CONORMAL BUNDLES, CONTACT HOMOLOGY, AND KNOT INVARIANTS

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ABSTRACT. Blah.

1. INTRODUCTION

String theory has provided a beautiful correspondence between enumerative geometry and knot invariants; for details, see the survey [8] or other papers in the present volume. BLAH BLAH BLAH From the point of view of symplectic geometry, there is a natural way to associate a knot invariant, derived by counting holomorphic curves, to BLAH.

Ooguri and Vafa [13] begin by considering a knot K in S^3; this knot gives rise to a natural Lagrangian submanifold $\mathcal{L}K$ in the symplectic manifold T^*S^3 which is the conormal bundle to K:

$$\mathcal{L}K = \{(x,\xi) \,|\, x \in K \text{ and } \langle \xi, v\rangle = 0 \text{ for all } v \in T_xK\} \subset T^*S^3.$$

There is a way of deforming T^*S^3 into the manifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$ by collapsing the zero section and then performing a "small resolution" of the resulting conifold singularity. This deformation gives rise to a Lagrangian submanifold $\tilde{\mathcal{L}}K \subset \mathcal{O}(-1)\oplus\mathcal{O}(-1)$ which corresponds to $\mathcal{L}K \subset T^*S^3$. There is then a correspondence between knot invariants of K derived from Chern–Simons theory, and relative Gromov–Witten enumerative invariants of $\tilde{\mathcal{L}}K$:

$$\begin{array}{ccc} T^*S^3 & \xrightarrow{\text{conifold transition}} & \mathcal{O}(-1)\oplus\mathcal{O}(-1) \\ \mathcal{L}K & \longleftrightarrow & \tilde{\mathcal{L}}K \\ \text{knot invariants} & \longleftrightarrow & \text{holomorphic invariants.} \end{array}$$

From the viewpoint of symplectic geometry, however, it is natural to look at enumerative holomorphic-curve invariants on the *left* side of this correspondence, before the conifold transition. This gives rise to the knot invariants which are the subject of the present paper.

We now describe how the symplectic knot invariants arise. Consider a general n-dimensional Riemannian manifold M with a compact submanifold $K \subset M$. The Ooguri–Vafa setup uses a knot K in $M = S^3$; for technical reasons, it is simpler in the symplectic setup to use $M = \mathbb{R}^3$ instead. There

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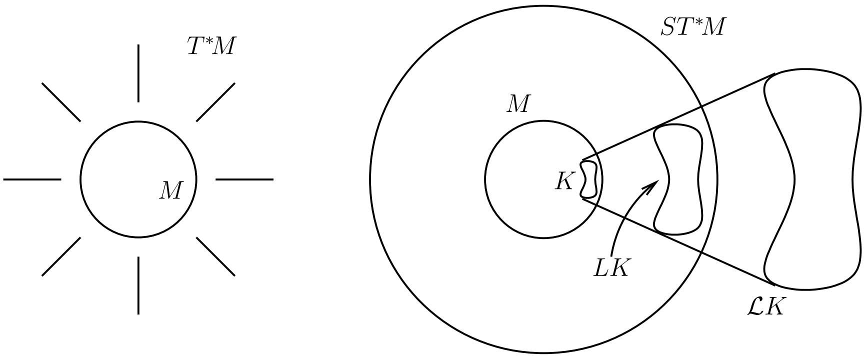


FIGURE 1. Schematic depiction of the cotangent bundle of M (left) and the conormals $\mathcal{L}K$ and LK (right). Note that $\mathcal{L}K$ intersects M in the knot K and ST^*M in the unit conormal LK.

is a canonical 1-form λ on the total space of T^*M which pairs the result of the projections $T(T^*M) \to TM$ and $T(T^*M) \to T^*M$, and the natural symplectic structure on T^*M is given by $\omega = -d\lambda$. (In local coordinates x_i on M and ξ^i in the corresponding cotangent directions, $\lambda = \xi^i dx_i$ and $\omega = dx_i \wedge d\xi^i$.) It is then easy to check that the conormal bundle $\mathcal{L}K$ is Lagrangian in T^*M; that is, $\mathcal{L}K$ has dimension n and $\omega|_{\mathcal{L}K} = 0$.

Geometrically, $\mathcal{L}K$ is somewhat awkward to work with, because it is noncompact. Instead, given a metric on M, we can work with the *unit* conormal bundle LK, which we define to be the intersection of $\mathcal{L}K$ with the cosphere bundle $ST^*M = \{(x,\xi) \in T^*M \,|\, \|\xi\| = 1\}$. The cosphere bundle ST^*M has a natural *contact form*; a contact form, the odd-dimensional analogue of a symplectic form, is a 1-form α such that $\alpha \wedge d\alpha^{n-1}$ is nowhere zero. In this case, the contact form is given by $\alpha = \lambda|_{ST^*M}$. With respect to this contact structure, the unit conormal LK is *Legendrian*, meaning that LK has dimension $n-1$ and $\alpha|_{LK} = 0$.

Ambient isotopy of K in M leads to an isotopy of LK in ST^*M through Legendrian submanifolds; hence a Legendrian-isotopy invariant of LK yields a smooth-isotopy invariant of K. Such an invariant in the Legendrian setting is provided by *Legendrian contact homology* [3], which in this case is the simplest nontrivial approximation to the Symplectic Field Theory [4] of Eliashberg, Givental, and Hofer. We will define Legendrian contact homology more carefully in the following section, but, roughly speaking, it counts holomorphic curves with boundary on the Legendrian submanifold.

The knot invariant given by Legendrian contact homology, which we call *knot contact homology* and denote by $HC_*(K)$, takes the form of a homology graded in degrees greater than or equal to 0. We will describe a combinatorial form for knot contact homology in Section BLAH. For more details, see [10, 11, 12].

Because of the similarity of the symplectic and string-theoretic pictures, one might expect that knot contact homology incorporates Chern–Simons knot invariants such as the Jones polynomial. Whether this is true remains to be seen, but knot contact homology is at least connected to "classical" knot invariants such as the Alexander polynomial and the A-polynomial [2]; through the latter, knot contact homology has ties to $SL(2,\mathbb{C})$ representations of the knot group.

2. Legendrian Contact Homology

Let V be a contact manifold with contact form α, and let $L \subset V$ be a Legendrian submanifold. The *Reeb vector field* R_α on V is uniquely defined by the conditions $\iota(R_\alpha)d\alpha = 0$, $\alpha(R_\alpha) = 1$. A *Reeb orbit* in V is a closed orbit of the flow under the Reeb vector field, and a *Reeb chord* of L is a path along the flow of the Reeb vector field which begins and ends on L. The *symplectization* of V is the symplectic manifold $W = V \times \mathbb{R}$ with symplectic form $\omega = d(e^t\alpha)$, where t is the coordinate on $\mathbb{R}$ and α is induced from V. We can give W an almost complex structure BLAH.

The Symplectic Field Theory of V is an algebraic structure derived from counting holomorphic curves in the symplectization of V which limit to Reeb orbits at $\pm\infty$. As a more manageable approximation, we can study Eliashberg and Hofer's contact homology [3], which counts genus 0 holomorphic curves with one end at $+\infty$ and an arbitrary number of ends at $-\infty$. More precisely, the curves in question are holomorphic maps from a multiply punctured sphere to $V \times \mathbb{R}$ such that a neighborhood of one puncture limits to a cylinder over a Reeb orbit as we approach $+\infty$ in $\mathbb{R}$, and neighborhoods of the other punctures limit to cylinders over Reeb orbits as we approach $-\infty$ in $\mathbb{R}$. These curves arise naturally when considering the compactification of the moduli space of holomorphic genus 0 curves in $V \times \mathbb{R}$. Out of these curves, one forms a graded complex whose homology is called contact homology and is an invariant of the contact structure.

Given a Legendrian submanifold $L \subset V$, one can construct a relative version of contact homology called *Legendrian contact homology*. For simplicity, we will assume that the ambient manifold V has no closed Reeb orbits; this holds, for instance, in the case $V = ST^*\mathbb{R}^3$. One then counts holomorphic disks which limit to Reeb chords of L at $\pm\infty$, and whose boundary lies on L.

More precisely, suppose that L has finitely many Reeb chords which we label $a_1, \dots, a_n$. The complex whose homology yields Legendrian contact homology is the tensor algebra $\mathcal{A}$ freely generated by $a_1, \dots, a_n$ over the group ring $\mathbb{Z}H_1(L)$; that is, $\mathcal{A}$ is the module over $\mathbb{Z}H_1(L)$ generated by (noncommutative) words in $a_1, \dots, a_n$, including the empty word. This algebra can be given a grading by assigning degrees to the generators a_i which are the Conley–Zehnder indices of the Reeb chords; see [5]. In order

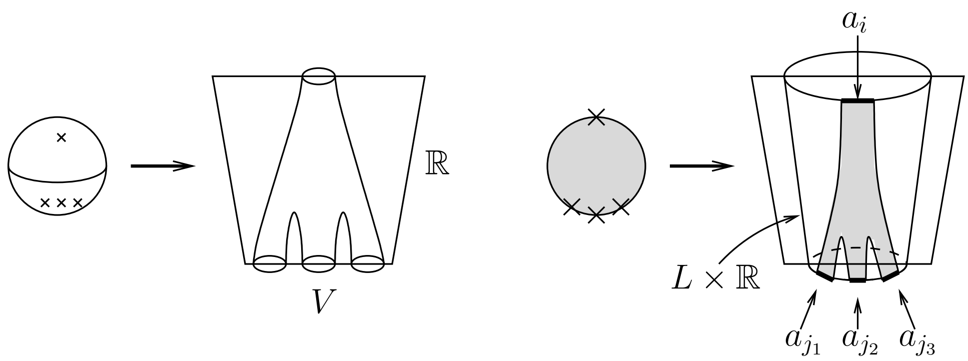


FIGURE 2. On the left, a holomorphic curve in $V \times \mathbb{R}$ contributing to the contact homology of V. The punctures are mapped to the cylindrical ends, which approach cylinders over Reeb orbits of V. On the right, a holomorphic disk in $V \times \mathbb{R}$ contributing to the Legendrian contact homology of L. Here the boundary of the disk, away from the punctures, is mapped to $L \times \mathbb{R}$, and the punctures are mapped to strips approaching Reeb chords of L. This disk contributes the term $a_{j_1} a_{j_2} a_{j_3}$ to ∂a_i.

to define a differential on $\mathcal{A}$, we need to define certain moduli spaces of holomorphic curves.

For each Reeb chord a_i, choose a "capping path" lying in L which joins the endpoints of the chord. Let D_k be a disk with boundary, minus boundary punctures $x, y_1, \ldots, y_k$ appearing in order around the boundary, and give D_k the complex structure induced from the unit disk in $\mathbb{C}$. For $A \in H_1(L)$, we define the moduli space $\mathcal{M}^A(a_i; a_{j_1}, \ldots, a_{j_k})$ to be the set of maps $f : D_k \to V \times \mathbb{R}$ such that:

- f is holomorphic and has finite energy with respect to $d\alpha$;
- f maps the boundary of D_k (without the punctures) to $L \times \mathbb{R}$;
- f maps a neighborhood of the puncture x to a strip approaching the strip $a_i \times \mathbb{R}$ at the $+\infty$ end of $\mathbb{R}$;
- f maps a neighborhood of the puncture y_i to a strip approaching the strip $a_{j_i} \times \mathbb{R}$ at the $-\infty$ end of $\mathbb{R}$;
- the image of the boundary of D_k, made into a closed curve by appending the appropriate capping paths, is in the homology class A.

Note that there is an $\mathbb{R}$ action on $\mathcal{M}^A(a_i; a_{j_1}, \ldots, a_{j_k})$ given by translation in the $\mathbb{R}$ direction. The moduli spaces that interest us are the ones that are rigid modulo this $\mathbb{R}$ action.

Define the differential of a_i as follows:

$$\partial a_i = \sum_{\dim \mathcal{M}^A(a_i; a_{j_1}, \ldots, a_{j_k}) = 1} \#(\mathcal{M}^A(a_i; a_{j_1}, \ldots, a_{j_k})/\mathbb{R})\, A\, a_{j_1} \ldots a_{j_k},$$

where $\#(\mathcal{M}^A(a_i; a_{j_1}, \dots, a_{j_k})/\mathbb{R})$ is the signed number of points in the 0-dimensional space $\mathcal{M}/\mathbb{R}$; extend ∂ to $\mathcal{A}$ via the Leibniz rule. Then $(\mathcal{A}, \partial)$ becomes a differential graded algebra, usually abbreviated in the subject as a DGA.

Theorem 2.1. *$\partial^2 = 0$, ∂ lowers degree by 1, and the graded homology $H_*(\mathcal{A}, \partial)$ is an invariant of the Legendrian isotopy class of L.*

It is currently a bit of a misnomer to label this result as a theorem, since the analytical details to prove it in full generality are still being worked out. There is a standard technique, dating back to Chekanov's pioneering work on Legendrian knots in standard contact $\mathbb{R}^3$ [1], for sidestepping these analytical issues by finding a purely combinatorial form for the differential graded algebra $(\mathcal{A}, \partial)$ and proving that this combinatorial algebra is invariant under Legendrian isotopy. This is the strategy used in [10, 11, 12] to define knot contact homology.

In the case of interest here, V is the contact manifold $ST^*\mathbb{R}^3 = \mathbb{R}^3 \times S^2$, which can also be viewed as the 1-jet space $J^1(S^2)$, and L is the Legendrian torus LK given by the unit conormal to the knot K in $ST^*\mathbb{R}^3$. Given a framing and an orientation of the K, we obtain longitude and meridian classes $\lambda, \mu \in H_1(LK)$ and hence an identification of $\mathbb{Z}H_1(LK)$ with $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$.

The Reeb vector field lies within the $\mathbb{R}^3$ fibers in $\mathbb{R}^3 \times S^2$; in the fiber lying over $\xi \in S^2$, it simply points in the direction of ξ. It follows that Reeb chords of LK correspond to "binormal chords": oriented line segments beginning and ending on K which are normal to K at both endpoints. If we parametrize K by S^1, then the distance function in $\mathbb{R}^3$ between points on K gives a map $d : S^1 \times S^1 \to \mathbb{R}$ whose nondiagonal critical points are binormal chords. One can then set up the theory so that the degree of a binormal chord in the contact homology DGA is the Morse index of the corresponding critical point of d. Hence the generators of the DGA all have degree 0, 1, or 2; it follows that the DGA is a nonnegatively graded algebra over the ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$.

We can view Legendrian contact homology in $ST^*\mathbb{R}^3$ as counting curves in $T^*\mathbb{R}^3$ as follows. The symplectization $ST^*\mathbb{R}^3 \times \mathbb{R}$ is symplectically diffeomorphic to $T^*\mathbb{R}^3 \setminus \mathbb{R}^3$, the cotangent bundle minus the zero section, via the map which sends $((x, \xi), t)$ to $(x, e^t \xi)$. The $+\infty$ end of the symplectization maps to the unbounded "end" of $T^*\mathbb{R}^3$, while the $-\infty$ end maps to the zero section. Under this identification, the holomorphic curves used to calculate Legendrian contact homology are maps of boundary-punctured disks to $T^*\mathbb{R}^3$ satisfying the following properties:

- the map sends the boundary of the disk to $\mathcal{L}K$;
- the map sends a neighborhood of one puncture to something which approaches, on the unbounded end of $T^*\mathbb{R}^3$, a cylindrical strip of the form $BC \times \{e^t \xi \,|\, t \gg 0\} \subset T^*\mathbb{R}^3$, where BC is a binormal chord and ξ is a covector dual to the direction of BC;

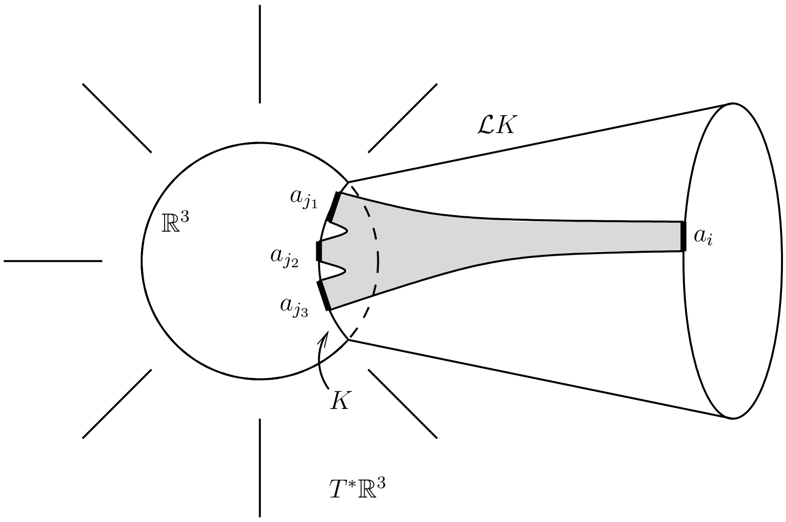


FIGURE 3. Holomorphic disk used in the computation of the Legendrian contact homology of LK. The ambient space is $T^*\mathbb{R}^3$. In this picture, the holomorphic disk has boundary on $\mathcal{L}K$, except for the strips which limit to the binormal chords $a_{j_1}, a_{j_2}, a_{j_3}$ on the zero section and the binormal chord a_i at infinity.

- the map sends neighborhoods of the other punctures to neighborhoods of binormal chords of K in the zero section of $T^*\mathbb{R}^3$.

See Figure 3.

We remark that the curves which produce Legendrian contact homology in this context are somewhat different from the curves which would be counted in relative Gromov–Witten theory; it is unknown what relation our invariant has to the Chern–Simons type invariants studied by Ooguri–Vafa et al. Also note that the almost complex structure on $T^*\mathbb{R}^3 \setminus \mathbb{R}^3$ inherited from the symplectization, which we use to define our holomorphic curves, cannot be extended to all of $T^*\mathbb{R}^3$.

This discussion of Legendrian contact homology should be viewed as motivation for the knot contact homology invariant which will be defined combinatorially in the following section. There is work in progress to verify that the combinatorial DGA actually yields Legendrian contact homology. Alternatively, one can directly show à la Chekanov that the combinatorial theory gives a topological knot invariant, without using its origin in symplectic geometry; this is the approach of [10, 11, 12], and will be our approach in the remaining sections.

3. Knot Contact Homology: Definition

Let K be an oriented knot in $\mathbb{R}^3$; this has a canonical framing given by the zero framing. As discussed in the previous section, the Legendrian contact homology of $LK \subset ST^*\mathbb{R}^3$, which is a knot invariant, is the homology of a complex defined in nonnegative degree over the ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$. We will now give a combinatorial definition of the complex, which is called the *framed knot DGA*.

Fix a knot diagram for K with n crossings. The framed knot DGA is the algebra $\mathcal{A}$ over $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ freely generated by the following generators:

- $\{a_{ij}\}_{1 \le i,j \le n,\ i \ne j}$ of degree 0;
- $\{b_{\alpha i}\}_{1 \le \alpha, i \le n}$ and $\{c_{i\alpha}\}_{1 \le \alpha, i \le n}$ of degree 1;
- $\{d_{\alpha\beta}\}_{1 \le \alpha, \beta \le n}$ and $\{e_\alpha\}_{1 \le \alpha \le n}$ of degree 2.

Here Greek subscripts represent crossings in the knot diagram, numbered arbitrarily from 1 to n, and Roman subscripts represent components of the knot diagram, from undercrossing to undercrossing, also numbered arbitrarily from 1 to n.

To define the differential on $\mathcal{A}$, we first need some auxiliary definitions. Each crossing i involves three components of the knot diagram, the overstrand o_i and the understrands l_i and r_i, distinguished as being on the left and right sides as the overstrand is traversed in the direction of the knot's orientation. Define ϵ_1 to be ± 1 depending on the sign of crossing 1, where sign is defined in terms of the knot's orientation in the usual way. (If l_1 follows r_1 when traversing the knot, then $\epsilon_1 = 1$; if r_1 follows l_1, then $\epsilon_1 = -1$.)

Let $\Psi^L, \Psi^R, \Psi^L_2, \Psi^R_1$ be the $n \times n$ matrices defined as follows:

$$(\Psi^L)_{\alpha i} = \begin{cases} \lambda^{-\epsilon_1} & \alpha = 1, i = r_1 \\ 1 & \alpha \ne 1, i = r_\alpha \\ \mu & i = l_\alpha \\ -a_{l_\alpha o_\alpha} & i = o_\alpha \\ 0 & \text{otherwise} \end{cases} \qquad (\Psi^R)_{i\alpha} = \begin{cases} \lambda^{\epsilon_1}\mu & \alpha = 1, i = r_1 \\ \mu & \alpha \ne 1, i = r_\alpha \\ 1 & i = l_\alpha \\ -a_{o_\alpha l_\alpha} & i = o_\alpha \\ 0 & \text{otherwise} \end{cases}$$

$$(\Psi^L_2)_{\alpha i} = \begin{cases} \mu & i = l_\alpha \\ -a_{l_\alpha o_\alpha} & i = o_\alpha \\ 0 & \text{otherwise} \end{cases} \qquad (\Psi^R_1)_{i\alpha} = \begin{cases} \lambda^{\epsilon_1}\mu & \alpha = 1, i = r_1 \\ \mu & \alpha \ne 1, i = r_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

Assemble generators of $\mathcal{A}$ into $n \times n$ matrices A, B, C, D as follows: $A_{ij} = \left\{\begin{smallmatrix} 1+\mu & i=j \\ a_{ij} & i \ne j \end{smallmatrix}\right.$; $B_{\alpha i} = b_{\alpha i}$; $C_{i\alpha} = c_{i\alpha}$; and $D_{\alpha\beta} = d_{\alpha\beta}$. Write ∂A for the matrix whose entries are the differentials of the corresponding entries of A, and similarly for ∂B, ∂C, and ∂D. Then the differential on generators is given by

$$\begin{aligned}
\partial A &= 0 \\
\partial B &= \Psi^L \cdot A \\
\partial C &= A \cdot \Psi^R \\
\partial D &= B \cdot \Psi^R - \Psi^L \cdot C \\
\partial e_i &= (B \cdot \Psi_1^R - \Psi_2^L \cdot C)_{ii};
\end{aligned}$$

extend via the Leibniz rule to obtain a differential on all of $\mathcal{A}$.

Proposition 3.1 ([12])**.** *$\partial^2 = 0$ and the homology $HC_*(K) = H_*(\mathcal{A}, \partial)$ depends only on the isotopy class of K, not the particular knot diagram. This homology is called the (framed) knot contact homology of K.*

There is an equivalence relation on semifree differential graded algebras known as *stable tame isomorphism* [1], under which $(\mathcal{A}, \partial)$ is independent of knot diagram; equivalent DGAs have isomorphic homology. Up to equivalence, we can refer to $(\mathcal{A}, \partial)$ as the framed knot DGA of K. For a proof of invariance, using a slightly different formulation of the framed knot DGA, see [12].

The above definition of the framed knot DGA is combinatorial and relatively simple but extremely opaque. Unfortunately, a purely topological interpretation for the DGA or its homology is presently lacking. Most current applications of knot contact homology use only its lowest-degree component, the degree 0 homology $HC_0(K)$, which does have a topological formulation.

Let $K \subset \mathbb{R}^3$ be an oriented knot equipped with the zero framing, and let l, m denote the homotopy classes of the longitude and meridian of K in $\pi_1(\mathbb{R}^3 \setminus K)$. Let $\mathcal{A}_K$ denote the tensor algebra over $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ freely generated by the set $\pi_1(\mathbb{R}^3 \setminus K)$; a monomial in $\mathcal{A}_K$ looks like $[\gamma_1][\gamma_2]\dots[\gamma_k]$, where $[\gamma_i]$ denotes the image of $\gamma_i \in \pi_1(\mathbb{R}^3 \setminus K)$ in $\mathcal{A}_K$. Define the *cord algebra* of K to be the quotient of $\mathcal{A}_K$ by the relations

- $[e] = 1 + \mu$;
- $[\gamma m] = [m\gamma] = \mu[\gamma]$ and $[\gamma l] = [l\gamma] = \lambda[\gamma]$ for $\gamma \in \pi_1(\mathbb{R}^3 \setminus K)$;
- $[\gamma_1\gamma_2] + [\gamma_1 m \gamma_2] = [\gamma_1][\gamma_2]$ for $\gamma_1, \gamma_2 \in \pi_1(\mathbb{R}^3 \setminus K)$.

If K is the unknot, then $\pi_1(\mathbb{R}^3 \setminus K) \cong \mathbb{Z}$ is generated by m, and the above relations yield $[e] = 1 + \mu$ and $[l] = [e] = \lambda[e]$; it follows easily that the cord algebra of the unknot is $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]/((\lambda - 1)(\mu + 1))$.

For general knots, the cord algebra is evidently a knot invariant, but seems intractable. On the other hand, degree 0 knot contact homology is readily computable in terms of generators and relations, but is not obviously a topological invariant. A key result in [12] (cf. [11]) states that these two constructions coincide.

Proposition 3.2 ([11, 12])**.** *The cord algebra of K is isomorphic to $HC_0(K)$. Given an n-crossing knot diagram of K, the cord algebra can be expressed as*

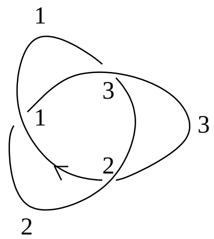


FIGURE 4. The left handed trefoil, with crossings and diagram components marked.

the tensor algebra $\mathcal{A}$ over $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ generated by $\{a_{ij}\}_{1 \leq i,j \leq n,\ i \neq j}$ modulo the $2n^2$ relations given by the entries of the matrices $\Psi^L \cdot A$, $A \cdot \Psi^R$.

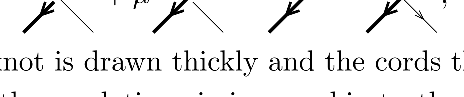
Suppose, for instance, that we wish to compute the cord algebra of the left handed trefoil shown in Figure 4. We have $\Psi^L = \begin{pmatrix} -a_{21} & \mu & \lambda \\ 1 & -a_{32} & \mu \\ \mu & 1 & -a_{13} \end{pmatrix}$, $\Psi^R = \begin{pmatrix} -a_{12} & \mu & 1 \\ 1 & -a_{23} & \mu \\ \lambda^{-1}\mu & 1 & -a_{31} \end{pmatrix}$, and $A = \begin{pmatrix} 1+\mu & a_{12} & a_{13} \\ a_{21} & 1+\mu & a_{23} \\ a_{31} & a_{32} & 1+\mu \end{pmatrix}$. When we equate the entries of $\Psi^L \cdot A$ and $A \cdot \Psi^R$ to zero, we find that $a_{31} = \lambda^{-1} a_{21}$, $a_{13} = \lambda a_{12}$, $a_{32} = a_{12}$, $a_{23} = a_{21}$, and $a_{23} = a_{13}$; it follows that we can substitute for everything in terms of a_{12}. If we write $x = a_{12}$, then the relations reduce to two polynomials in x, and the cord algebra becomes

$$HC_0(\text{LH trefoil}) \cong \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}][x]/(\lambda x^2 - \lambda x - \mu^2 - \mu, \lambda x^2 - \mu x - \mu - 1).$$

This is different from the cord algebra of the unknot (set $\lambda = \mu = 1$), and also from the cord algebra of the right hand trefoil. The next section discusses general methods for distinguishing knot contact homologies and relations to classical invariants.

The term "cord algebra" comes from another interpretation of $HC_0(K)$. Fix a point $*$ on the knot K. A *cord* of K is a path in $\mathbb{R}^3$ which begins and ends on $K \setminus \{*\}$ and does not intersect K in the interior of the path. Let $\tilde{\mathcal{A}}_K$ denote the tensor algebra over $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ generated by homotopy classes of cords. In $\tilde{\mathcal{A}}_K$, mod out by several relations:

- the contractible homotopy class is equal to $1 + \mu$;
- if two cords are related by pulling an endpoint along K across $*$, then one is λ times the other (the precise choice is given by orientations);
- if two cords are related by pushing an interior point in the cord through K, then at the moment when the interior point crosses K, the cord breaks into two, and the four cords involved are related by

  where the knot is drawn thickly and the cords thinly.

Then $\tilde{\mathcal{A}}_K$ modulo these relations is isomorphic to the cord algebra of K. See [12] for details.

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