PLANE CURVES AND CONTACT GEOMETRY

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ABSTRACT. We apply contact homology to obtain new results in the problem of distinguishing immersed plane curves without dangerous self-tangencies.

1. INTRODUCTION

The purpose of this manuscript is to show that contact geometry, and in particular Legendrian knot theory and contact homology, can be used to give new information about plane curves without dangerous self-tangencies. Throughout, the term "plane curve" will refer to an immersion $S^1 \to \mathbb{R}^2$ up to orientation-preserving reparametrization, i.e., an oriented immersed plane curve in $\mathbb{R}^2$.

Definition 1. A self-tangency of a plane curve is *dangerous* if the orientations on the tangent directions to the curve agree at the tangency. Two plane curves without dangerous self-tangencies are *safely homotopic* if they are homotopic through plane curves without dangerous self-tangencies.

A generic homotopy of plane curves may contain three types of singularities, of which one is the dangerous self-tangency; see Figure 1. Arnold [1, 2] initiated the study of plane curves up to safe homotopy, in particular introducing a function J^+ on plane curves without dangerous self-tangencies. In the literature, any function of plane curves without dangerous self-tangencies which does not change under safe homotopy is called a *J^+-type invariant*.

The key point of interest of plane curves without dangerous self-tangencies is their close link to contact geometry, first noted by Arnold. There is a natural way to associate to any such plane curve a Legendrian knot in $\mathcal{J}^1(S^1)$, the 1-jet space of S^1, which is a contact manifold. We call this the *conormal knot* of the plane curve. For details, see Section 2.1.

The conormal knot is a special case of a construction which associates a Legendrian submanifold to any embedded submanifold of any manifold, or to any immersed submanifold without dangerous self-tangencies. This construction has recently been applied to construct new invariants of knots in S^3, and potentially yields interesting isotopy invariants of arbitrary submanifolds; see [5] or [12] for an introduction.

There are several well-known J^+-type invariants of plane curves, all arising from the conormal knot construction. The simplest is the Whitney index, or the degree of the Gauss map of the plane curve. This is invariant under

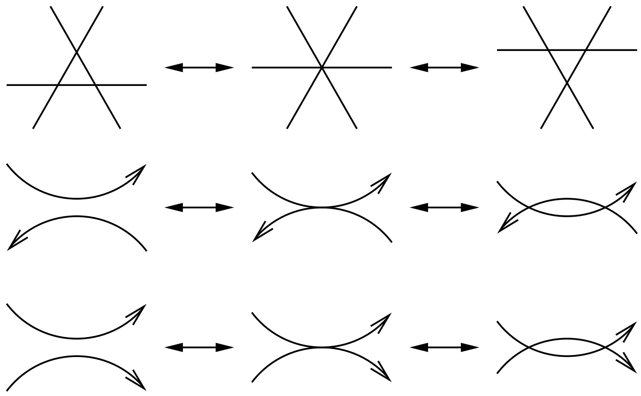

FIGURE 1. Singularities ("perestroikas") encountered in homotopies of plane curves: (a) triple point; (b) safe self-tangency; (c) dangerous self-tangency.

safe homotopy since it is invariant more generally under regular homotopy; it also counts the number of times the conormal knot winds around the base of the solid torus $\mathcal{J}^1(S^1)$.

A more nontrivial J^+-type invariant, as observed by Arnold, is simply the knot type of the conormal knot in the solid torus. More interesting still, since the conormal is Legendrian, the contact planes along the conormal knot give it a framing, and so the framed knot type of the conormal knot is invariant under safe homotopy. The framing is measured by a number which is Arnold's original J^+ invariant.

To the author's knowledge, all previous work on J^+-type invariants is based on studying the framed knot type of the conormal knot. For instance, Goryunov [9] examined the space of finite type invariants of plane curves without dangerous self-tangencies, and Chmutov, Goryunov, and Murakami [4] introduced a J^+-type invariant in the form of a HOMFLY polynomial for the framed conormal knot.

On the other hand, two safely homotopic plane curves have conormal knots which are isotopic not just as framed knots, but as Legendrian knots. We will see that the Legendrian type of the conormal knot gives a finer classification of plane curves than the framed knot type. The fact (essentially) that Legendrian isotopy is a subtler notion than framed isotopy was famously demonstrated by Chekanov [3] for knots in $\mathbb{R}^3$, using a combinatorial form of Legendrian contact homology [6]. In this paper, we show that contact homology gives a similar result in our case.

Theorem 1 (see Propositions 3 and 4)**.** *There are (arbitrarily many) plane curves with the same framed conormal knot type which are not safely homotopic.*

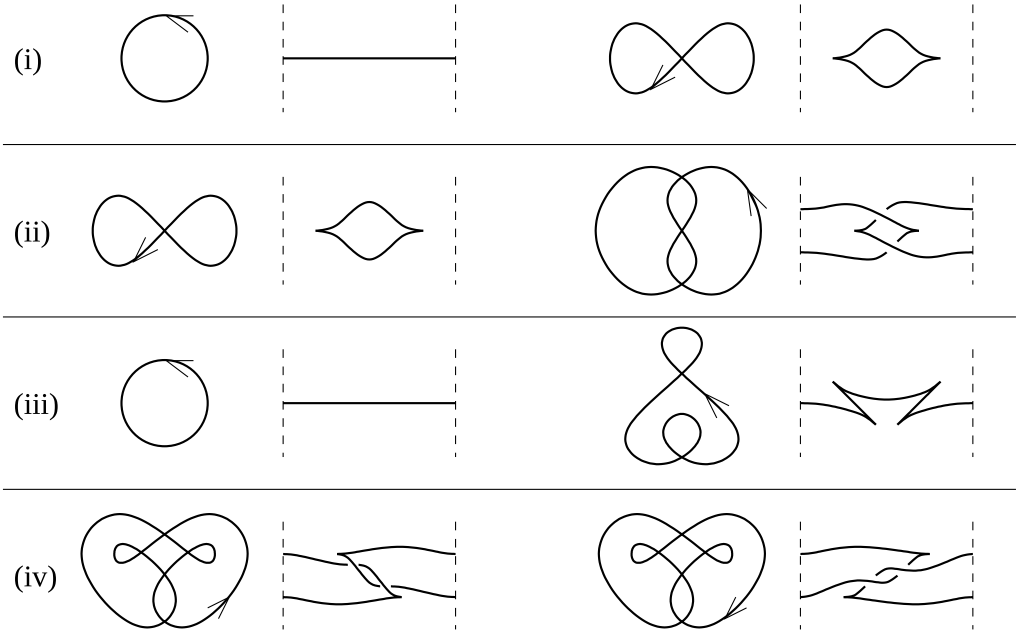

Figure 2. Pairs of plane curves, along with their conormal knots, that are distinguished by increasingly subtle invariants: (i) Whitney index; (ii) conormal knot type; (iii) framed conormal knot type (Arnold's J^+ invariant); (iv) Legendrian conormal knot type.

In the language of Legendrian knot theory, we can rephrase this result: there are arbitrarily many plane curves whose conormal knots all have the same classical invariants but are not Legendrian isotopic.

An example of a pair of plane curves satisfying the conditions in Theorem 1 is given by the bottom line of Figure 2. Note that this "pair" is actually the same plane curve but with different orientations. Proposition 3 uses contact homology to distinguish between these curves.

We review definitions in Section 2.1, and present an algorithm for drawing conormal knots in Section 2.2. Section 2.3 gives the proof of our main result, Theorem 1. In Section 2.4, we show that contact homology gives new information about loops of plane curves as well.

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2. Results and Proofs

2.1. **The conormal knot.** Let C be a plane curve. At each point $x \in C$, the orientation on C determines two unit vectors, v_x in the direction of C and w_x given by rotating v_x 90° counterclockwise.

Definition 2. The *conormal knot* of C is the subset of the unit cotangent bundle $ST^*\mathbb{R}^2$ given by

$$\{\xi \in ST^*\mathbb{R}^2 \,|\, \xi \text{ lies over some } x \in C \text{ and } \langle \xi, v_x \rangle = 0, \langle \xi, w_x \rangle = 1\}.$$

The conormal knot inherits an orientation from the orientation on C, since each point on C yields one point in the conormal knot.

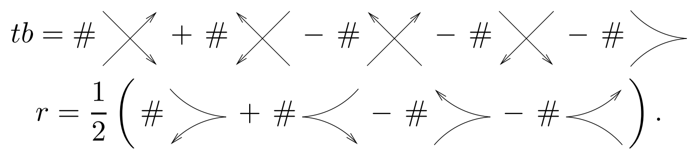
Here the metric on the fibers of $T^*\mathbb{R}^2$ used to define $ST^*\mathbb{R}^2$ is dual to the standard metric on $\mathbb{R}^2$. If C has no dangerous self-tangencies, then its conormal knot is embedded in $ST^*\mathbb{R}^2$, and so it makes sense to use the term "knot." We remark that the conormal knot is actually one half of the usual unit conormal bundle over the plane curve; the orientation of the plane curve, along with the orientation of $\mathbb{R}^2$, induces a coorientation on the curve, which picks out half of the conormal bundle.

The space $ST^*\mathbb{R}^2$ has a natural contact structure given by the kernel of the 1-form $\alpha = p_1\, dq_1 + p_2\, dq_2$, where q_1, q_2 are coordinates on $\mathbb{R}^2$ and p_1, p_2 are dual coordinates in the cotangent fibers. It is easy to check that the conormal knot K of any plane curve is Legendrian with respect to this contact structure, i.e., that $\alpha|_K = 0$.

Topologically, $ST^*\mathbb{R}^2 \cong S^1 \times \mathbb{R}^2$ is a solid torus, and it will be more useful for us to view it as the 1-jet space $\mathcal{J}^1(S^1) \cong T^*S^1 \times \mathbb{R}$. If we set coordinates θ, y, z on $\mathcal{J}^1(S^1) \cong (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R} \times \mathbb{R}$, then $\mathcal{J}^1(S^1)$ has a natural contact form $\alpha = dz - y\, d\theta$. We can identify $ST^*\mathbb{R}^2$ and $\mathcal{J}^1(S^1)$ by setting $\theta = \arg(p_1 + ip_2)$ (the argument of the vector (p_1, p_2)), $z = q_1 \cos\theta + q_2 \sin\theta$, $y = -q_1 \sin\theta + q_2 \cos\theta$; this map identifies the contact structures as well.

It is convenient to picture a Legendrian knot in $\mathcal{J}^1(S^1)$ in terms of its *front*, or projection to $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$ given by the θz coordinates. In the subject, "front" is sometimes used in a different sense, namely as a cooriented plane curve with cusps; for clarity, we will avoid this connotation. A generic Legendrian knot has a front whose only singularities are double points and cusps. We can recover a Legendrian knot from its front by setting $y = dz/d\theta$; in particular, there is no need to specify over- and undercrossing information for a front. We depict $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$ by letting θ be the horizontal axis and z the vertical axis, and drawing dashed vertical lines to represent the identified lines $\theta = 0$ and $\theta = 2\pi$. See Figure 2 for examples of fronts in $\mathcal{J}^1(S^1)$.

Any front in $\mathcal{J}^1(S^1)$ has three "classical" invariants under Legendrian isotopy. The first is the knot type of the front in $\mathcal{J}^1(S^1)$, obtained by smoothing cusps and resolving crossings in the usual way. The other two are the Thurston–Bennequin number tb and rotation number r:



We note that the Thurston–Bennequin number in $\mathcal{J}^1(S^1)$ was first introduced by Tabachnikov [14].

We now examine the front K of the conormal knot of a plane curve C. There is a simple description for K: any point (q_1, q_2) on C, with unit tangent vector $(\cos\varphi, \sin\varphi)$, gives the point $(\theta, z) = (\varphi + \pi/2, -q_1 \sin\varphi + q_2 \cos\varphi)$ in K, and K is obtained by allowing (q_1, q_2) to range over C. Points of inflection of C correspond to cusps of K, and it is easy to check that any right cusp of K is traversed upwards and any left cusp downwards; just draw a neighborhood of an inflection point of C.

As for the classical Legendrian invariants of K, since K has equal numbers of left and right cusps, it follows that $r(K) = 0$. The Thurston–Bennequin number of K measures framing and is essentially Arnold's J^+ invariant: $tb(K) = J^+(K) + n(K)^2 - 1$, where $n(K)$ is the winding number of K around S^1. Hence the framed knot type of K determines all classical information about K.

2.2. **Drawing the conormal knot front.** We have already discussed how to define the conormal knot front of a plane curve, but the definition is not very useful computationally. Here we present an algorithm for easily obtaining a front isotopic to the conormal knot front.

Call a plane curve *rectilinear* if it is completely composed of line segments parallel to either coordinate axis, along with arbitrarily small smoothing 90° corners, and no two line segments lie on the same (horizontal or vertical) line. Clearly any plane curve is isotopic to a rectilinear curve, and so it suffices to describe the conormal front for any rectilinear curve.

For ease of notation, label the coordinate axes x and y rather than q_1 and q_2. To each line segment L in a rectilinear plane curve, we associate the following point in $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$:

- $(\pi/2, y)$ if L is in the $+x$ direction and y is the y coordinate of L;
- $(\pi, -x)$ if L is in the $+y$ direction and x is the x coordinate of L;
- $(3\pi/2, -y)$ if L is in the $-x$ direction and y is the y coordinate of L;
- $(0, x)$ if L is in the $-y$ direction and x is the x coordinate of L.

Next, "connect the dots" by joining the points corresponding to line segments which share an endpoint. Finally, smooth the result, rounding corners and placing cusps where necessary. See Figure 3.

Proposition 2. *The resulting front in $\mathcal{J}^1(S^1)$ is Legendrian isotopic to the front of the rectilinear plane curve.*

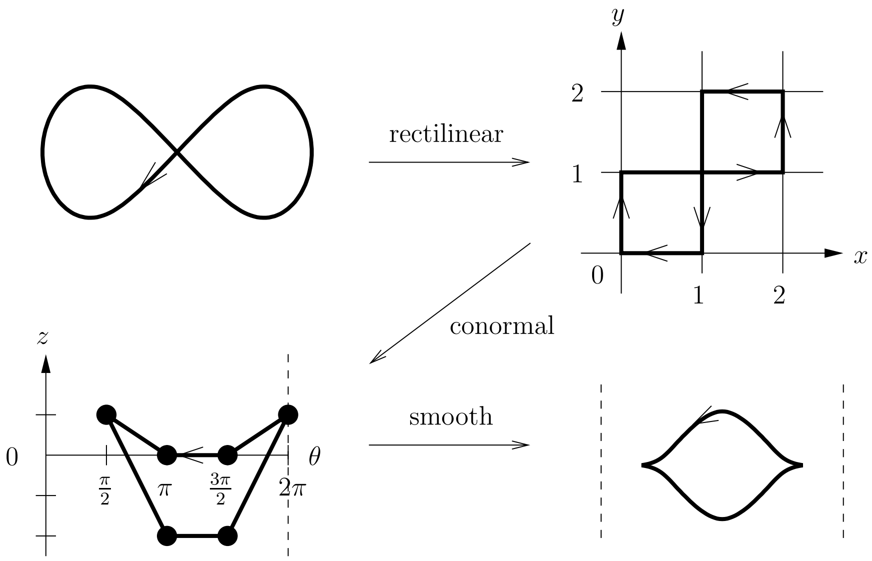


FIGURE 3. Algorithm for obtaining the conormal knot from a plane curve.

Proof. It is clear that the conormal front for the rectilinear curve passes through the points given in the algorithm above, since they comprise the conormal for the line segments of the rectilinear curve. The conormals of the smoothing corners interpolate between these points. The conormal front for a smoothing corner at the point (x, y) is given by $\{(\theta, x \cos\theta + y \sin\theta)\}$ for some range of θ in an interval of length $\pi/2$. Hence the conormal fronts for any two smoothing corners intersect either once or not at all. It follows that, up to Legendrian isotopy, the conormals for the smoothing corners can be approximated by the line segments joining points in the algorithm above. □

2.3. **Nonhomotopic plane curves.** We can use the algorithm from the previous section to show that there are plane curves whose conormal knots have the same framed knot type but which are not Legendrian isotopic.

Proposition 3. *The plane curves in the bottom line of Figure 2 have the same framed conormal knot type but are not safely homotopic.*

Proof. The two plane curves give conormal knots which are topologically Whitehead links; see Figure 4. Both conormal knots have $tb = -3$ (equivalently, $J^+ = -2$).

We claim that the two conormal knots are not Legendrian isotopic. Indeed, applying the Legendrian satellite construction (see the appendix of [13]) to the conormal fronts and the stabilized unknot in $\mathbb{R}^3$ yields two familiar Legendrian knots in $\mathbb{R}^3$: these are called "Eliashberg knots" in [7]

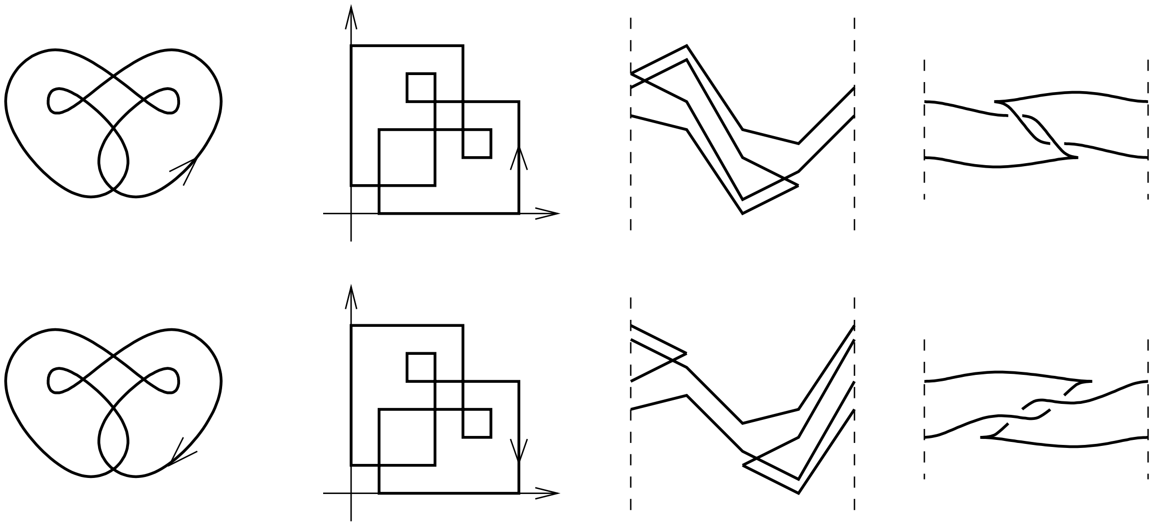


FIGURE 4. Nonhomotopic plane curves, their rectilinear approximations, conormals, and smoothed conormal fronts.

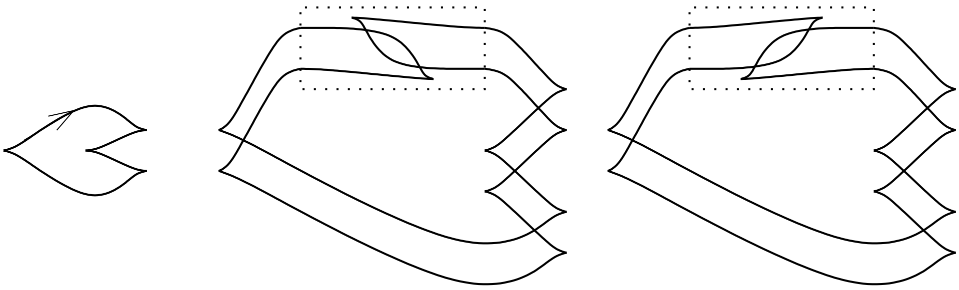


FIGURE 5. The Legendrian satellites of the conormal knots from Figure 4 to the stabilized unknot produce nonisotopic Legendrian knots.

and labeled $E(2,3)$ and $E(1,4)$. See Figure 5. The two knots can be distinguished by their contact homology differential graded algebras [3]; in particular, $E(2,3)$ has Poincaré polynomial $2t+t^{-1}$ and $E(1,4)$ has Poincaré polynomial t^3+t+t^{-3}. It follows that the two conormal knots in $\mathcal{J}^1(S^1)$ are not Legendrian isotopic, as desired. $\square$

We can use the plane curves from Proposition 3 to produce an arbitrarily large family of plane curves whose conormal knots have the same classical invariants but are not Legendrian isotopic. For $r,s \geq 0$, let $C_{r,s}$ be the plane curve shown in Figure 6, which can be viewed as a connected sum of the plane curves from Proposition 3. (Note however that the connected sum operation on plane curves is not well-defined.)

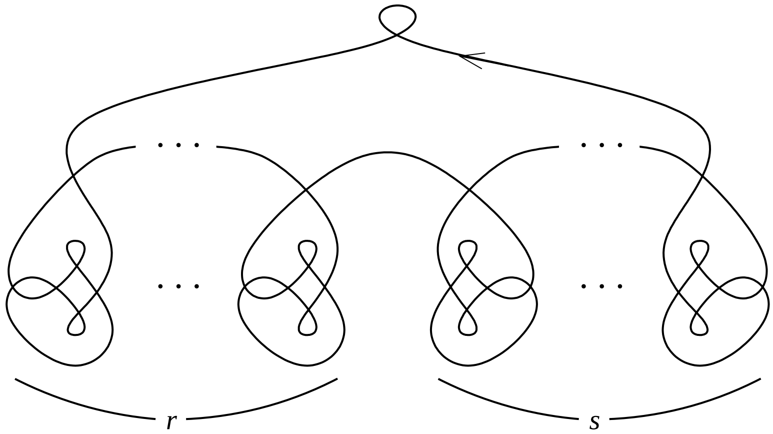


FIGURE 6. The "connected sum" plane curve $C_{r,s}$.

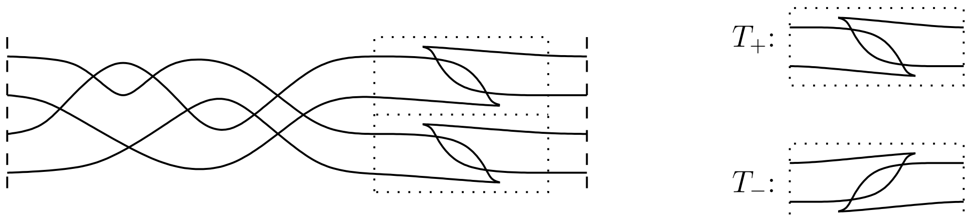


FIGURE 7. The conormal knot for $C_{2,0}$. To obtain the conormal knots for $C_{1,1}$ and $C_{0,2}$, replace one or both of the boxed tangles T_+ by T_-.

Proposition 4. *For fixed $n \geq 1$, the n plane curves $C_{r,s}$, $r + s = n$, have the same framed conormal knot type but are not safely homotopic.*

Sketch of proof. The details of the proof require some working familiarity with computations in contact homology for Legendrian knots in standard contact $\mathbb{R}^3$, along the lines of [8, 11]; we provide an outline here and leave the details to the reader.

We can use the algorithm from Section 2.1 to find the conormal fronts for $C_{r,s}$. When $r + s = n$ is fixed, the conormal fronts for $C_{r,s}$ are identical except for n tangles. Of these tangles, r are given by the tangle T_+ defined in Figure 7, and s by T_-. The situation for $n = 2$ is shown in Figure 7; the picture for $n > 2$ is very similar. Note that the conormal fronts for $C_{r,s}$ are all isotopic as framed knots.

Now consider the Legendrian satellite $K_{r,s}$ of the conormal front for $C_{r,s}$ to the stabilized unknot, as in the proof of Proposition 3. We distinguish between the knots $K_{r,s}$ using Poincaré polynomials for contact homology.

It is easy to show that the Chekanov–Eliashberg differential graded algebra for $K_{r,s}$ has a graded augmentation, for instance because it has a ruling

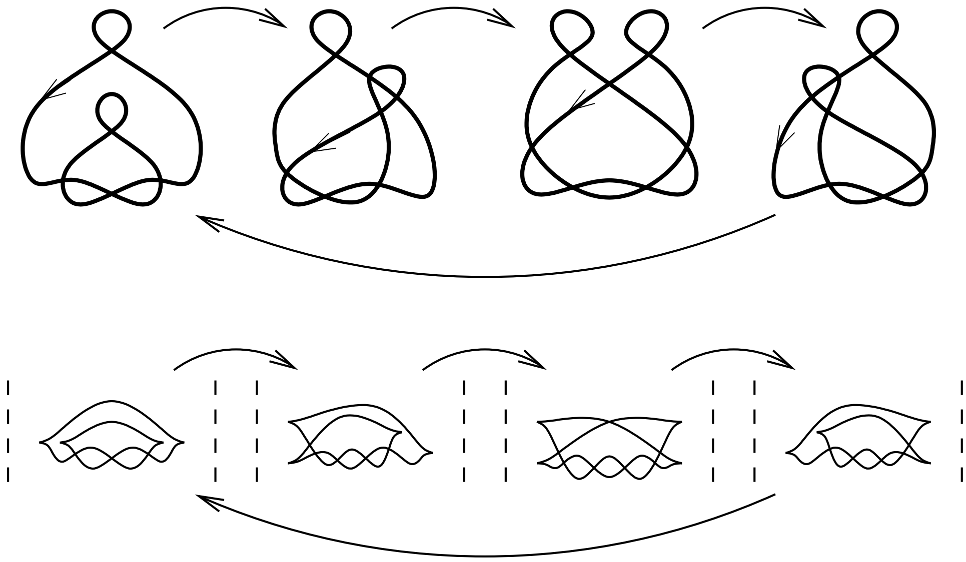


FIGURE 8. A nontrivial loop γ of plane curves, and the corresponding loop $\tilde{\gamma}$ of conormal knots.

[8]. An examination of Maslov indices shows that all crossings in $K_{r,s}$ have degrees $0, \pm 1, \pm 2$, except for the crossings in the tangles $T_\pm$; the two crossings in any T_+ have degree 1 and -1, while the two crossings in any T_- have degree 3 and -3. Since the (linearized) differential of the degree 3 crossing in any T_- is 0, we conclude that any Poincaré polynomial for $K_{r,s}$ has t^3 coefficient equal to s. It follows that the Legendrian knots $K_{r,s}$, $r + s = n$, are not Legendrian isotopic, and thus that the plane curves $C_{r,s}$ are not safely homotopic. $\square$

2.4. **Loops of plane curves.** Here we consider loops in the space of plane curves. Let $\mathcal{C}$ denote the space of plane curves, and let $\mathcal{D} \subset \mathcal{C}$ be the discriminant of plane curves with dangerous self-tangencies. We will present a loop which is contractible in $\mathcal{C}$ but noncontractible in $\mathcal{C} \setminus \mathcal{D}$.

Consider the loop γ in $\mathcal{C} \setminus \mathcal{D}$ pictured in Figure 8. This induces a loop $\tilde{\gamma}$ of Legendrian knots in $\mathcal{J}^1(S^1)$, also shown in Figure 8. As a loop of (framed) knots in $\mathcal{J}^1(S^1)$, $\tilde{\gamma}$ is contractible; this follows from the contractibility of the corresponding loop of trefoils in S^3, which itself follows from work of Hatcher (see [10]). By using the contact condition and contact homology, we can do even better: a result of Kálmán [10] shows that $\tilde{\gamma}$ is nontrivial when considered as a loop of Legendrian knots.

Proposition 5. *The loop γ is contractible in $\mathcal{C}$, but has order at least 5 in $\pi_1(\mathcal{C} \setminus \mathcal{D})$.*

Proof. It is straightforward to check that γ is contractible in $\mathcal{C}$. Note that by the h-principle, $\mathcal{C}$ is weakly homotopy equivalent to the space of free loops in $S^1 \times \mathbb{R}^2$, which is not simply connected. However, γ can be represented by a loop of *based* loops in $S^1 \times \mathbb{R}^2$, and the space of based loops is simply connected since $\pi_2(S^1 \times \mathbb{R}^2) = 0$.

Now consider γ as a loop in $\mathcal{C} \setminus \mathcal{D}$. The loop $\tilde{\gamma}$ of Legendrian knots in $\mathcal{J}^1(S^1)$ lifts to an identical-looking loop $\tilde{\gamma}'$ of Legendrian knots in the universal cover $\mathbb{R}^3$ with the standard contact structure. (Just ignore the dashed lines in Figure 8.) A special case of Theorem 1.2 in [10] states that $\tilde{\gamma}'$ has order at least 5 in the Legendrian category; hence $\tilde{\gamma}$ does as well. $\square$

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