Manifolds of Positive Scalar Curvature

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Does a given manifold carry a Riemannian metric with strictly positive scalar curvature?

We will survey results addressing this question. Notation: a manifold is *psc* if it carries a metric with positive scalar curvature. Large portions of this survey were shamelessly stolen from Lawson and Michelsohn [5].

1 Who cares?

Asking that a manifold be psc is, in some sense, the weakest of all possible curvature constraints. Incidentally, note that any compact manifold in dimension at least 3 carries a metric with strictly negative scalar curvature [1].

Psc manifolds arise in general relativity. Schoen and Yau use them in their proof of the positive mass/energy conjecture/theorem, a famous ex-conjecture in general relativity which states that the total energy (including contributions from both matter and gravity) of a nontrivial isolated system is always positive, given certain "nice" conditions (spacetime is asymptotically flat, matter with positive mass density is the only source of gravitational fields).

2 General topological obstructions

In a celebrated paper from 1963, Lichnerowicz [7] gave the first known topological obstruction to a manifold being psc, using the Lichnerowicz-Weitzenböck formula for the Dirac Laplacian

$$\not{D}^2 = \nabla^*\nabla + S/4,$$

where S is the scalar curvature. On an even-dimensional manifold X, the complex spinor bundle $\not{S}_{\mathbb{C}}$ splits into a direct sum of two subbundles $\not{S}^+$ and $\not{S}^-$, and the Dirac operator sends sections of one into sections of the other: $\not{D} : \Gamma(\not{S}^+) \longrightarrow \Gamma(\not{S}^-)$. More generally, there is a Dirac operator $\not{D} : \Gamma(\not{S}_{\mathbb{C}}) \longrightarrow \Gamma(\not{S}_{\mathbb{C}})$.

Proposition 1 (Lichnerowicz) *If X is a spin psc manifold, then X has no harmonic spinors, i.e.,* $\ker \not{D} = 0$. *In fact, the same holds if X has nonnegative scalar curvature which is not identically zero.*

PROOF. Suppose $\sigma \in \Gamma(\$^+)$ satisfies $D\!\!\!/\,\sigma = 0$. Then the Lichnerowicz-Weitzenböck formula yields

$$\int_X S\|\sigma\|^2 = -(\nabla^*\nabla\sigma, \sigma) = -\|\nabla\sigma\|^2,$$

whence the proposition. □

Now the Atiyah-Singer index theorem applied to $D\!\!\!/$ shows in this case that ind $D\!\!\!/ = \hat{A}(X)$; $\hat{A}(X)$ is the $\hat{A}$-genus, a topological invariant which for four-manifolds is equal to$-\sigma(X)/8 = -p_1(X)/24$.

Corollary 2 (Lichnerowicz) *A compact spin psc manifold X of dimension $4m$ satisfies $\hat{A}(X) = 0$.*

Note that, for instance, the $K3$ surface $V(x^4 + y^4 + z^4 + w^4) \subset \mathbb{C}P^3$ is not psc.

In 1974, Hitchin [4] generalized the Lichnerowicz result by applying the Atiyah-Singer index theorem to a family of operators, and using the Weitzenböck formula again. The $\hat{A}$-genus is a special case of the Atiyah-Milnor-Singer invariant

$$\hat{\mathcal{A}} : \Omega_*^{\mathrm{Spin}} \longrightarrow KO^{-*}(\mathrm{pt});$$

in fact, $\hat{\mathcal{A}} = \hat{A}$ in dimensions $4m$.

Proposition 3 (Hitchin) *A compact spin manifold X which is psc must satisfy $\hat{\mathcal{A}}(X) = 0$.*

By Bott periodicity, the ring $KO^{-*}(\mathrm{pt})$ is $\mathbb{Z}$ in dimensions 0 and 4 (mod 8), $\mathbb{Z}/2$ in dimensions 1 and 2 (mod 8), and 0 otherwise. Dimensions 0 and 4 (mod 8) are covered by the $\hat{A}$-genus; the others give rise to the following result.

Corollary 4 (Hitchin) *Every compact spin manifold in dimension 1 or 2 (mod 8) is homeomorphic to a manifold which is not psc.*

PROOF. Half of the exotic spheres Σ in dimensions 1 or 2 (mod 8) (namely, those which do not bound spin manifolds) satisfy $\hat{\mathcal{A}}(\Sigma) \neq 0$. If X is a compact spin manifold in dimension 1 or 2 (mod 8), then one of $\hat{\mathcal{A}}(X)$ and $\hat{\mathcal{A}}(X\#\Sigma) = \hat{\mathcal{A}}(X) + \hat{\mathcal{A}}(\Sigma)$ is not psc. □

3 Compact simply connected manifolds

In [3], Gromov and Lawson essentially classified all compact simply connected psc manifolds. To construct large families of manifolds which were psc, they used the result below (Proposition 5) concerning surgeries of manifolds. Recall that on a manifold of dimension n, a surgery in dimension i is the following construction. Suppose we are given an embedded sphere S^i with trivial normal bundle, so that there is a neighborhood of S^i diffeomorphic to $S^i \times D^{n-i}$, with boundary $S^i \times S^{n-i-1}$; then replace $S^i \times D^{n-i}$ by $D^{i+1} \times S^{n-i-1}$, glued along their (identical) boundary.

Proposition 5 was also proved by Schoen and Yau [10] using minimal submanifolds and solutions to partial differential equations.

Proposition 5 (Gromov–Lawson, Schoen–Yau) *Any manifold obtained from a compact psc manifold by surgeries in codimension at least 3 is also psc. In particular, the connected sum of two compact psc manifolds is psc.*

Outline of Proof. We will only treat the case of connected sums. The idea is as follows: given a compact psc manifold M, change the metric in a small ball B around a point p so that it agrees with the original metric of M near the boundary of B, while looking like the metric of $S^{n-1}(\varepsilon) \times \mathbb{R}$ near p, where $S^{n-1}(\varepsilon)$ is the standard Euclidean sphere with radius ε.

Give the cartesian product $D \times \mathbb{R}$ the induced product metric. Consider the hypersurface $H = \{(x,t) : (\|x\|, t) \in \gamma\} \subset D \times \mathbb{R}$, where γ is some curve in $\mathbb{R}^2 = \{(r,t)\}$ which begins by coming down along the positive r axis, and ends by going out along the line $r = \varepsilon$. Now give H the metric induced from D. Then H begins with the metric induced from M, and ends with the metric induced from $\tilde{S}^{n-1}(\varepsilon) \times \mathbb{R}$, where $\tilde{S}^{n-1}(\varepsilon)$ is the sphere in M of points distance ε from p. By choosing this curve carefully, we can ensure that positive scalar curvature is preserved. If ε is small enough, then along the straight line $r = \varepsilon$, we can gradually change the metric from the one induced from $\tilde{S}^{n-1}(\varepsilon) \times \mathbb{R}$ to the one induced from $S^{n-1}(\varepsilon) \times \mathbb{R}$; the verification of this involves computing principal curvatures of, and the metric on, $\tilde{S}^{n-1}(\varepsilon)$, to order ε. □

Using this result and facts from h-cobordism theory, Gromov and Lawson were then able to give very general conditions on when a simply connected manifold is psc.

Proposition 6 (Gromov–Lawson) *Let X be a compact simply connected spin n-manifold with $n \geq 5$. If X is spin cobordant to a psc manifold, then X is psc.*

Lemma 7 ([5]) *A simply connected manifold X of dimension at least 5 is spin if and only if every 2-sphere embedded in X has trivial normal bundle.*

Proof. Since $H_2(X) \cong \pi_2(X)$ by Hurewicz, $H_2(X; \mathbb{Z}/2) = H_2(X) \otimes \mathbb{Z}_2$ is generated by embedded 2-spheres. Suppose $\iota : S^2 \hookrightarrow X$ is such an embedding. Then

$$\iota^* w_2(X) = w_2(\iota^* TX) = w_2(TS^2 \oplus \nu S^2) = w_2(\nu S^2),$$

and hence

$$< w_2(X), \iota_*[S^2] >=< \iota^* w_2(X), [S^2] >=< w_2(\nu S^2), [S^2] > .$$

Thus $w_2(X) = 0$ if and only if $w_2(\nu S^2) = 0$ for all such embedded S^2's. Now since νS^2 is orientable and dim $\nu S^2 \geq 3$, $w_2(\nu S^2) = 0$ if and only if νS^2 is trivial. □

Proof of proposition 6. Let X satisfy the conditions of the proposition, so that there is a compact spin manifold W with $\partial W = X \amalg X_0$, where X_0 is psc. It suffices to show that X can be obtained from X_0 by a series of surgeries in codimension at least 3.

By performing surgery on embedded circles in X_0, we can kill off $\pi_1(X_0)$; similarly, we can assume that W is simply connected. Now by Lemma 7, all embedded spheres in

W have trivial normal bundle, so that we can kill off $\pi_2(W)$ by surgeries. It follows that $H_k(W, X) = 0$ for $k = 0, 1, 2$. Then from the universal coefficient theorem on cohomology

$$0 \longrightarrow \mathrm{Ext}^1(H_{k-1}(W,X), \mathbb{Z}) \longrightarrow H^k(W,X) \longrightarrow \mathrm{Hom}(H_k(W,X), \mathbb{Z})$$

and Poincaré duality (see [8]) $H_{n-k}(W, X_0) \cong H^k(W, X)$, we see that $H_{n-k}(W, X_0) = 0$ for $k = 0, 1$, and 2, and $H_{n-3}(W, X_0)$ is torsion-free.

There is a result of Smale [11] which is roughly as follows: if M is simply connected in dimension at least 5, and the free part and the torsion part of $H_i(M)$ is generated by β_i and α_i elements, respectively, then there is a nondegenerate Morse function on M with $\beta_i + \alpha_i + \alpha_{i-1}$ critical points of index i. Adapted to this case, the result implies that there is a Morse function $f : W \longrightarrow [0, 1]$ with $f|_{X_0} = 0$ and $f|_{X_1} = 1$, all of whose critical points lie in the interior of W and have index at most $n - 3$.

There is a result in Morse theory (see [8, p. 29]) which states that a level set just above a critical point of index i is obtained from a level set just below the critical point by performing surgery in dimension i. We conclude that X can be constructed from X_0 by a series of surgeries in codimension at least 3. □

Proposition 8 (Gromov–Lawson) *Every compact simply connected n-manifold with $n \geq 5$ which is not spin is psc.*

Proof. Let X be such a manifold. This time, there is a compact oriented manifold W with $\partial W = X \amalg X_0$, where X_0 is psc. As before, we may assume that X_0 and W are simply connected. Then the second Stiefel-Whitney class gives a homomorphism $w_2 : \pi_2(W) \cong H_2(W) \longrightarrow \mathbb{Z}_2$. Now by Lemma 7, w_2 measures the nontriviality of the normal bundles to the embedded spheres in $\pi_2(W)$, so by surgery on S^2's, we can kill ker w_2. On the other hand, since X is not spin, the restriction of w_2 to $\pi_2(X)$ is nontrivial; hence we may assume that w_2 is an isomorphism. This implies that the map $\pi_2(X) \longrightarrow \pi_2(W) \cong \mathbb{Z}_2$ is surjective, so that $H_2(W, X) = 0$. The proof now follows the proof of the previous proposition. □

Corollary 9 (Gromov–Lawson) *Every compact simply connected manifold of dimension 5, 6, or 7 is psc.*

Proof. $\Omega_n^{\mathrm{spin}} = 0$ for $n = 5, 6$, or 7. □

Thus for spin manifolds, psc descends to a condition on the spin cobordism ring Ω_*^{spin}. The Hitchin result (Proposition 3) shows that a necessary condition for a class in Ω_*^{spin} to be psc is that it have $\hat{\mathcal{A}}$=0. Gromov and Lawson conjectured in [3] that this is a sufficient condition, and proved that it is sufficient up to torsion: if $\hat{\mathcal{A}}(X) = 0$, then some connected sum $X \# X \# \cdots \# X$ is psc. Further partial results were obtained by Miyazaki, who showed that $X \# X \# X \# X$ suffices, and Rosenberg, who showed that the conjecture was true in dimensions ≤ 23; these results used explicit psc representatives of classes in Ω_*^{spin}. Using techniques from stable homotopy theory, Stolz [12] finally settled the conjecture in 1990.

Proposition 10 (Gromov–Lawson–Stolz) *A compact simply connected spin manifold X of dimension at least 5 is psc if and only if $\hat{\mathcal{A}}(X) = 0$.*

This completes the classification of compact simply connected manifolds.

4 Manifolds with nontrivial fundamental group

The seminal paper in this direction is Gromov and Lawson [2]. First note that the cartesian product of any manifold with S^2 is psc, so that in dimension at least 6, the psc condition places no restriction on the fundamental group of the manifold. On the other hand, the fundamental group can guarantee that a manifold is not psc. Using minimal surfaces, Schoen and Yau [9] proved the first result of this type: the torus T^3, and more generally any 3-manifold whose fundamental group contains the fundamental group of a surface of positive genus, is not psc.

A couple of definitions are necessary. Given $\varepsilon > 0$, a C^1 map $f : X \longrightarrow Y$ is defined to be ε-contracting if $\|f_* v\| \leq \varepsilon \|v\|$ for all $v \in TX$. A compact n-manifold is enlargeable if, for every $\varepsilon > 0$, it has an orientable riemannian cover which admits an ε-contracting map onto the standard n-sphere $S^n(1)$ with constant curvature 1, which is constant outside some compact set and of non-zero degree. (Intuitively, the covering manifold is "bigger" than $S^n(1)$ by a factor of at least $1/\varepsilon$.) For instance, the flat torus $\mathbb{R}^n/\mathbb{Z}^n$ is enlargeable: it is covered by $\mathbb{R}^n/(k\mathbb{Z})^n$ (k a positive integer), which maps onto $S^n(1)$ by wrapping the n-disk inscribed in $[0, k]^n$ around $S^n(1)$, so that the boundary of the n-disk is mapped to one point, and sending everything else to that same point.

There are large classes of manifolds which are enlargeable. Any 3-manifold whose fundamental group contains the fundamental group of a surface of positive genus is enlargeable, as is any compact manifold admitting a metric of nonpositive sectional curvature. Any compact solvmanifold is also enlargeable; a solvmanifold is by definition diffeomorphic to G/Γ where G is a solvable Lie group and Γ is a discrete subgroup.

Proposition 11 (Gromov–Lawson) *No enlargeable spin manifold (or enlargeable manifold whose relevant covers are spin) is psc; in fact, any metric with nonnegative scalar curvature on such a manifold must be flat.*

Corollary 12 *Any compact manifold which carries a metric of nonpositive (negative) sectional curvature cannot carry a metric of positive (nonnegative) scalar curvature.*

5 Group actions

Here is one more result, for variety.

Proposition 13 (Lawson–Yau [6]) *Any compact manifold which admits an effective differentiable action by a compact, connected, nonabelian Lie group is psc.*

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