bbbk^{\mathrm{tb}(\Lambda)+1} & * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

By hypothesis, for $i < 0$, $\mathrm{LCH}^{\epsilon}_{i}(\Lambda; \Bbbk) = 0$. So, Formula (1) implies that $\mathrm{LCH}^{\epsilon}_{i}(\Lambda; \Bbbk) = 0$ for $i > 1$ and Formula (2) implies that $\mathrm{LCH}^{\epsilon}_{1}(\Lambda; \Bbbk) = \Bbbk$. Finally, the Euler characteristic of $\mathrm{LCH}^{\epsilon}_{*}(\Lambda)$ is equal to the writhe minus the number of right cusps which, in turn, is $\mathrm{tb}(\Lambda)$, so $\mathrm{LCH}^{\epsilon}_{0}(\Lambda; \Bbbk)$ must have dimension $\mathrm{tb}(\Lambda) + 1$. ☐

Positivity of Λ can sometimes be deduced from a presentation of Λ as a braid closure. For instance, it is immediate from the formula for gradings of Reeb chords (see Section 2.1) that rainbow closures of positive braids are positive; in the atlas, this means $m(3_1)$, $m(5_1)$, $m(7_1)$, 8_{19}, 10_{124}, 15^n_{41185} (all torus knots), 10_{139}, and 12^n_{242} are positive. More generally, given an *admissible* positive braid in the sense of [2, Definition 2.5], the result of multiplying by Δ^{-2} and taking the closure gives a positive Legendrian; see [2, Section 5.1]. (By [2, Proposition 2.7], any positive braid containing a half-twist is admissible.) If the topological knot type has a unique representative with maximal Thurston–Bennequin number, and appears in the atlas, then the representative in the atlas must be positive. Alternatively, it is sometimes possible to check directly that a Legendrian knot is positive. Via such considerations, the following knots are also positive:

$$m(5_2)B,\ m(7_2)B,\ m(7_2)D,\ 7_3B,\ 7_4,\ m(7_5)B,\ 9_{49},\ 10_{128}B,$$
$$m(10_{142})B,\ m(10_{145}),\ m(10_{161}),\ 12^n_{591}.$$

This leaves 35 Legendrian knots in the atlas which admit (graded) augmentations but may or may not have torsion. We gathered some computational evidence that none of these knots admits augmentations with torsion:

- The dimension of $\mathrm{LCH}^{*}_{\epsilon}(\Lambda; \Bbbk)$ is the same for all $\mathbb{Z}/2$-valued and $\mathbb{Z}/3$-valued augmentations ϵ (where $\Bbbk = \mathbb{Z}/2$ or $\mathbb{Z}/3$). It follows from the universal coefficient theorem that if ϵ is a $\mathbb{Z}$-valued augmentation for which $\mathrm{LCH}^{*}_{\epsilon}(\Lambda)$ has 2-torsion or 3-torsion, then in fact $\mathrm{LCH}^{*}_{\epsilon}(\Lambda)$ has a copy of $\mathbb{Z}/6$.
- The Bockstein map induced by the short exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ vanishes for all augmentations for these knots. So,

any 2^n-torsion would have to arise as copies of $\mathbb{Z}/4$ rather than $\mathbb{Z}/2$ (and so the 6 in the previous point is, in fact, 12).

(For the 24 of these knots with crossing number ≤ 9, the dimension of $\mathrm{LCH}^*_\epsilon(\Lambda; \mathbb{Z}/5)$ was also constant, agreeing with the dimension over $\mathbb{Z}/2$ and $\mathbb{Z}/3$, for all $\mathbb{Z}/5$-valued augmentations ϵ, so any 2-, 3-, or 5-torsion would have to arise as $\mathbb{Z}/60$-torsion. For more complicated knots, the number of $\mathbb{Z}/5$-valued augmentations became too large for our naïve programs to check.)

Our limited computations support positive answers to the following questions:

Question 2.10. *Suppose a Legendrian knot Λ has augmentations ϵ_1 and ϵ_2 to some field $\Bbbk$ so that $\mathrm{LCH}^{\epsilon_1}_*(\Lambda; \Bbbk) \ncong \mathrm{LCH}^{\epsilon_2}_*(\Lambda; \Bbbk)$. Does it follow that Λ admits a $\mathbb{Z}$-valued augmentation ϵ so that $\mathrm{LCH}^\epsilon_*(\Lambda)$ has torsion?*

Question 2.11. *Suppose a Legendrian knot Λ has a $\mathbb{Z}$-valued augmentation ϵ so that $\mathrm{LCH}^\epsilon_*(\Lambda)$ has torsion. Does it follow that for every prime p there is a $\mathbb{Z}$-valued augmentation ϵ_p so that $\mathrm{LCH}^{\epsilon_p}_*(\Lambda)$ contains p-torsion? That for every integer n there is a $\mathbb{Z}$-valued augmentation ϵ_n so that $\mathrm{LCH}^{\epsilon_n}_*(\Lambda)$ has a $\mathbb{Z}/n$-summand?*

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