

ON TRANSVERSE INVARIANTS FROM KHOVANOV HOMOLOGY

ROBERT LIPSHITZ, LENHARD NG, AND SUCHARIT SARKAR

ABSTRACT. In [Pla06], O. Plamenevskaya associated to each transverse knot K an element of the Khovanov homology of K . In this paper, we give two refinements of Plamenevskaya's invariant, one valued in Bar-Natan's deformation (from [BN05]) of the Khovanov complex and another as a cohomotopy element of the Khovanov spectrum (from [LSa]). We show that the first of these refinements is invariant under negative flypes and SZ moves; this implies that Plamenevskaya's class is also invariant under these moves. We go on to show that for small-crossing transverse knots K , both refinements are determined by the classical invariants of K .

1. INTRODUCTION

Transverse links have emerged as central objects of study in three-dimensional contact geometry. We will restrict our attention to transverse links in standard contact \mathbb{R}^3 : by a *transverse link*, we mean a knot or link in \mathbb{R}^3 that is everywhere transverse to the 2-plane field $\ker(dz - y dx)$. One typically studies transverse links up to *transverse isotopy*, or isotopy through a one-parameter family of transverse links.

Up to transverse isotopy, transverse links have two *classical invariants*: the underlying topological link type and the self-linking number $sl \in \mathbb{Z}$. (Strictly speaking, for a multi-component link, each component has a self-linking number; see Remark 2.1.) A natural question is whether there are transverse links that have the same classical invariants but are not transversely isotopic. The answer is yes, with the earliest examples given by Etnyre–Honda [EH05] and Birman–Menasco [BM06b], but the question is surprisingly subtle. By contrast, the corresponding question for Legendrian links, which are also central to contact geometry, was answered considerably earlier by Chekanov [Che02].

One approach to distinguishing transverse links with the same classical invariants is to introduce further, more refined invariants of transverse links. The first candidate for such an invariant was introduced by Plamenevskaya [Pla06]: to a transverse link of topological type K , this associates a distinguished class in the Khovanov homology of K . Since

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Plamenevskaya’s groundbreaking work, transverse invariants of a similar flavor have been discovered by Wu [Wu08] in \mathfrak{sl}_n Khovanov–Rozansky homology, and by Ozsváth–Szabó–Thurston [OSzT08] and Lisca–Ozsváth–Stipsicz–Szabó [LOSS09] in knot Floer homology. (Knot contact homology produces a transverse invariant of a somewhat different flavor, called transverse homology [EENS, Ng11].)

A priori, it might be the case that some of these invariants are determined by the smooth link type and self-linking number. An invariant of transverse links is called *effective* if it achieves different values for some pair of transverse links with the same classical invariants. It is known that the invariant in knot Floer homology is effective [NOT08], as is transverse homology [Ng11].

It has long been an open question whether Plamenevskaya’s original invariant in Khovanov homology is effective. One goal of this paper is to shed some light on this question, although we do not resolve it. To produce candidates for distinct transverse links with the same classical invariants, two techniques are commonly considered: negative braid flypes on braids and *SZ* moves on Legendrian links. (There are other techniques as well, but these two seem to be the most successful for small-crossing knots.) In fact, we will show that negative braid flypes and *SZ* moves are equivalent for transverse knots; see Proposition 2.8.

In Theorem 4.15, we show that the Plamenevskaya invariant is unchanged by negative braid flypes, and thus by *SZ* moves as well. Our result can be seen as evidence that the Plamenevskaya invariant may not be effective; by contrast, both the *HF*K transverse invariant and transverse homology can distinguish transverse knots related by these moves. There are transverse links that are known not to be related by these moves, but the simplest one known to the authors is topologically a certain cable of the trefoil knot [ELT12] and (we believe) outside the reach of current technology for computing the Plamenevskaya invariant.

In a different direction, we give three refinements of the Plamenevskaya invariant, which could be effective even if the original invariant is not. The first of these is a filtered version of the Plamenevskaya invariant, living in Bar-Natan’s (or, if one prefers, Lee’s) deformation of the Khovanov complex [BN05, Lee05]; see Theorem 4.2. In fact, this filtered invariant comes in two versions, and subtracting the two versions gives another transverse invariant, *a priori* incomparable to the filtered or original Plamenevskaya invariants; see Theorem 4.5. Finally, a space-level version $\mathcal{X}_{Kh}(K)$ of Khovanov homology was recently constructed [LSa], and the (original) Plamenevskaya invariant admits a refinement as an element of the stable cohomotopy (rather than cohomology) groups of $\mathcal{X}_{Kh}(K)$; see Theorem 5.1. This leads to a number of computable auxiliary invariants; see Section 5.3.

Unfortunately, we have also been unable to show that any of these refinements is effective. In particular, the filtered invariants are unchanged by negative flypes and *SZ* moves. There are also simpler structural results that mean that for small-crossing knots, the refinements have no non-classical information; see Sections 4.5 and 5.3.

However, there are some indications that some of the refinements may stand a better chance of being effective than the original Plamenevskaya invariant by itself. For instance,

over a ring where 2 is invertible, the original invariant agrees for any two transverse knots that become the same after one stabilization (see Proposition 4.13 for the exact result), while we do not know if this is true for the filtered invariant. We know even less about the behavior of the cohomotopy invariant; it might even be able to distinguish transverse knots related by negative flypes.

We remain optimistic that, with more work on computational tools, for more complicated knots, both the filtered Plamenevskaya invariant and the cohomotopy Plamenevskaya invariant will turn out to be effective. For the latter, we include some discussion in Section 5.3 of some possible ways that effectiveness might be tested.

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2. SOME CONSTRUCTIONS IN CONTACT GEOMETRY

Here we review some well-known facts involving transverse knots and links, referring the reader to the survey paper [Etn05] for general background on transverse knots and links, and to [NT09, KN10] for further background on some of the specifics that we give here. By a *transverse link* (respectively, *Legendrian link*) we mean a knot or a link in \mathbb{R}^3 that is everywhere transverse (respectively, tangent) to the standard contact structure $\ker(dz - y dx)$; by a transverse knot (respectively, *Legendrian knot*) we mean a transverse link (respectively, Legendrian link) of one component. There are two well-known correspondences, one between transverse links and equivalence classes of braids, and another between transverse links and equivalence classes of Legendrian links. We describe each of these in turn, and then present a result linking them.

2.1. Transverse links and braids. By work of Bennequin [Ben83], any braid can be naturally viewed as a transverse link (whose topological link type is the closure of the braid), and every transverse link arises in this way (up to transverse isotopy) from some braid. The transverse Markov Theorem [OS03, Wri02] states that under this correspondence, transverse links up to transverse isotopy can be identified with braids up to conjugation and positive braid stabilization and destabilization,

$$B \longleftrightarrow B\sigma_m,$$

where B is an element of the braid group B_m and $B\sigma_m \in B_{m+1}$. We refer to conjugation and positive (de)stabilization as *transverse Markov moves*. The self-linking number of a transverse link T can be expressed in terms of a corresponding braid B as $sl(T) = w - m$, where w is the writhe of B (the sum of the exponents of the braid word) and m is the braid index of B .

Remark 2.1. For a transverse link $T = T_1 \cup \cdots \cup T_r$ with $r \geq 2$ components, the self-linking number of T is related to the self-linking number of its components as follows:

$$sl(T) = \sum_i sl(T_i) + 2 \sum_{i < j} lk(T_i, T_j),$$

where lk represents the (topological) linking number. This follows from the formula $sl(T) = w - m$ above, or from any of the standard definitions of self-linking number in contact geometry (cf. Section 2.2). The r -tuple $(sl(T_1), \dots, sl(T_r)) \in \mathbb{Z}^r$ is invariant under transverse isotopy, and in some sense should be considered the true “self-linking number” associated to T . However, for consistency we will refer to the single integer $sl(T)$ as the self-linking number.

There is also a notion of negative braid stabilization and destabilization,

$$B \longleftrightarrow B\sigma_m^{-1},$$

for $B \in B_m$. This descends to a well-defined operation on transverse links called *transverse stabilization*, which decreases self-linking number by 2. Given any two transverse links representing the same topological link type, one can perform some number of transverse stabilizations to each to obtain transversely isotopic links.

Birman and Menasco introduced a class of flype operations on certain braids that preserves the braid index as well as the topological link type of the braid closure; see [BM93] for 3-braids, generalized in [BM06a] for arbitrary braid index. In this paper, we will use “flype” to mean the following.

Definition 2.2. Let $A, B \in B_m$ be braids and $k \in \mathbb{Z}$. We say that the braids

$$A\sigma_m^k B\sigma_m, \quad A\sigma_m B\sigma_m^k,$$

which are elements in B_{m+1} , are related by a *positive flype*. Similarly, we say that

$$A\sigma_m^k B\sigma_m^{-1}, \quad A\sigma_m^{-1} B\sigma_m^k$$

are related by a *negative flype*. See Figure 2.1.

(Note that Birman and Menasco’s original definition of flypes in [BM06a] is actually a more general, weighted version of the flypes that we consider here.) In addition to link type and braid index, both flypes preserve the self-linking number of the corresponding transverse link.

One can express a negative flype as a composition of conjugation, one negative braid stabilization, and one negative braid destabilization. Since this will be important to us, we write down a precise sequence of braid moves here, labeled by the relevant braid operations as well as the corresponding Reidemeister moves on the diagram for the closed braid (cf.

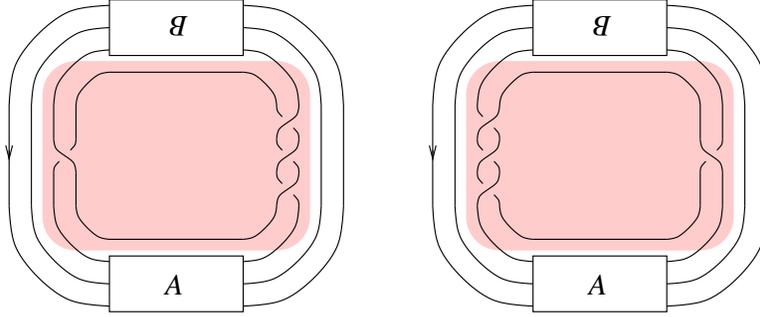


FIGURE 2.1. **Negative flype.** Pictured here: the closures of braids $A\sigma_3^3B\sigma_3\sigma_3^{-1}$ (left) and $A\sigma_3^{-1}B\sigma_3^3$ (right), for $A, B \in B_3$. Note that the diagrams are related by a standard topological flype in the shaded regions.

[BM06a, Figure 5]):

$$\begin{aligned}
(2.1) \quad A\sigma_m^k B\sigma_m^{-1} &\rightarrow A\sigma_m^{-1}\sigma_m^{k+1} B\sigma_m^{-1} && \text{(Reidemeister II)} \\
&\rightarrow A\sigma_m^{-1}\sigma_{m+1}^{-1}\sigma_m^{k+1} B\sigma_m^{-1} && \text{(negative braid stabilization; Reidemeister I)} \\
&\rightarrow A\sigma_{m+1}^{k+1}\sigma_m^{-1}\sigma_{m+1}^{-1} B\sigma_m^{-1} && \text{(braid relation; Reidemeister III)} \\
&\rightarrow \sigma_{m+1}^{k+1} A\sigma_m^{-1} B\sigma_{m+1}^{-1}\sigma_m^{-1} && \text{(braid relation—far commutativity)} \\
&\rightarrow A\sigma_m^{-1} B\sigma_{m+1}^{-1}\sigma_m^{-1}\sigma_{m+1}^{k+1} && \text{(braid conjugation)} \\
&\rightarrow A\sigma_m^{-1} B\sigma_m^{k+1}\sigma_{m+1}^{-1}\sigma_m^{-1} && \text{(braid relation; Reidemeister III)} \\
&\rightarrow A\sigma_m^{-1} B\sigma_m^{k+1}\sigma_m^{-1} && \text{(negative braid destabilization; Reidemeister I)} \\
&\rightarrow A\sigma_m^{-1} B\sigma_m^k && \text{(Reidemeister II).}
\end{aligned}$$

Since these moves involve negative braid (de)stabilization, they do not yield a transverse isotopy, and indeed there are many examples of negative flypes producing distinct transverse links; see e.g. [BM06b, KN10].

In a completely analogous way, one can express a positive flype as a composition of braid conjugation, one positive braid stabilization, and one positive braid destabilization. In this case it follows from the transverse Markov theorem that positive flypes preserve transverse type.

Definition 2.3. Two transverse links T, T' are *flype-equivalent* if there are transverse links $T_0 = T, T_1, \dots, T_k = T'$ such that for each $i = 1, \dots, k$, T_{i-1} and T_i can be represented by braids that differ by a flype.

Equivalently, transverse links are flype-equivalent if they have braid representatives that are related by a sequence of the following braid moves: braid conjugation, positive braid (de)stabilization, and negative flypes. (Note in particular that since positive flypes can be

expressed in terms of conjugation and positive (de)stabilization, they can be omitted here and in Definition 2.3 if desired.)

If B_1, B_2 are braids related by a negative flype and T_1, T_2 are the corresponding transverse links, then by (2.1), $S(T_1)$ and $S(T_2)$ are transversely isotopic, where S represents transverse stabilization. Thus we have the following result.

Proposition 2.4. *Let T, T' be transverse links. Then the following properties satisfy (1) \Rightarrow (2) \Rightarrow (3):*

- (1) T, T' are flype-equivalent;
- (2) $S(T)$ and $S(T')$ are transversely isotopic;
- (3) $sl(T) = sl(T')$ and T, T' are topologically isotopic.

It is known that (3) does not necessarily imply (2): by [ELT12], there are transverse knots representing certain cables of torus knots that have the same self-linking number but require an arbitrarily large number of stabilizations to become transversely isotopic. We do not know if (2) necessarily implies (1), although it seems unlikely.

2.2. Transverse and Legendrian links. Another approach to transverse links is through Legendrian links. Any (oriented) Legendrian link can be C^0 perturbed to a well-defined transverse link, its *(positive) transverse pushoff*, and Legendrian links that are Legendrian isotopic have transverse pushoffs that are transversely isotopic. Conversely, any transverse link can be C^0 perturbed to a Legendrian link called a *Legendrian approximation*, though the Legendrian approximation is only well-defined up to negative Legendrian stabilizations. (Recall that there are two stabilization operations on Legendrian links, positive and negative Legendrian stabilization $L \mapsto S_{\pm}(L)$, that are well-defined on Legendrian isotopy classes: in the front projection to the xz plane, each stabilization replaces a piece of the Legendrian link by a two-cusped zigzag, where the cusps are oriented downwards (respectively, upwards) for positive (respectively, negative) stabilization.)

Thus there is a many-to-one correspondence between Legendrian links and transverse links, under which transverse links up to transverse isotopy correspond to Legendrian links up to Legendrian isotopy and negative Legendrian stabilization and destabilization. If L is a Legendrian link and T is its transverse pushoff, then the classical invariants of L and T , the Thurston–Bennequin and rotation numbers $tb(L), rot(L)$ and the self-linking number $sl(T)$, are related by $sl(T) = tb(L) - rot(L)$. Furthermore, the transverse stabilization $S(T)$ is the transverse pushoff of $S_+(L)$.

Definition 2.5. Two Legendrian links L, L' are *SZ-equivalent* if there are Legendrian links $L_0 = L, L_1, \dots, L_k = L'$ such that for each $i = 1, \dots, k$, L_{i-1} and L_i are Legendrian isotopic to Legendrian links whose xz projections are identical except for one of the tangle replacements shown in Figure 2.2. Similarly define *SZ₋-equivalent* (respectively, *SZ₊-equivalent*) if we restrict to only the tangle replacement on the left (respectively, right) of Figure 2.2.

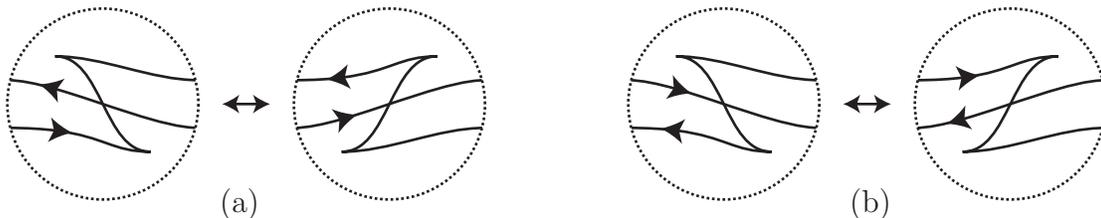


FIGURE 2.2. An SZ_- move (left) and an SZ_+ move (right) on Legendrian links.

Two transverse links are *SZ-equivalent* if they are the transverse pushoffs of *SZ*-equivalent Legendrian links.

Note that *SZ*-equivalent Legendrian links are topologically isotopic and have the same Thurston–Bennequin and rotation numbers. (One can generalize the move shown in Figure 2.2 to arbitrary orientations, in which case tb is preserved but rot is not necessarily preserved.) Indeed, the tangle replacements in Figure 2.2, which have previously appeared in the literature (e.g., [EN03]) though without the name “*SZ* moves”, are a key tool in constructing Legendrian links with the same topological type and classical invariants that are not necessarily Legendrian isotopic. The Chekanov 5_2 knots [Che02] are *SZ*-equivalent; *SZ* moves also appear in some guise in [EFM01, NOT08, ENV13], among other papers.

Accordingly, *SZ*-equivalent transverse links are topologically isotopic and have the same self-linking number, and so they provide good candidates for possibly distinct transverse links that share the same classical invariants. An observation of [EFM01], in our language, states that if L, L' are SZ_+ -equivalent (respectively, SZ_- -equivalent), then $S_+(L)$ and $S_+(L')$ (respectively, $S_-(L)$ and $S_-(L')$) are Legendrian isotopic. Thus we have the following result.

Proposition 2.6. *Let L, L' be Legendrian links, with transverse pushoffs T, T' . If L, L' are SZ_- -equivalent, then T, T' are transversely isotopic. Furthermore, the following properties satisfy (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4):*

- (1) L, L' are *SZ*-equivalent;
- (2) T, T' are *SZ*-equivalent;
- (3) $S(T), S(T')$ are transversely isotopic (recall S denotes transverse stabilization);
- (4) $sl(T) = sl(T')$ and T, T' are topologically isotopic.

Remark 2.7. Note that a *negative* stabilization of a braid results in a transverse stabilization of the corresponding transverse link, while a *positive* stabilization of a Legendrian link produces a transverse stabilization for the transverse pushoff.

2.3. Relation between equivalences. We have discussed two techniques to produce candidates for transverse links that are topologically equivalent and have the same self-linking number but may not be transversely isotopic: negative flypes for braids (Section 2.1) and SZ_+ moves for Legendrian links (Section 2.2). Here we show that these two techniques are identical.

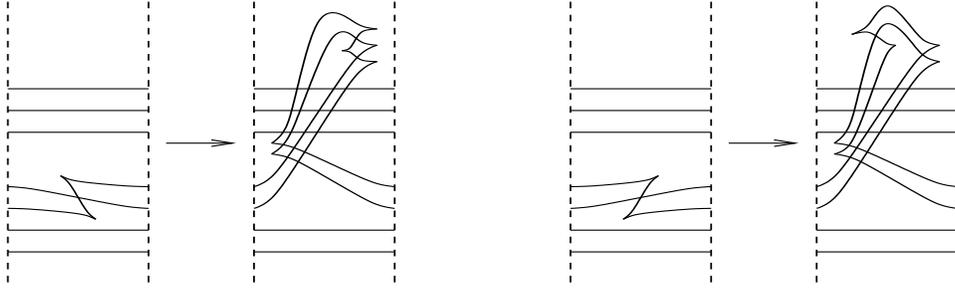


FIGURE 2.3. Changing two fronts related by an SZ_+ move. Each is changed by a Legendrian isotopy.

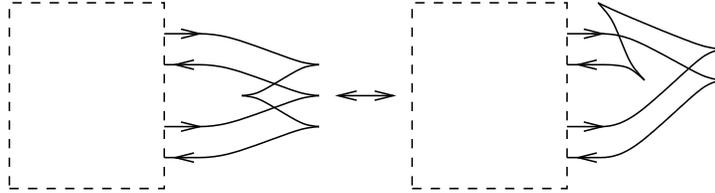


FIGURE 2.4. Fronts related by an improved SZ_+ move. The diagrams are assumed to coincide inside the dashed boxes.

Proposition 2.8. *Transverse links are flype-equivalent if and only if they are SZ -equivalent.*

Note that since positive flypes and SZ_- moves do not change transverse type, one can replace “ SZ -equivalent” by “ SZ_+ -equivalent”, and similarly restrict to negative flypes, in the statement of Proposition 2.8.

Proof of Proposition 2.8. We use grid diagrams as an intermediary between braids, Legendrian links, and transverse links; see [NT09] for the necessary background. For our purposes, a grid diagram is a link diagram consisting entirely of non-collinear horizontal and vertical line segments, with vertical segments crossing over horizontal segments wherever they intersect.

First suppose that two transverse links are the pushoffs of Legendrian links whose front projections are related by an SZ_+ move. By applying the Legendrian isotopies shown in Figure 2.3, we may assume that the fronts of the Legendrian links are related by the “improved” SZ_+ move shown in Figure 2.4. Now represent these fronts by grid diagrams as in the left-hand diagrams in Figure 2.5, which agree within the dashed boxes. (Here we use that grid diagrams can be viewed as fronts for Legendrian links by rotating them 45° counterclockwise and smoothing corners.)

Given a grid diagram G representing a Legendrian link L , we can produce a braid representing the transverse pushoff of L , as follows. Replace any vertical segment oriented

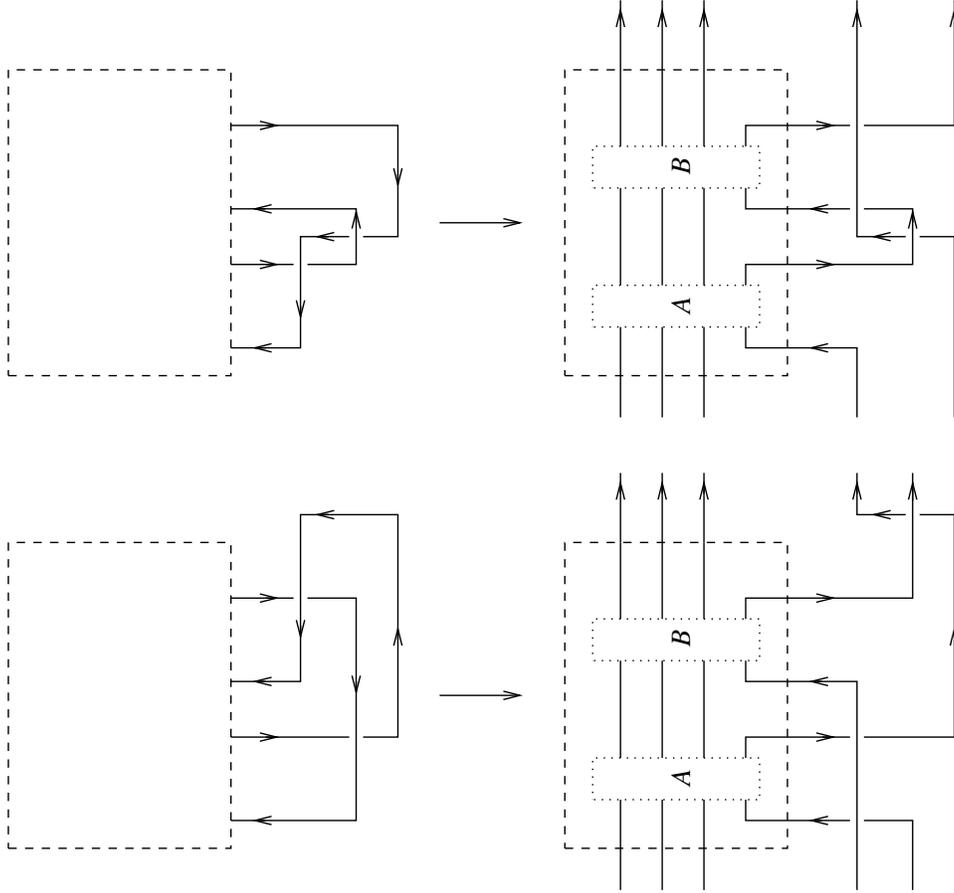


FIGURE 2.5. **Grid and braid diagrams for an improved SZ_+ move.**
 Left: grid diagrams related by an improved SZ_+ move; right: the braids obtained from these diagrams.

downwards (from point p to point q , say) by two half-infinite vertical segments, one pointing upwards from p , and the other pointing upwards to q , and again impose the condition that vertical segments pass over horizontal segments. The result is the braid, read bottom to top; in the language of [NT09], this is $B_{\uparrow}(G)$. When we apply this procedure to the grid diagrams on the left of Figure 2.5, we obtain the braids on the right of Figure 2.5. (Strictly speaking, some of the vertical segments in the right diagrams should be perturbed slightly to avoid collinearity.) These braids are of the form $A\sigma_m^2 B\sigma_m^{-1}$ and $\sigma_m A\sigma_m^{-1} B\sigma_m$ for some $A, B \in B_m$, and are thus related by a negative flype (along with conjugation).

Conversely, suppose that two transverse links are represented by braids differing by a negative flype $A\sigma_m^k B\sigma_m^{-1} \longleftrightarrow A\sigma_m^{-1} B\sigma_m^k$; in the figures, we assume that $k = 3$, but the case

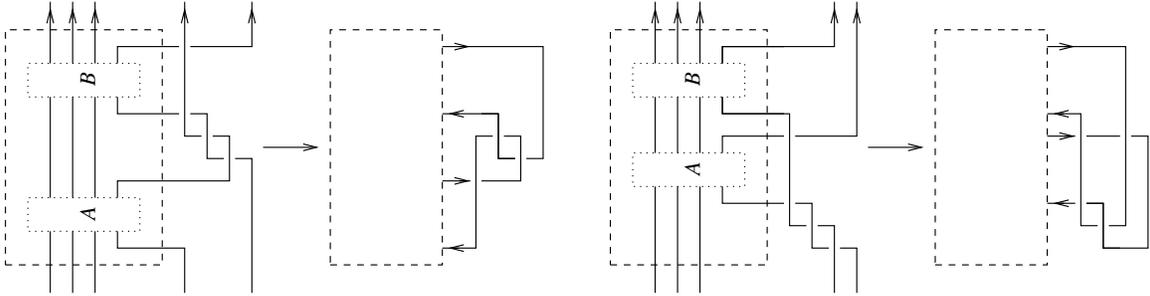


FIGURE 2.6. **Braid and grid diagrams for a negative flype.** As usual, the diagrams coincide within corresponding dashed boxes.

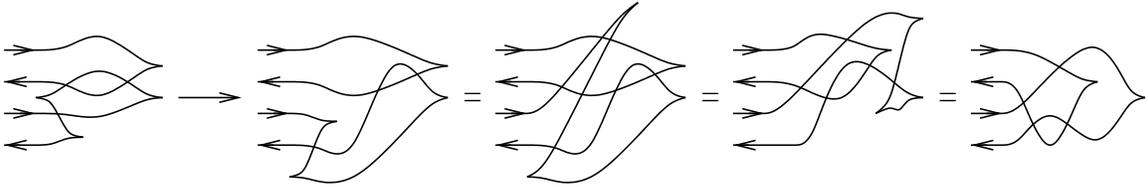


FIGURE 2.7. **An SZ_- move followed by Legendrian isotopy.**

of general k is nearly identical. Draw the braids as in Figure 2.6. Reversing the procedure from above yields the grid diagrams shown in Figure 2.6. These grid diagrams, in turn, correspond to Legendrian links that are related by an SZ_+ move and Legendrian isotopy. See Figure 2.7. \square

3. SOME CONSTRUCTIONS FROM KHOVANOV HOMOLOGY

In this section, we collect some standard constructions from Khovanov homology. Specifically, in Section 3.1 we recall the Bar-Natan deformation of the Khovanov complex, and in Section 3.2 we recall the homotopy equivalences of Bar-Natan complexes induced by Reidemeister moves.

3.1. The Bar-Natan deformation of Khovanov homology. The Bar-Natan deformation of the Khovanov complex (from [BN05]) comes from a 2-dimensional Frobenius algebra $V = \mathbb{Z}\langle x_-, x_+ \rangle$, with multiplication m given by

$$x_+ \otimes x_+ \mapsto x_+ \quad x_+ \otimes x_- \mapsto x_- \quad x_- \otimes x_+ \mapsto x_- \quad x_- \otimes x_- \mapsto x_-,$$

and comultiplication Δ by

$$x_- \mapsto x_- \otimes x_- \quad x_+ \mapsto x_+ \otimes x_- + x_- \otimes x_+ - x_+ \otimes x_+.$$

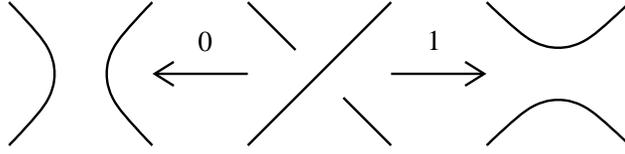


FIGURE 3.1. Resolutions of a crossing.

Without the terms in **red** this is $H^*(S^2)$, which underlies Khovanov homology. It is also useful to introduce the variable $x_+ = x_- - x_+$;⁽ⁱ⁾ with respect to the basis $\{x_-, x_+\}$ the multiplication and comultiplication diagonalize to

$$(3.1) \quad \begin{array}{ccccccc} x_+ \otimes x_+ & \xrightarrow{m} & -x_+ & & x_+ \otimes x_- & \xrightarrow{m} & 0 & & x_- \otimes x_+ & \xrightarrow{m} & 0 & & x_- \otimes x_- & \xrightarrow{m} & x_- \\ & & & & x_+ & \xrightarrow{\Delta} & x_+ \otimes x_+ & & x_- & \xrightarrow{\Delta} & x_- \otimes x_- & & & & \end{array}$$

Let K be an oriented link diagram with n crossings. Ordering the crossings of K induces an identification of the set of complete resolutions of K with $\{0, 1\}^n$. (Our convention for which is the 0-resolution and which is the 1-resolution is given in Figure 3.1.) Let $\mathcal{C}(K)$ be the chain complex generated by pairs (v, x) where $v \in \{0, 1\}^n$ and x is a labeling of each circle in the complete resolution K_v corresponding to v by x_+ or x_- . (That is, as a vector space, $\mathcal{C}(K)$ is isomorphic to the Khovanov complex $\mathcal{C}_{Kh}(K)$.) The differential is defined exactly as for the Khovanov complex, except that we use Bar-Natan's Frobenius algebra (above) in place of Khovanov's Frobenius algebra.

Gradings will be of some importance. Let n_- (respectively, n_+) be the number of negative (respectively, positive) crossings in K . Given $v \in \{0, 1\}^n$ let $|v| = \sum v_i$ denote the weight of v . Then the *homological grading* on $\mathcal{C}(K)$ (or $\mathcal{C}_{Kh}(K)$) is given by

$$(3.2) \quad \text{gr}_h(v, x) = -n_- + |v|.$$

The differential on $\mathcal{C}(K)$ (obviously) increases gr_h by 1.

The quantum grading on the Khovanov complex becomes a *quantum filtration* (or *q-filtration*) on the complex $\mathcal{C}(K)$; it is given by

$$(3.3) \quad \text{gr}_q(v, x) = n_+ - 2n_- + |v| + \#\{Z \in K_v \mid x(Z) = x_+\} - \#\{Z \in K_v \mid x(Z) = x_-\}.$$

That is, up to the normalization $n_+ - 2n_-$, $\text{gr}_q(v, x)$ is given by the weight of v plus the number of x_+ 's in x minus the number of x_- 's in x . The differential on $\mathcal{C}(K)$ satisfies

$$\text{gr}_q(\delta(v, x)) \geq \text{gr}_q(v, x).$$

Let $\mathcal{F}_m \mathcal{C}(K)$ denote the part of $\mathcal{C}(K)$ in filtration $\geq m$, i.e.,

$$\mathcal{F}_m \mathcal{C}(K) = \text{span}\{(v, x) \mid \text{gr}_q(v, x) \geq m\}.$$

⁽ⁱ⁾This definition of x_+ is the negative of the one used in [LSb].

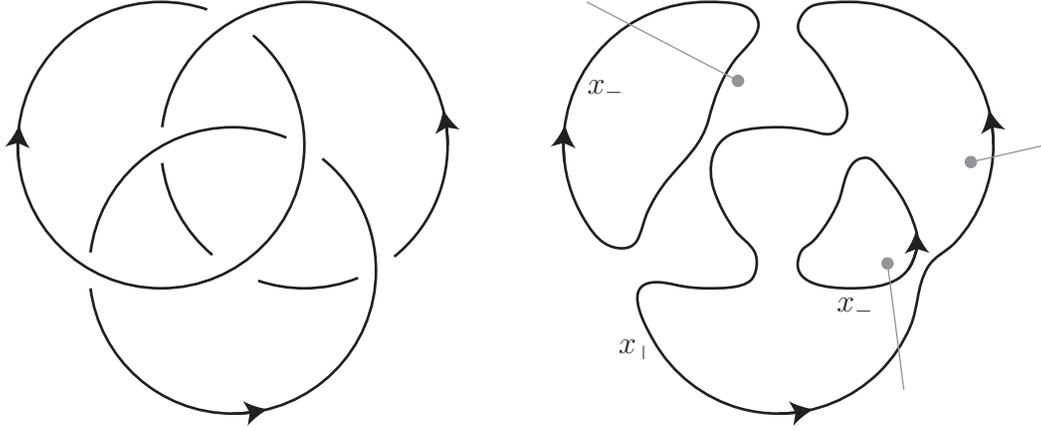


FIGURE 3.2. **Canonical generators of the Bar-Natan complex.** Left: an oriented link. Right: the corresponding oriented resolution with the associated Bar-Natan-Lee-Rasmussen-Turner generator. (The points q_C and arcs A are indicated in gray.)

Let $\mathcal{C}^n(K)$ be the part of $\mathcal{C}(K)$ in homological grading n , i.e.,

$$\mathcal{C}^n(K) = \text{span}\{(v, x) \mid \text{gr}_h(v, x) = n\}.$$

Following Lee [Lee05] (cf. [Ras10]), Turner showed that the homology of $\mathcal{C}(K)$ is $(\mathbb{Z} \oplus \mathbb{Z})^{\otimes |K|}$ (where $|K|$ denotes the number of components of K) [Tur06]. Moreover, the generators of $H^*(\mathcal{C}(K))$ correspond to orientations of K , as follows. Given an orientation o of K , there is a corresponding complete resolution K_o of K , the *oriented resolution*. Each circle C in K_o inherits an orientation from K . Fix a point p_C on C and let q_C be the result of pushing p_C slightly to the left of C (with respect to the orientation of C). Let A be an arc from q_C to ∞ , transverse to all of the circles in K_o . If A crosses an even number of circles then label C by x_- ; if A crosses an odd number of circles, label C by x_+ . See Figure 3.2. This labeling is the cycle $\psi(o)$ of $\mathcal{C}(K)$ corresponding to o . Note that this depended on an arbitrary universal choice: we could equally well exchange x_- and x_+ in the definition.

Remark 3.1. Lee [Lee05] considered another deformation of the Khovanov complex, corresponding to the Frobenius structure

$$\begin{aligned} x_+ \otimes x_+ &\xrightarrow{m} x_+ & x_+ \otimes x_- &\xrightarrow{m} x_- & x_- \otimes x_+ &\xrightarrow{m} x_- & x_- \otimes x_- &\xrightarrow{m} x_+, \\ x_- &\xrightarrow{\Delta} x_- \otimes x_- + x_+ \otimes x_+ & x_+ &\xrightarrow{\Delta} x_+ \otimes x_- + x_- \otimes x_+. \end{aligned}$$

Mackaay-Turner-Vaz showed that over a ring R in which 2 is invertible, the Lee deformation is twist-equivalent (in the sense of [Kho06b]) to the Bar-Natan deformation [MTV07]. In particular, the results below apply to the Lee deformation as well, if we work over a ring

in which 2 is invertible. This gives no (obvious) additional transverse information, but can be useful for studying the Plamenevskaya invariant; see Proposition 4.13.

3.2. The maps induced by Reidemeister moves. For us, a *filtered chain complex with distinguished generators* is a chain complex generated freely over \mathbb{Z} by a generating set, where each element of the generating set carries a homological grading gr_h and a filtration grading gr_q , such that the differential δ increases the homological grading by 1 and does not decrease the filtration grading. The Bar-Natan chain complex $\mathcal{C}(K)$ (see Section 3.1) is an example of a filtered chain complex with distinguished generators: the generators in which each circle is labeled by either x_+ or x_- .

The following is a standard cancellation lemma about filtered chain complexes with distinguished generators. Since we make repeated use of it for writing down the Reidemeister maps and for proving locality properties, we give a proof of it.

Lemma 3.2. *Let (C, δ_C) be a filtered chain complex with distinguished generators. Let α and β be two of the generators such that $\mathbb{Z}\langle\alpha, \beta\rangle$ is a subcomplex (respectively, a quotient complex) of C , where the coefficient $\langle\delta_C\alpha, \beta\rangle$ of β in $\delta_C(\alpha)$ is ± 1 , and $\text{gr}_q(\alpha) = \text{gr}_q(\beta)$. Let (D, δ_D) be the quotient complex (respectively, the subcomplex) of C generated by the remaining generators, with δ_D given by applying δ_C and then setting to 0 any generators not in D (respectively restricting δ_C to D). Then the quotient map $f: C \rightarrow D$ (respectively, the inclusion map $g: D \rightarrow C$) induces a filtered chain homotopy equivalence; that is, there is a filtered chain map $g: D \rightarrow C$ (respectively, $f: C \rightarrow D$) and filtered homotopies $h: C \rightarrow C$, $k: D \rightarrow D$, such that $h \circ \delta_C + \delta_C \circ h = \text{Id}_C - g \circ f$ and $k \circ \delta_D + \delta_D \circ k = \text{Id}_D - f \circ g$; and in fact one can take $k = 0$.*

Proof. Assume, after replacing β by $-\beta$ if necessary, that $\langle\delta_C\alpha, \beta\rangle = 1$. In the first case, when $\mathbb{Z}\langle\alpha, \beta\rangle$ is a subcomplex of C , define $g: D \rightarrow C$ as

$$g(x) = x - \langle\delta_C x, \beta\rangle\alpha,$$

and in the second case, when $\mathbb{Z}\langle\alpha, \beta\rangle$ is a quotient complex of C , define $f: C \rightarrow D$ as

$$f(x) = \begin{cases} 0 & \text{if } x = \alpha, \\ -\sum_y \langle\delta_C\alpha, y\rangle y & \text{if } x = \beta, \\ x & \text{otherwise.} \end{cases}$$

In either case, define $k = 0$ and $h: C \rightarrow C$ as

$$h(x) = \begin{cases} \alpha & \text{if } x = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that these maps have the desired properties. \square

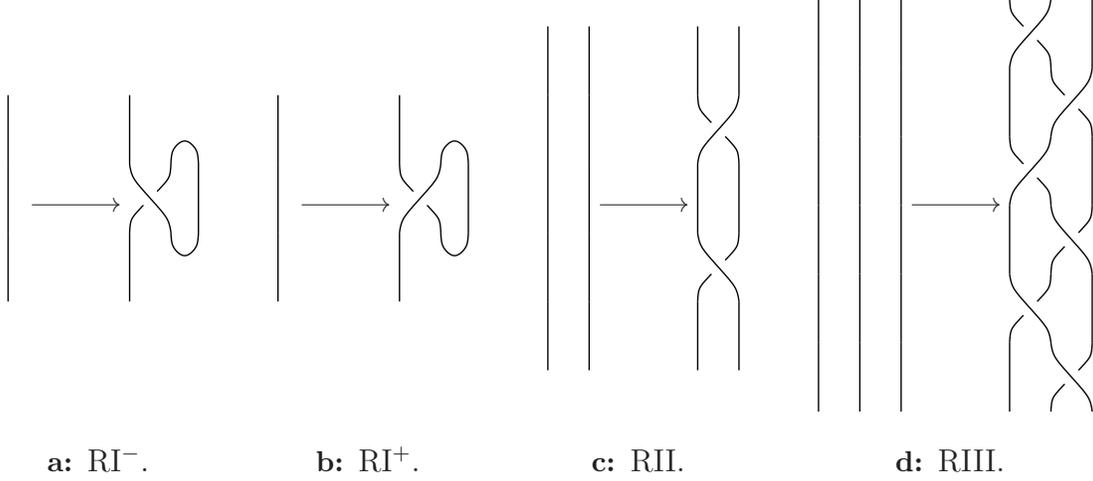


FIGURE 3.3. **The four Reidemeister moves.** (a) is the Reidemeister I negative stabilization, (b) is the Reidemeister I positive stabilization, (c) is the Reidemeister II move, and (d) is the braid-like Reidemeister III move. Among these four, (b), (c), and (d) may be viewed as the transverse Markov-Reidemeister moves.

Remark 3.3. Lemma 3.2 fails (even in the non-filtered case) if one does not work with distinguished generators: the two step chain complex $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ admits a quotient map to $\mathbb{Z}/2\mathbb{Z}$ which does not induce a chain homotopy equivalence.

Now, let K and K' be two link diagrams representing isotopic links. Then K and K' can be related by a sequence of Reidemeister moves (Figure 3.3); and a sequence of Reidemeister moves connecting K to K' induces a filtered chain homotopy equivalence between $\mathcal{C}(K)$ and $\mathcal{C}(K')$. We discuss these filtered chain homotopy equivalences next.

3.2.1. *Negative stabilization and destabilization.* Consider the negative stabilization shown in Figure 3.3a. Let c denote the new crossing in the stabilized diagram K' , and let $(K')_0$ (respectively, $(K')_1$) denote the result of replacing c by its 0-resolution (respectively, 1-resolution). Then $(K')_0$ is isomorphic to the unstabilized diagram K , while $(K')_1$ is isomorphic to the disjoint union of K and an unknot U_0 . There is a subcomplex D of $\mathcal{C}(K')$ spanned by the generators in $\mathcal{C}((K')_1)$ in which U_0 is labeled by x_+ ; D can be identified, preserving the bigrading, with $\mathcal{C}(K)$ by forgetting the component U_0 . The map associated to stabilization is the inclusion map

$$\mathcal{C}(K) \cong D \rightarrow \mathcal{C}(K').$$

The generators of $\mathcal{C}(K')$ not in D cancel in pairs, via the arrows $x \mapsto x \otimes x_-$. Since D is obtained from $\mathcal{C}(K')$ by cancellations of the form described in Lemma 3.2, the destabilization map is given by its homotopy inverse.

3.2.2. Positive stabilization and destabilization. Next, consider the positive stabilization shown in Figure 3.3b. Once again, let c denote the new crossing in the stabilized diagram K' , and let $(K')_0$ (respectively, $(K')_1$) denote the result of replacing c by its 0-resolution (respectively, 1-resolution). This time, $(K')_1$ is isomorphic to the unstabilized diagram K , while $(K')_0$ is isomorphic to the disjoint union of K and an unknot U_0 . There is a subcomplex E of $\mathcal{C}(K')$ spanned by the generators in $\mathcal{C}((K')_1)$ and the generators in $\mathcal{C}((K')_0)$ in which U_0 is labeled by x_+ ; and $\mathcal{C}(K')/E$ is identified with $\mathcal{C}(K)$ by forgetting the component U_0 . The map associated to destabilization is the projection

$$\mathcal{C}(K') \rightarrow \mathcal{C}(K')/E \cong \mathcal{C}(K).$$

Since E can be contracted by cancellations of the form described in Lemma 3.2, the stabilization map is given by the homotopy inverse to the destabilization map.

3.2.3. Reidemeister II. The maps giving Reidemeister II invariance can be described quite explicitly. Assume K' is obtained from K by the move from Figure 3.3c. Order the two new crossings in K' and for any $v \in \{0, 1\}^2$, let $(K')_v$ denote the partial resolution of K' at the two new crossings corresponding to v . Choose the ordering of the new crossings so that $(K')_{10} = K$. Then $(K')_{01}$ has an unknot component, say U_0 , contained in the isotopy region. Let E be the subcomplex of $\mathcal{C}(K')$ spanned by the generators in $\mathcal{C}((K')_{11})$ and the generators in $\mathcal{C}((K')_{01})$ where U_0 is labeled by x_+ ; therefore, $\mathcal{C}((K')_{10})$ is a subcomplex of $\mathcal{C}(K')/E$ and we have the following projection and inclusion maps:

$$\mathcal{C}(K') \xrightarrow{\pi} \mathcal{C}(K')/E \xleftarrow{\iota} \mathcal{C}((K')_{10}) \cong \mathcal{C}(K).$$

Both the complexes E and $(\mathcal{C}(K')/E)/\mathcal{C}((K')_{10})$ can be contracted by cancellations of the form described in Lemma 3.2, and therefore both π and ι have filtered homotopy inverses, say π^{-1} and ι^{-1} . The map from $\mathcal{C}(K')$ to $\mathcal{C}(K)$ is $\iota^{-1} \circ \pi$ and the map from $\mathcal{C}(K)$ to $\mathcal{C}(K')$ is $\pi^{-1} \circ \iota$.

3.2.4. Reidemeister III. The maps giving the usual Reidemeister III invariance can also be written explicitly, but doing so is somewhat tedious. Instead, we adopt the following indirect argument, which is similar in spirit to the previous argument. As in [Bal11, Section 7.3], since we have already proved Reidemeister II invariance, it suffices to prove invariance under the braid-like Reidemeister III move of Figure 3.3d. So, suppose K and K' differ by a braid-like Reidemeister III move, where K' has 6 more crossings than K . Order the six new crossings of K' , and for any $v \in \{0, 1\}^6$, let $(K')_v$ denote the resolution at the new crossings corresponding to v . Choose the ordering of the new crossings in K' so that $(K')_{111000} = K$. It is shown in the proof of [LSa, Proposition 6.4] that there is a contractible subcomplex

E of $\mathcal{C}(K')$ so that $\mathcal{C}((K')_{111000})$ is a subcomplex of $\mathcal{C}(K')/E$ and $(\mathcal{C}(K')/E)/\mathcal{C}((K')_{111000})$ is contractible. (Actually, in [LSa, Proposition 6.4], the corresponding statement for the Khovanov chain complex $(\mathcal{C}_{Kh}, \delta_{Kh})$, which is the associated graded object of the filtered Bar-Natan chain complex (\mathcal{C}, δ) , is proved; however, by looking at the homological gradings, we see that the subcomplexes and quotient complexes for \mathcal{C}_{Kh} remain subcomplexes and quotient complexes for \mathcal{C} as well, and therefore, the statement for the associated graded object implies the statement for the filtered complex.) This gives a diagram

$$\mathcal{C}(K') \xrightarrow{\pi} \mathcal{C}(K')/E \xleftarrow{\iota} \mathcal{C}((K')_{111000}) \cong \mathcal{C}(K).$$

Furthermore, it is shown in the proof of [LSa, Proposition 6.4] that both the acyclic complexes E and $(\mathcal{C}(K')/E)/\mathcal{C}((K')_{111000})$ can be contracted by sequences of elementary cancellations of the form described in Lemma 3.2. Therefore, both ι and π are filtered homotopy equivalences, and the homotopy equivalences f and g are gotten by inverting (up to filtered homotopy) either ι or π .

3.2.5. *Locality of the invariance maps.* We conclude this subsection with the observation that the Reidemeister maps are local in a particular sense, a fact that we will need in Section 4.4. (This is well-known. Other versions of locality are exploited, for instance, in [BN05] and [Kho06a].)

Proposition 3.4. *Suppose that K_1 is a link diagram and $T_1 \subset K_1$ is a tangle. Let T_2 be a tangle diagram representing a tangle isotopic to T_1 , and let $K_2 = (K_1 \setminus T_1) \cup T_2$ be the result of replacing T_1 with T_2 in K_1 . Let $f: \mathcal{C}(K_1) \rightarrow \mathcal{C}(K_2)$ and $g: \mathcal{C}(K_2) \rightarrow \mathcal{C}(K_1)$ be the filtered maps induced by an isotopy from T_1 to T_2 , as above, and let h and k denote the filtered homotopies, so that $\text{Id} - g \circ f = \delta \circ h + h \circ \delta$ and $\text{Id} - f \circ g = \delta \circ k + k \circ \delta$.*

Order the crossings of K_i ($i = 1, 2$) so that the crossings in $K_i \setminus T_i$ come before the crossings in T_i ; and so that the orderings of the crossings in $K_1 \setminus T_1 = K_2 \setminus T_2$ agree. Suppose that there are n crossings in $K_i \setminus T_i$, and n_i crossings in T_i .

Then:

- *Given a generator $((u, v), x) \in \mathcal{C}_{Kh}(K_1)$, with $(u, v) \in \{0, 1\}^n \times \{0, 1\}^{n_1} \cong \{0, 1\}^{n+n_1}$, $f((u, v), x)$ lies over the sub-cube $\{u\} \times \{0, 1\}^{n_2}$ of $\{0, 1\}^{n+n_2}$; and $h((u, v), x)$ lies over the sub-cube $\{u\} \times \{0, 1\}^{n_1}$ of $\{0, 1\}^{n+n_1}$.*

Moreover, if C is a circle in the resolution $(K_1)_{(u,v)}$ disjoint from T_1 , and C' is the corresponding circle in some resolution $(K_2)_{(u,w)}$, then the label $f((u, v), x)(C')$ of C' in $f((u, v), x)$ and the label $h((u, v), x)(C)$ of C in $h((u, v), x)$ are the same as the label $x(C)$ of C in x .

- *Given a generator $((u, w), y) \in \mathcal{C}_{Kh}(K_2)$, with $(u, w) \in \{0, 1\}^n \times \{0, 1\}^{n_2} \cong \{0, 1\}^{n+n_2}$, $g((u, w), y)$ lies over the sub-cube $\{u\} \times \{0, 1\}^{n_1}$ of $\{0, 1\}^{n+n_2}$; and $k((u, w), y)$ lies over the sub-cube $\{u\} \times \{0, 1\}^{n_2}$ of $\{0, 1\}^{n+n_2}$.*

Moreover, if C is a circle in the resolution $(K_2)_{(u,w)}$ disjoint from T_2 , and C' is the corresponding circle in some resolution $(K_2)_{(u,w)}$, then the label $g((u, w), y)(C')$ of C'

in $g((u, w), y)$ and the label $k((u, w), y)(C)$ of C in $k((u, w), y)$ are the same as the label $y(C)$ of C in y .

Proof. This is immediate from the form of the maps given in Sections 3.2.1–3.2.4; the key point is that the subcomplexes and quotient complexes, and the arrows one cancels (Lemma 3.2) to prove that the inclusion and projection maps are filtered homotopy equivalences, are all given locally. \square

4. THE FILTERED PLAMENEVSKAYA INVARIANT

In this section we define the (filtered extension of the) Plamenevskaya invariant (Section 4.1) and prove it is invariant under transverse isotopies (Section 4.2). We then make some observations on its behavior under negative stabilization (Section 4.3) before proving our main theorem, invariance under flypes and *SZ* moves (Section 4.4). After this, we make a few further observations guaranteeing that the filtered Plamenevskaya invariant does no better than the classical invariants at distinguishing transverse representatives of low-crossing knots (Section 4.5).

4.1. The definition of the invariant. Let K be a transverse link in S^3 (with respect to the standard contact structure ξ_{std}), presented as the closure of an (oriented) braid. Abusing notation slightly, we will also use K to denote the corresponding link diagram.

Definition 4.1. Recall from Section 3.1 that associated to each orientation o of K is a cycle $\psi(o)$ in $\mathcal{C}(K)$. The *positive* (respectively, *negative*) *filtered Plamenevskaya invariant* of K is the generator $\psi^+(K) := \psi(o)$ (respectively, $\psi^-(K) := \psi(-o)$) corresponding to the usual orientation of K as a transverse link (respectively, the opposite orientation to the usual one). Since most of the results in this paper will apply to both $\psi^+(K)$ and $\psi^-(K)$, we will use $\psi(K)$ to denote either $\psi^+(K)$ or $\psi^-(K)$. The invariant $\psi(K)$ lies in homological grading

$$\mathrm{gr}_h(\psi(K)) = -n_- + n_+ = 0.$$

With respect to the q -filtration, the lowest filtered part of $\psi(\pm o)$ is when each circle is labeled by x_- ; this lies in filtration

$$\mathrm{gr}_q(\psi(K)) = n_+ - n_- - m = sl(K).$$

(Here m denotes the braid index.) We thus regard $\psi(K)$ as an element of $\mathcal{F}_{sl(K)}\mathcal{C}^0(K)$.

The sense in which $\psi(K)$ is an invariant of the transverse isotopy class of K is made precise in Theorem 4.2. Note that it follows from (the argument in) [Lee05] that $\psi(K) \in \mathcal{C}^0(K)$ is a cycle; this can also be seen directly from the fact that the only differentials from the oriented resolution correspond to merge cobordisms, and $m(x_+ \otimes x_-) = m(x_- \otimes x_+) = 0$, cf. Formula (3.1).

The filtered Plamenevskaya invariant induces a number of auxiliary invariants. For any $p \leq 0 < q$, let

$$\psi_{p,q}(K) \in \mathcal{F}_{sl(K)+2p}\mathcal{C}^0(K)/\mathcal{F}_{sl(K)+2q}\mathcal{C}^0(K)$$

denote the image of $\psi(K)$ under the obvious chain map $\mathcal{F}_{sl(K)}\mathcal{C} \rightarrow \mathcal{F}_{sl(K)+2p}\mathcal{C}/\mathcal{F}_{sl(K)+2q}\mathcal{C}$, and let

$$[\psi_{p,q}(K)] \in H^0(\mathcal{F}_{sl(K)+2p}\mathcal{C}(K)/\mathcal{F}_{sl(K)+2q}\mathcal{C}(K))$$

denote the image of $\psi_{p,q}$ in the homology. Note: $\psi_{0,\infty}(K) = \psi(K)$; $[\psi_{-\infty,\infty}^+]$ and $[\psi_{-\infty,\infty}^-]$ are two of the generators of $H^*(\mathcal{C})$ corresponding to the orientations o and $-o$ respectively; and the element

$$\psi_{0,1}(K) \in \mathcal{F}_{sl(K)}\mathcal{C}^0(K)/\mathcal{F}_{sl(K)+2}\mathcal{C}^0(K) = \mathcal{C}_{Kh}^{0,sl(K)}(K)$$

is the transverse invariant defined by Plamenevskaya [Pla06]. In particular, $\psi_{0,1}^+(K) = \psi_{0,1}^-(K)$. Indeed, since the lowest-filtration parts of $\psi^+(K)$ and $\psi^-(K)$ are the same (every circle decorated by x_-), the difference element $\psi^{diff}(K) := \psi^+(K) - \psi^-(K)$ is (a cycle) in $\mathcal{F}_{sl(K)+2}\mathcal{C}(K)$. For $p \leq 1 < q$, let $\psi_{p,q}^{diff}(K)$ denote the image of $\psi^{diff}(K) \in \mathcal{F}_{sl(K)+2}\mathcal{C}(K)$ in the subquotient $\mathcal{F}_{sl(K)+2p}\mathcal{C}/\mathcal{F}_{sl(K)+2q}\mathcal{C}$. Then we have $\psi_{1,\infty}^{diff}(K) = \psi^{diff}(K) \in \mathcal{F}_{sl(K)+2}\mathcal{C}(K)$, while $\psi_{0,\infty}^{diff}(K) = \psi^+(K) - \psi^-(K) \in \mathcal{F}_{sl(K)}\mathcal{C}(K)$ is determined by the invariants $\psi^+(K)$ and $\psi^-(K)$.

As an aside, note that the Rasmussen invariant $s(K)$ has the following description:

$$s(K) = sl(K) - 1 + 2(\min\{q \mid [\psi_{-\infty,q}^-] \neq 0\}).$$

In particular, this implies the upper bound on the self-linking number first observed by Plamenevskaya [Pla06, Section 7]:

$$s(K) \geq sl(K) - 1 + 2 = sl(K) + 1.$$

4.2. Invariance under transverse isotopies. In this subsection we prove transverse invariance of $\psi(K)$ and $\psi^{diff}(K)$. We start with $\psi(K)$:

Theorem 4.2. *Suppose that K and K' are diagrams for closed braids, and that the corresponding transverse links are transversely isotopic. Then the filtered homotopy equivalence $f: \mathcal{C}(K) \rightarrow \mathcal{C}(K')$, induced by some sequence of transverse Markov moves connecting K to K' , satisfies*

$$f(\psi(K)) = \pm\psi(K') + \delta\phi$$

(for one choice of $+$ or $-$), where $\phi \in \mathcal{F}_{sl(K)}\mathcal{C}^{-1}(K')$. Moreover, the filtered homotopy equivalence f and the element ϕ can be chosen to be local in the sense of Proposition 3.4.

Corollary 4.3. *The image $[\psi_{p,q}]$ of $\psi_{p,q}$ in homology $H^0(\mathcal{F}_{sl(K)+2p}\mathcal{C}(K)/\mathcal{F}_{sl(K)+2q}\mathcal{C}(K))$ is an invariant of the transverse link K .*

The case $p = 0, q = 1$ is [Pla06, Theorem 2]. The general case is not significantly different, but we spell out the proof.

Proof of Theorem 4.2. By the transverse Markov theorem [Wri02, OS03], it suffices to prove that $\psi(K)$ is invariant under (braid-like) Reidemeister II and III moves and positive stabilizations and destabilizations (see Figures 3.3b–3.3d). The relevant maps are described in Section 3.2.2–3.2.4. We discuss each move briefly in turn.

For positive destabilization, with notation as in Section 3.2.2, note that $\psi(K)$ lies in $\mathcal{C}((K')_0)$, and has U_0 labeled by either x_- or x_+ (and the other components of the oriented resolution labeled alternately by x_+ and x_-). In either case, $\psi(K')$ survives to $\mathcal{C}(K')/D$ and is identified with $\psi(K')$ under the identification $\mathcal{C}(K')/D \cong \mathcal{C}(K)$. Positive stabilization is the homotopy inverse to positive destabilization, and hence also preserves ψ (up to the boundary of an element of $\mathcal{F}_{sl(K)}\mathcal{C}(K)$).

Reidemeister II and III are easier. For Reidemeister II, with notation as in Section 3.2.3, $\psi(K')$ lies in $\mathcal{C}((K')_{10})$, and so $\pi(\psi(K'))$ is exactly $\iota(\psi(K))$. Since the two filtered homotopy equivalences are obtained by inverting either π or ι (up to homotopy), they respect ψ . Similarly, for Reidemeister III, with notation as in Section 3.2.4, $\psi(K')$ lies in $\mathcal{C}((K')_{111000})$, so again $\pi(\psi(K')) = \iota(\psi(K))$, and hence the filtered homotopy equivalences respect ψ .

The fact that the map f and element ϕ can be chosen to be local follows from Proposition 3.4. \square

Remark 4.4. The above proof of Theorem 4.2 actually shows more generally that maps induced by transverse isotopy preserve any linear combination $\alpha\psi^+ + \beta\psi^-$ (not just ψ^+ and ψ^-), up to a boundary in $\mathcal{F}_{sl(K)}\mathcal{C}$ and an overall sign.

Now we turn to the invariant $\psi^{diff}(K)$. By Remark 4.4, $\psi^+(K) - \psi^-(K) = \psi_{0,\infty}^{diff}(K)$ is an invariant in $\mathcal{F}_{sl(K)}\mathcal{C}(K)$; but in fact more is true:

Theorem 4.5. *The element $\psi^{diff}(K) \in \mathcal{F}_{sl(K)+2}\mathcal{C}(K)$ is an invariant of the transverse link K ; more precisely, if $f: \mathcal{C}(K) \rightarrow \mathcal{C}(K')$ is the filtered homotopy equivalence induced by some sequence of transverse Markov moves connecting closed braids K and K' , then*

$$f(\psi^{diff}(K)) = \pm\psi^{diff}(K') + \delta\phi$$

for some $\phi \in \mathcal{F}_{sl(K)+2}\mathcal{C}^{-1}(K')$.

Proof. As in the proof of Theorem 4.2, we check invariance under positive (de)stabilization and braid-like Reidemeister II and III moves. For positive destabilization, we verified in the proof of Theorem 4.2 that $\psi^+(K')$ (respectively, $\psi^-(K')$) maps to $\psi^+(K)$ (respectively, $\psi^-(K)$) on the nose. It follows that the destabilization map takes their difference $\psi^{diff}(K')$ to $\psi^{diff}(K)$. The stabilization map is a filtered homotopy inverse to the destabilization map, and hence takes $\psi^{diff}(K)$ to $\psi^{diff}(K')$ up to the boundary of an element of $\mathcal{F}_{sl(K)+2}\mathcal{C}(K)$. This proves invariance under positive (de)stabilization. The proof of Reidemeister II and III invariance is exactly as in Theorem 4.2 (with ψ^{diff} in place ψ). \square

Remark 4.6. The reason for the \pm signs in Theorems 4.2 and 4.5 is perhaps not obvious: the Reidemeister maps we wrote down respect ψ (respectively, ψ^{diff}) exactly, not just up to

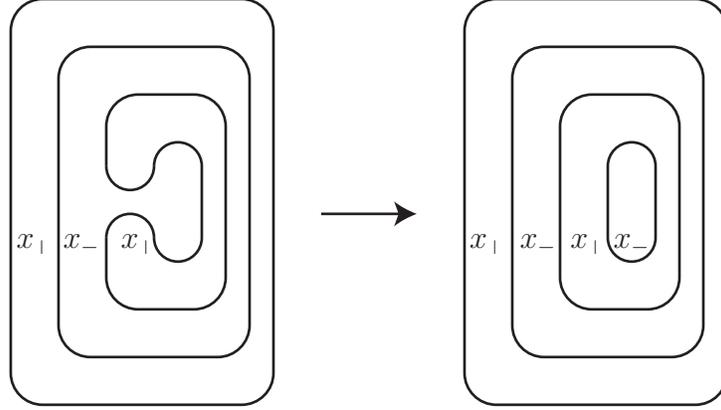


FIGURE 4.1. **Invariance under negative stabilization.** The generator θ' is shown on the left, and $\psi(K')$ on the right, both in the case that $x = x_-$.

sign. The \pm signs arise because the maps on Khovanov homology (and indeed, the Khovanov homology groups themselves) are only well-defined up to an overall sign.

4.3. Behavior under negative stabilization.

Proposition 4.7. *Let K be a closed braid in S^3 and let K' be the result of negatively stabilizing K once, so $sl(K') = sl(K) - 2$. Let $\psi(K) \in \mathcal{F}_{sl(K)}\mathcal{C}(K)$ and $\psi(K') \in \mathcal{F}_{sl(K')}\mathcal{C}(K')$ be the corresponding filtered Plamenevskaya invariants and let $f: \mathcal{C}(K) \rightarrow \mathcal{C}(K')$ and $g: \mathcal{C}(K') \rightarrow \mathcal{C}(K)$ be the filtered chain maps giving Reidemeister I invariance (cf. Section 3.2.1). Then there are elements $\theta \in \mathcal{F}_{sl(K')}\mathcal{C}^{-1}(K)$ and $\theta' \in \mathcal{F}_{sl(K')}\mathcal{C}^{-1}(K')$, local in the sense of Proposition 3.4, such that*

$$g(\psi(K')) = \pm\psi(K) + \delta\theta \text{ in } \mathcal{F}_{sl(K')}\mathcal{C}(K)$$

and

$$f(\psi(K)) = \pm\psi(K') + \delta\theta' \text{ in } \mathcal{F}_{sl(K')}\mathcal{C}(K').$$

Proof. With notation as in Section 3.2.1, $\psi(K')$ lies in $\mathcal{C}((K')_1)$ but not in the subcomplex D . Let $x \in \{x_-, x_+\}$ denote the label of U_0 in $\psi(K')$, and let $\theta' \in \mathcal{C}((K')_0)$ be the element which labels the circles alternately by x_- and x_+ , starting by labeling the inner-most circle oppositely from x . (So, θ' and x give the same labels to circles occurring both in $(K')_0$ and $(K')_1$; see Figure 4.1.) Assume the crossings in K' are ordered so that the new crossing c is the first, and the other crossings are ordered as in K . Then $\psi(K') - \delta\theta'$ lies in D ; and under the identification $D \cong \mathcal{C}(K)$, it is identified with $\psi(K)$ (respectively, $-\psi(K)$) if x equals x_- (respectively, x_+). The element θ' lies in filtration $sl(K')$, verifying that f has the specified form. (Note that θ' is local in the sense of Proposition 3.4.) It follows that the filtered homotopy inverse g to f also respects ψ , as an element of $\mathcal{F}_{sl(K')}\mathcal{C}$; the fact that the cancellations are local implies that we can take θ to be local, as well. \square

Remark 4.8. In a similar vein to Remark 4.4, observe that (up to an overall sign, and relative boundary in $\mathcal{F}_{sl(K')}\mathcal{C}$) the negative stabilization map sends the linear combination $\alpha\psi^+(K) + \beta\psi^-(K)$ to $\alpha\psi^+(K') - \beta\psi^-(K')$, and the negative destabilization map sends the linear combination $\alpha\psi^+(K') + \beta\psi^-(K')$ to $\alpha\psi^+(K) - \beta\psi^-(K)$.

Corollary 4.9. *If transverse links K and K' become transversely isotopic after performing k transverse (negative) stabilizations then $\psi_{p,\infty}(K) = \pm\psi_{p,\infty}(K')$ for any $p \leq -k$. That is, if $f: \mathcal{C}(K) \rightarrow \mathcal{C}(K')$ is the map associated to an isotopy from K to K' corresponding to k transverse stabilizations, then a transverse isotopy, and then k transverse destabilizations, then $f(\psi(K)) = \pm\psi(K') + \delta\theta$ for some $\theta \in \mathcal{F}_{sl(K)-2k}\mathcal{C}^{-1}(K')$.*

Proof. This is immediate from Theorem 4.2 (which covers transverse isotopies) and Proposition 4.7 (which covers negative stabilizations and destabilizations). \square

The following was proved by Rasmussen [Ras10] in the case of the Lee deformation; we will use it in Section 4.4:

Corollary 4.10. *Reidemeister moves preserve filtered Plamenevskaya generators up to homotopy. That is, let K and K' be diagrams for isotopic links and let $f: \mathcal{C}(K) \rightarrow \mathcal{C}(K')$ be the map of Bar-Natan complexes induced by a sequence of Reidemeister moves connecting K and K' . Then $f(\psi(K)) = \pm\psi(K') + \delta\theta$ for some $\theta \in \mathcal{C}^{-1}(K')$.*

Proof. This follows immediately from Corollary 4.9. Note that in general we do not have any control over the filtration level in which θ lies. \square

Corollary 4.11. *If a transverse link K can be negatively destabilized k times, then $[\psi_{0,k}(K)] = 0$.*

Proof. Let K' be a transverse link so that $S^k(K') = K$. Let $f: \mathcal{C}(K') \rightarrow \mathcal{C}(K)$ be the filtered chain map corresponding to k negative stabilizations. Proposition 4.7 ensures the existence of some $\theta \in \mathcal{F}_{sl(K)}\mathcal{C}(K)$ so that

$$f(\psi(K')) = \pm\psi(K) + \delta\theta.$$

However, since f is a filtered map, $f(\psi(K'))$ lies in $\mathcal{F}_{sl(K')}\mathcal{C}(K) = \mathcal{F}_{sl(K)+2k}\mathcal{C}(K)$. Therefore, letting $\theta_{0,k}$ denote the image of θ in $\mathcal{F}_{sl(K)}\mathcal{C}(K)/\mathcal{F}_{sl(K)+2k}\mathcal{C}(K)$, we have

$$\psi_{0,k}(K) = \mp\delta(\theta_{0,k})$$

and hence $[\psi_{0,k}(K)] = 0$. \square

Remark 4.12. Note that, by the s -invariant upper bound on self-linking number [Pla06], we get

$$sl(K') = sl(K) + 2k \leq s(K) - 1 = sl(K) - 2 + 2(\min\{q \mid [\psi_{-\infty,q}(K)] \neq 0\}),$$

or

$$\min\{q \mid [\psi_{-\infty,q}(K)] \neq 0\} \geq k + 1.$$

Corollary 4.11 furnishes a (possibly) stronger inequality:

$$\min\{q \mid [\psi_{0,q}(K)] \neq 0\} \geq k + 1.$$

We conclude this subsection with an observation regarding the original Plamenevskaya invariant $\psi_{0,1}$ and the property (2) from Proposition 2.4.

Proposition 4.13. *Let K and K' be transverse links so that $S(K) = S(K')$. Fix a ring R in which 2 is invertible. Let $f: \mathcal{C}_{Kh}(K; R) \rightarrow \mathcal{C}_{Kh}(K'; R)$ be the chain map on the Khovanov chain complexes over R associated to an isotopy from K to K' corresponding to a single negative stabilization, then a transverse isotopy, and then a single negative destabilization. Then $f(\psi_{0,1}(K)) = \alpha\psi_{0,1}(K') + \delta_{Kh}(\phi)$ for some $\phi \in \mathcal{C}_{Kh}^{-1,sl(K)}(K'; R)$ and some unit α in R .*

Proof. Consider the Lee filtered chain complex $(\mathcal{C}_{Lee}, \delta_{Lee})$ from [Lee05]. Let $\psi_{Lee}(K)$ be the Lee generator corresponding the usual orientation of K ; it is a cycle lying in filtration $\mathcal{F}_{sl(K)}\mathcal{C}_{Lee}(K)$, and its lowest filtration term is also $\psi_{0,1}(K)$.

However, by [MTV07, Proposition 2.2], over the ring R , the Bar-Natan filtered complex (\mathcal{C}, δ) is twist equivalent to the Lee filtered complex $(\mathcal{C}_{Lee}, \delta_{Lee})$ (see Remark 3.1). Therefore, over the ring R , Corollary 4.9 also holds for the Lee complex. That is, we have

$$f(\psi_{Lee}(K)) = \alpha\psi_{Lee}(K') + \delta_{Lee}(\theta)$$

for some $\theta \in \mathcal{F}_{sl(K)-2}\mathcal{C}_{Lee}(K')$ and some unit $\alpha \in R$.

Since δ_{Lee} preserves the quantum grading mod 4, there is a quotient map

$$\mathcal{F}_{sl(K)-2}\mathcal{C}_{Lee} \twoheadrightarrow Kh^{*,sl(K)}.$$

Letting $\phi \in Kh^{-1,sl(K)}$ denote the image of θ under this quotient map, we get the desired identity:

$$f(\psi_{0,1}(K)) = \alpha\psi_{0,1}(K') + \delta_{Kh}(\phi). \quad \square$$

Remark 4.14. Although both the original Plamenevskaya invariant $\psi_{0,1}$ and the filtered version ψ are invariant under negative flypes (see Theorem 4.15 below), Proposition 4.13 suggests one way in which we might be hopeful that the filtered invariant could be effective even if the original invariant were not. To our knowledge, it is conceivable that there could be transverse knots that become the same after one stabilization, but are not related by a sequence of negative flypes; compare conditions (1) and (2) in Proposition 2.4. Such knots are indistinguishable using $\psi_{0,1}$ by Proposition 4.13, but could possibly be distinguished using the filtered invariant ψ .

4.4. Invariance under negative flypes. In this subsection we prove that the filtered Plamenevskaya invariant is unchanged by negative flypes (Theorem 4.15), and that the difference invariant ψ^{diff} is unchanged by a subclass of negative flypes (Theorem 4.19).

Theorem 4.15. *The filtered Plamenevskaya invariant is invariant under negative flypes. That is, if K and K' are related by a negative flype and $f: \mathcal{C}(K) \rightarrow \mathcal{C}(K')$ is the map*

associated to the sequence of Reidemeister moves (2.1) then $f(\psi(K)) = \pm\psi(K') + \delta\phi$ (for one choice of $+$ or $-$), where $\phi \in \mathcal{F}_{sl(K)}\mathcal{C}(K')$.

Corollary 4.16. *The filtered Plamenevskaya invariant is invariant under flype and SZ equivalence.*

Before proving Theorem 4.15, we fix some notation. For the rest of the subsection, fix braids A and B as in Definition 2.2 and let K (respectively, K') be the braid closure of $A\sigma_m^k B\sigma_m^{-1}$ (respectively, $A\sigma_m^{-1} B\sigma_m^k$).

Let $n + |k| + 1$ be the number of crossings of K . Order the crossings of K and K' so that the first n crossings lie in $A \cup B$ and the $(n + 1)^{\text{st}}$ crossing is σ_m^{-1} . Let $u \in \{0, 1\}^n$ correspond to the oriented resolution of $A \cup B$; that is, u is 0 at each σ_i and 1 at each σ_i^{-1} .

Let $f: \mathcal{C}(K) \rightarrow \mathcal{C}(K')$ be the filtered chain homotopy equivalence induced by the sequence of moves (2.1). By Corollary 4.10,

$$f(\psi(K)) = \pm\psi(K') + \delta\phi$$

for some $\phi \in \mathcal{C}^{-1}(K')$. We want to show that $\phi \in \mathcal{F}_{sl(K)}\mathcal{C}(K')$.

Lemma 4.17. *With the notations from above, the following statements hold.*

- (1) *If $k \geq 0$, the elements $\psi(K)$ and $\psi(K')$ lie over the vertex $u \times (1, 0, 0, \dots, 0) \in \{0, 1\}^{n+|k|+1}$; otherwise, they lie over the vertex $u \times (1, 1, 1, \dots, 1) \in \{0, 1\}^{n+|k|+1}$.*
- (2) *If $k \geq 0$, the element $f(\psi(K))$ lies over the vertices $u \times (\epsilon_1, \dots, \epsilon_{|k|+1})$ where exactly one of the ϵ_i is 1 (and the rest are 0); otherwise, it lies over the vertex $u \times (1, \dots, 1)$.*
- (3) *If $k \geq 0$, the element ϕ lies over the vertex $u \times (0, 0, 0, \dots, 0) \in \{0, 1\}^{n+|k|+1}$; otherwise, it lies over the vertices $u \times (\epsilon_1, \dots, \epsilon_{|k|+1})$ where exactly one of the ϵ_i is 0 (and the rest are 1).*

Proof. The first part of the statement is immediate from the definitions: the oriented resolution corresponds to taking the 1-resolution at each σ_i and the 0-resolution at each σ_i^{-1} . The second follows from the first part, locality of the map f (Proposition 3.4) and the fact that f respects the homological grading. The third part follows from locality (Proposition 3.4) and the fact that ϕ lies in homological grading 1 lower than $\psi(K')$. \square

Lemma 4.18. *If $k \geq 0$, then over the vertex $u \times (0, 0, \dots, 0)$, $\mathcal{F}_{sl(K)}\mathcal{C}$ is all of \mathcal{C} ; that is, if $(v, x) \in \mathcal{C}$ with $v = u \times (0, 0, 0, \dots, 0)$, then $\text{gr}_q(v, x) \geq sl(K)$. Similarly, if $k < 0$, and if $(v, x) \in \mathcal{C}$ with $v = u \times (\epsilon_1, \dots, \epsilon_{|k|+1})$ where exactly one of the ϵ_i is 0, then $\text{gr}_q(v, x) \geq sl(K)$.*

Proof. The minimal quantum grading occurs when all of the circles are decorated by x_- . In each of the above cases, there are m circles in K_v (here $(m + 1)$ is the braid index). The weight $|v|$ is $n_- - 1$, so Formula (3.3) gives

$$\text{gr}_q(v, (x_-, \dots, x_-)) = n_+ - 2n_- + n_- - 1 - m = n_+ - n_- - (m + 1) = sl(K),$$

as claimed. \square

Proof of Theorem 4.15. As noted earlier, the element ϕ is given by Corollary 4.10; we must show that $\phi \in \mathcal{F}_{sl(K)}\mathcal{C}(K')$. But this follows from the third part of Lemma 4.17 in conjunction with Lemma 4.18. \square

The story for the invariant ψ^{diff} is slightly subtler. The flype isotopy from Formula (2.1) involves a single negative stabilization followed by a transverse isotopy followed by a single negative destabilization. Up to an overall sign, and relative boundary, the negative stabilization sends $\psi^+ - \psi^-$ to $\psi^+ + \psi^-$ (Remark 4.8), the transverse isotopy preserves $\psi^+ + \psi^-$ (Remark 4.4), and the negative destabilization sends $\psi^+ + \psi^-$ to $\psi^+ - \psi^-$. Therefore,

$$f(\psi^{diff}(K)) = \pm\psi^{diff}(K') + \delta\phi$$

for some $\phi \in \mathcal{C}^{-1}(K')$. Furthermore, as in the proof of Theorem 4.15, the homological grading and locality of ϕ force ϕ to lie in filtration level $\mathcal{F}_{sl(K)}\mathcal{C}(K')$. This shows that the invariant $\psi_{0,\infty}^{diff}$ is preserved under a general flype. However, for special types of flypes, more can be said.

Theorem 4.19. *With braids A and B as in Definition 2.2, consider the flype isotopy between the braid closures K and K' of $A\sigma_m^k B\sigma_m^{-1}$ and $A\sigma_m^{-1} B\sigma_m^k$. Furthermore, assume $k \geq 0$. If $f: \mathcal{C}(K) \rightarrow \mathcal{C}(K')$ is the map associated to such a flype, then $f(\psi^{diff}(K)) = \pm\psi^{diff}(K') + \delta\phi$ (for one choice of $+$ or $-$), where $\phi \in \mathcal{F}_{sl(K)+2}\mathcal{C}(K')$.*

Proof. From the discussion above, we see that $f(\psi^{diff}(K)) = \pm\psi^{diff}(K') + \delta\phi$ for some $\phi \in \mathcal{F}_{sl(K)}\mathcal{C}^{-1}(K')$. Indeed, reusing earlier notations and proofs, we see that the homological grading and locality of ϕ forces it to lie over the vertex $v = u \times (0, 0, \dots, 0)$. Let x be the labeling that labels each of the m circles in $(K')_v$ by x_- . Then $\phi_0 = (v, x)$ is the unique generator over v that lies in quantum grading $sl(K)$. Therefore, ϕ can be written as $\alpha\phi_0 + \phi_1$ for some $\alpha \in \mathbb{Z}$ and some $\phi_1 \in \mathcal{F}_{sl(K)+2}\mathcal{C}^{-1}(K')$. It follows that

$$(4.1) \quad \alpha\delta(\phi_0) = f(\psi^{diff}(K)) \mp \psi^{diff}(K') - \delta(\phi_1).$$

If $\psi_{0,1}(K')$ denotes the original Plamenevskaya generator (which lies over the oriented resolution and labels all the circles by x_-), then $\langle \delta\phi_0, \psi_{0,1} \rangle = \pm 1$. However, all the terms of the right side of the Equation (4.1) lie in filtration level $\mathcal{F}_{sl(K)+2}\mathcal{C}(K')$. Since $\psi_{0,1}(K')$ is a homogenous element in quantum grading $sl(K)$, this forces $\alpha = 0$. Hence, $\phi = \phi_1 \in \mathcal{F}_{sl(K)+2}\mathcal{C}^{-1}(K')$. \square

Remark 4.20. We do not know if Theorem 4.19 holds generally without the assumption $k \geq 0$. However, for all of the examples we know where negative flypes produce putatively distinct transverse knots, k is at least 2.

4.5. Triviality for simple links. In Section 4.4, we saw that the filtered Plamenevskaya invariant ψ remains invariant under negative flypes. In this subsection, we will see that ψ cannot distinguish different transverse representatives of any link with particularly simple Khovanov homology.

Proposition 4.21. *Let K and K' be transverse links with the same topological link type and with $sl(K) = sl(K')$. Further assume $H^{-1}(\mathcal{C}(K)/\mathcal{F}_{sl(K)}\mathcal{C}(K)) = 0$. If $f: \mathcal{C}(K') \rightarrow \mathcal{C}(K)$ is the filtered chain map corresponding to some sequence of Reidemeister moves connecting K' to K , then there exists some $\phi \in \mathcal{F}_{sl(K)}\mathcal{C}^{-1}(K)$ so that*

$$f(\psi(K')) = \pm\psi(K) + \delta\phi.$$

Proof. Consider the following short exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}_{sl(K)}\mathcal{C}^{-1}(K) & \xrightarrow{\iota} & \mathcal{C}^{-1}(K) & \xrightarrow{\pi} & \mathcal{C}^{-1}(K)/\mathcal{F}_{sl(K)}\mathcal{C}^{-1}(K) \longrightarrow 0 \\ & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ 0 & \longrightarrow & \mathcal{F}_{sl(K)}\mathcal{C}^0(K) & \xrightarrow{\iota} & \mathcal{C}^0(K) & \xrightarrow{\pi} & \mathcal{C}^0(K)/\mathcal{F}_{sl(K)}\mathcal{C}^0(K) \longrightarrow 0. \end{array}$$

Since f is a filtered map, $f(\psi(K')) \mp \psi(K) \in \mathcal{F}_{sl(K)}\mathcal{C}^0(K)$. By Corollary 4.10, there exists $\theta \in \mathcal{C}^{-1}(K)$, so that

$$\delta\theta = \iota(f(\psi(K')) \mp \psi(K)).$$

Therefore,

$$\delta(\pi(\theta)) = \pi(\delta\theta) = \pi(\iota(f(\psi(K')) \mp \psi(K))) = 0,$$

and hence $[\pi(\theta)] \in H^{-1}(\mathcal{C}(K)/\mathcal{F}_{sl(K)}\mathcal{C}(K))$ represents some homology element. Furthermore, the connecting homomorphism $H^{-1}(\mathcal{C}(K)/\mathcal{F}_{sl(K)}\mathcal{C}(K)) \rightarrow H^0(\mathcal{F}_{sl(K)}\mathcal{C}(K))$ maps $[\pi(\theta)]$ to $[f(\psi(K')) \mp \psi(K)]$. Since $H^{-1}(\mathcal{C}(K)/\mathcal{F}_{sl(K)}\mathcal{C}(K)) = 0$, $[\pi(\theta)] = 0$ and hence $[f(\psi(K')) \mp \psi(K)] = 0$ in $H^0(\mathcal{F}_{sl(K)}\mathcal{C}(K))$. Therefore, there exists some $\phi \in \mathcal{F}_{sl(K)}\mathcal{C}^{-1}(K)$ so that

$$f(\psi(K')) \mp \psi(K) = \delta\phi. \quad \square$$

Corollary 4.22. *Let K be a topological link, and let $\overline{sl}(K)$ be the maximal self-linking number among all transverse representatives of K . If $Kh^{-1,j}(K; \mathbb{Z}) = 0$ for all $j < \overline{sl}(K)$, then the filtered Plamenevskaya invariant does not distinguish transverse representatives of K with the same self-linking number.*

Proof. This follows immediately from Proposition 4.21. Since the Khovanov chain complex is the associated graded object of the Bar-Natan chain complex, $\bigoplus_{j < q} Kh^{-1,j}(K) = 0$ implies $H^{-1}(\mathcal{C}(K)/\mathcal{F}_q\mathcal{C}(K)) = 0$. \square

The maximal self-linking number is known for all knots through 11 crossings [Ng12], and Corollary 4.22 holds for all topological knot types up to 11 crossings except the ones in the following list (with $m(K)$ denoting the topological mirror of K):

$$\begin{aligned} & m(8_{20}), m(10_{125}), m(10_{126}), m(10_{130}), m(10_{141}), m(10_{143}), m(10_{148}), 10_{155}, \\ & m(10_{159}), 11n_{22}, 11n_{26}, 11n_{40}, 11n_{46}, m(11n_{50}), 11n_{51}, 11n_{54}, m(11n_{65}), \\ & 11n_{71}, m(11n_{75}), m(11n_{87}), m(11n_{127}), m(11n_{132}), m(11n_{138}), 11n_{146}, \\ & m(11n_{159}), m(11n_{172}), m(11n_{176}), 11n_{178}, 11n_{184}. \end{aligned}$$

We currently do not know of any distinct transverse representatives with the same self-linking number, flype-equivalent or otherwise, of any of the above knot types.

5. A COHOMOTOPY REFINEMENT OF THE GRADED PLAMENEVSKAYA INVARIANT

In [LSa] a space-level refinement of Khovanov homology was given. That is, let K be a link diagram. For each j there is a (formal desuspension of a) suspension spectrum of a CW complex $\mathcal{X}_{Kh}^j(K)$ so that $\tilde{H}^i(\mathcal{X}_{Kh}^j(K)) \cong Kh^{i,j}(K)$ and the homotopy type of $\mathcal{X}_{Kh}^j(K)$ is a link invariant.

In this section we give a space-level refinement of the Plamenevskaya invariant. That is:

Theorem 5.1. *Associated to a braid diagram K is a map*

$$\Psi(K): \mathcal{X}_{Kh}^{sl(K)}(K) \rightarrow \mathbb{S},$$

where \mathbb{S} is the sphere spectrum. The induced map on cohomology

$$\Psi(K)^*: \mathbb{Z} = \tilde{H}^0(\mathbb{S}) \rightarrow \tilde{H}^0(\mathcal{X}_{Kh}^{sl(K)}(K)) \cong Kh^{0,sl(K)}(K)$$

sends a generator of \mathbb{Z} to the graded Plamenevskaya invariant $[\psi_{0,1}(K)]$.

If K' is another braid diagram representing the same transverse link type then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{X}_{Kh}^{sl(K)}(K) & \xrightarrow{\Psi(K)} & \mathbb{S} \\ \Phi \downarrow \simeq & & \downarrow \simeq \\ \mathcal{X}_{Kh}^{sl(K)}(K') & \xrightarrow{\Psi(K')} & \mathbb{S}. \end{array}$$

(Here, Φ is the homotopy equivalence induced by a sequence of transverse Markov moves connecting K and K' , and the map $\mathbb{S} \rightarrow \mathbb{S}$ is a self-homotopy equivalence of the sphere spectrum.)

In other words, there is a transverse invariant $\Psi(K) \in \pi_s^0(\mathcal{X}_{Kh}^{sl(K)}(K))$, well-defined up to sign and automorphisms of $\mathcal{X}_{Kh}^{sl(K)}(K)$.

5.1. The definition of the invariant. Recall that the space $\mathcal{X}_{Kh}^j(L)$ is defined by feeding a framed flow category $\mathcal{C}_{Kh}^j(L)$ into the Cohen–Jones–Segal machine [CJS95] (see also [LSa, Definition 3.23]). We use $|\mathcal{C}|$ for the result of applying the Cohen–Jones–Segal construction to a framed flow category; note that this is not the same as taking the geometric realization of the category (even after viewing it as a topological category). We will need the following properties of this construction:

- (1) The space $|\mathcal{C}|$ has one cell for each object of \mathcal{C} (in addition to a basepoint).
- (2) If \mathcal{C}' is a full, upwards-closed sub-category of \mathcal{C} then there is a quotient map $|\mathcal{C}| \rightarrow |\mathcal{C}'|$ (gotten by collapsing the cells corresponding to objects not in \mathcal{C}' to the basepoint).
- (3) The category $\mathcal{C}_{Kh}^j(L)$ has one object for each generator (v, x) of $\mathcal{C}_{Kh}(L)$ with quantum grading j .

- (4) The space $\text{Hom}((v, x), (w, y))$ is non-empty if and only if (v, x) and (w, y) can be connected by a sequence of differentials in $\mathcal{C}_{Kh}(L)$.

Consider the graded Plamenevskaya generator (v, x) . That is, v corresponds to the oriented resolution and x labels each circle in v by x_- . By Property (3), this generator corresponds to an object (v, x) of $\mathcal{C}_{Kh}^{sl}(L)$. The generator (v, x) is a cycle, and hence by Property (4) the object (v, x) by itself is an upwards-closed subcategory of $\mathcal{C}_{Kh}^{sl}(L)$. By Property (1), the realization $|\{(v, x)\}|$ of the subcategory $\{(v, x)\}$ is a sphere S^N . By Property (2), there is a quotient map $|\mathcal{C}_{Kh}^{sl}(L)| \rightarrow |\{(v, x)\}| = S^N$. Formally desuspending N times gives the *cohomotopy Plamenevskaya invariant* $\Psi(K): \mathcal{X}_{Kh}^{sl(K)}(K) \rightarrow \mathbb{S}$.

5.2. Invariance under transverse isotopies. We turn now to invariance of the cohomotopy Plamenevskaya invariant, i.e., the proof of Theorem 5.1. First, one more piece of notation: the map $\Psi(K)$ is induced by a particular cell $e(K) \in |\mathcal{C}_{Kh}(K)|$, with the property that no higher-dimensional cells are attached to $e(K)$. The map $\Psi(K)$ is an umkehr (wrong way) map associated to $e(K)$, gotten by collapsing $|\mathcal{C}_{Kh}(K)| \setminus \text{interior}(e(K))$ to a point.

Proof of Theorem 5.1. We have already defined the map Φ , and it is obvious from the construction that the induced map on cohomology is the graded Plamenevskaya invariant. It remains to prove invariance. As in the proof of Theorem 4.2, we will check invariance under Reidemeister II, braid-like Reidemeister III, and positive stabilization. But first we need to verify independence of the auxiliary data used to construct $\mathcal{X}_{Kh}(K)$.

Fix a braid closure K . Recall that $\mathcal{X}_{Kh}(K)$ depends on the following auxiliary choices:

- An ordering of the crossings of L .
- A sign assignment s for the cube $\mathcal{C}(n)$.
- A neat embedding ι and a framing φ for the cube flow category $\mathcal{C}_C(n)$ relative to s .
- A framed neat embedding of the Khovanov flow category $\mathcal{C}_{Kh}(L)$ relative to some \mathbf{d} . This framed neat embedding is a perturbation of (ι, φ) .
- Integers A, B and real numbers ϵ, R used in the construction of the CW complex.

It is shown in [LSa, Proposition 6.1] that up to homotopy equivalence, the space $\mathcal{X}_{Kh}(K)$ is independent of these auxiliary choices. The maps coming from changing ϵ and R take each cell homeomorphically to the corresponding cell, and hence commute with the maps e and, hence, Φ (see [LSa, Lemma 3.25]). The maps for changing A, B and \mathbf{d} again take each cell by a degree 1 map to the corresponding cell (see [LSa, Lemma 3.26]) so again commute with e and so Φ . Any two perturbations of ι can be connected by a 1-parameter family. The corresponding map again is a homeomorphism taking cells to corresponding cells (this again comes from Lemma [LSa, Lemma 3.25]), so again commutes with e and Φ . The same applies to changing ι .

The proof of independence of the sign assignment goes as follows. Let K' be the disjoint union of K with a 1-crossing unknot U , so that the 0-resolution of U has two components (say). The cube of resolutions of K' has the cube of resolutions of K as two faces, say

f_0 and f_1 , corresponding to taking the 0 and 1 resolution at U , respectively. Given sign assignments s_0 and s_1 for K there is a sign assignment s for K' such that $s|_{f_i} = s_i$. Consider the subcomplex Y of $|\mathcal{C}_{Kh}(K'; s)|$ in which the circle(s) corresponding to U are labeled by x_+ . The subcomplex Y is contractible (because $\tilde{H}^*(Y) = 0$), and there is a cofibration sequence

$$|\mathcal{C}_{Kh}(K; s_0)| \rightarrow Y \rightarrow |\mathcal{C}_{Kh}(K; s_1)|.$$

The Puppe construction then gives the desired homotopy equivalence. Now, there is a map $\Xi: Y \rightarrow \mathbb{D}^{N+1}$ (for an appropriate N) so that the following diagram commutes:

$$\begin{array}{ccccc} |\mathcal{C}_{Kh}(K; s_0)| & \longrightarrow & Y & \longrightarrow & |\mathcal{C}_{Kh}(K; s_1)| \\ \Psi(K; s_0) \downarrow & & \Xi \downarrow & & \downarrow \Psi(K; s_1) \\ S^N & \longrightarrow & S^N \cup \mathbb{D}^{N+1} = \mathbb{D}^{N+1} & \longrightarrow & S^{N+1} = \mathbb{D}^{N+1} / \partial \mathbb{D}^{N+1}. \end{array}$$

Specifically, the map Ξ is given by sending the cell in f_0 corresponding to $e(K)$ to S^N and the cell in f_1 corresponding to $e(K)$ to \mathbb{D}^{N+1} . It follows that the Puppe map $|\mathcal{C}_{Kh}(K; s_1)| \rightarrow \Sigma|\mathcal{C}_{Kh}(K; s_0)|$ commutes with Ψ .

Finally, changing the ordering of the crossings has the same effect as a particular change of sign assignment, giving invariance under this as well. This completes the proof of invariance under the auxiliary choices.

Next, we turn to Reidemeister invariance. It is convenient in all cases to use the “subcomplex of quotient complex” trick that we used in Theorem 4.2 (where the complexes are, in fact, drawn from Sections 3.2.2–3.2.4).

We start with Reidemeister II. Suppose K' is obtained from K by a Reidemeister II move introducing two new crossings. Let n be the number of crossings in K . In [LSa, Proposition 6.3], the first and third authors constructed an upwards-closed subcategory \mathcal{C}_1 of $\mathcal{C}_{Kh}(K')$ so that $|\mathcal{C}_1|$ is contractible. Let \mathcal{C}_2 denote the complementary, downwards-closed subcategory of $\mathcal{C}_{Kh}(K')$. A further downwards-closed subcategory \mathcal{C}_3 of \mathcal{C}_2 was constructed so that $|\mathcal{C}_3|$ is contractible and so that the complement \mathcal{C}_4 of \mathcal{C}_3 is exactly $\mathcal{C}_{Kh}(K)$. Indeed, there is a vertex $u \in \{0, 1\}^2$ so that resolving the two new crossings of K' according to u gives K ; and \mathcal{C}_4 is the sub-cube $\{0, 1\}^n \times \{u\} \subset \{0, 1\}^{n+2}$. In particular, after making compatible choices of framed embeddings for the flow categories, there is a map $\mathbb{D}^n \rightarrow |\mathcal{C}_2|$ making the following diagram commute (on the nose):

$$\begin{array}{ccccc} & & \mathbb{D}^n & & \\ & e(K') \swarrow & \downarrow & \searrow e(K) & \\ |\mathcal{C}_{Kh}(K')| & \xleftarrow[\leftarrow{|\mathcal{C}_2|}]{\simeq} & |\mathcal{C}_2| & \xrightarrow[|\mathcal{C}_3|]{\simeq} & |\mathcal{C}_{Kh}(K)|. \end{array}$$

Reidemeister II invariance of Ψ follows by replacing $e(K)$ and $e(K')$ by the umkehr maps $\Psi(K): |\mathcal{C}_{Kh}(K)| \rightarrow \mathbb{D}^n / \partial \mathbb{D}^n$ and $\Psi(K'): |\mathcal{C}_{Kh}(K')| \rightarrow \mathbb{D}^n / \partial \mathbb{D}^n$.

The proof of braid-like Reidemeister III invariance is essentially the same; only the definitions of the \mathcal{C}_i change (see [LSa, Proposition 6.4]).

Finally, stabilization invariance is slightly easier. Suppose that K' is obtained from K by a positive stabilization. The proof of [LSa, Proposition 6.2] gives an upwards-closed subcategory \mathcal{C}_1 of $\mathcal{C}_{Kh}(K')$ so that $|\mathcal{C}_1|$ is contractible, and the complementary downwards-closed subcategory \mathcal{C}_2 of $\mathcal{C}_{Kh}(K')$ is identified with $\mathcal{C}_{Kh}(K)$. It is immediate from the definition of \mathcal{C}_1 that the image of $e(K') : \mathbb{D}^N \rightarrow |\mathcal{C}_{Kh}(K')|$ lies inside $|\mathcal{C}_2|$; and agrees with $e(K) : \mathbb{D}^N \rightarrow |\mathcal{C}_{Kh}(K)|$. So, again, Ψ is invariant. This concludes the proof. \square

Remark 5.2. A careful reader will observe that in the construction of $\mathcal{X}_{Kh}(K)$, one also made a global choice of ladybug matching (see [LSa, Definition 5.6]). For each of the two choices of ladybug matchings \mathbf{m}_1 and \mathbf{m}_2 , one gets spectra $\mathcal{X}_{Kh}^j(K; \mathbf{m}_i)$ and a transverse invariant $\Psi(K; \mathbf{m}_i) \in \pi_s^0(\mathcal{X}_{Kh}^{sl(K)}(K; \mathbf{m}_i))$. Although we show in [LSa, Proposition 6.5] that the spectra $\mathcal{X}_{Kh}^j(K; \mathbf{m}_1)$ and $\mathcal{X}_{Kh}^j(K; \mathbf{m}_2)$ are (stably) homotopy equivalent, we do not know if these homotopy equivalences carry $\Psi(K; \mathbf{m}_1)$ to $\Psi(K; \mathbf{m}_2)$. So, whenever we talk about $\Psi(K)$ without referencing the choice of ladybug matching, it is implicit that we have already made some (global) choice of ladybug matching.

On the other hand, let T be a transverse link and $B = \sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \cdots \sigma_{i_\ell}^{\epsilon_\ell}$ a braid representing T . Then $B^* = \sigma_{i_\ell}^{\epsilon_\ell} \cdots \sigma_{i_2}^{\epsilon_2} \sigma_{i_1}^{\epsilon_1}$ represents a potentially different transverse link T^* , called the *transverse mirror* of T [NT09]. The proof of [LSa, Proposition 6.5] does show that there is a homotopy equivalence between $\mathcal{X}_{Kh}(K; \mathbf{m}_1)$ and $\mathcal{X}_{Kh}(K^*; \mathbf{m}_2)$ that carries $\Psi(K; \mathbf{m}_1)$ to $\Psi(K^*; \mathbf{m}_2)$.

5.3. Consequences and computable invariants. We conclude by stating some immediate properties of the cohomotopy refinement of the Plemenevskaya invariant, and suggesting some further (computable) auxiliary invariants.

Corollary 5.3. *The graded Plamenevskaya invariant $[\psi_{0,1}(K)]$ lies in the image of the co-Hurewicz map $\pi_s^0(\mathcal{X}_{Kh}^{sl(K)}(K)) \rightarrow \tilde{H}_s^0(\mathcal{X}_{Kh}^{sl(K)}(K))$. In particular, if $\pi_s^0(\mathcal{X}_{Kh}^{sl(K)}(K)) = 0$ then $[\psi_{0,1}(K)] = 0$.*

Corollary 5.4. *If the co-Hurewicz map $\pi_s^0(\mathcal{X}_{Kh}^{sl(K)}(K)) \rightarrow \tilde{H}_s^0(\mathcal{X}_{Kh}^{sl(K)}(K))$ is injective then the cohomotopy Plamenevskaya invariant $\Psi(K)$ is determined by the graded Plamenevskaya invariant $[\psi_{0,1}(K)]$. In particular, if $Kh^{i,sl(K)}(K; \mathbb{Z}) = 0$ for all $i > 0$ then $\Psi(K)$ is determined by $[\psi_{0,1}(K)]$.*

Proof. The first part is immediate since $\Psi(K)$ maps to $[\psi_{0,1}(K)]$ under the co-Hurewicz map. The second part follows from the first part and the Hopf classification theorem (which is dual to Hurewicz theorem; see [Spa56]) which asserts that if $Kh^{i,sl(K)}(K; \mathbb{Z}) = \tilde{H}^i(\mathcal{X}_{Kh}^{sl(K)}(K); \mathbb{Z}) = 0$ for all $i > 0$, then the co-Hurewicz map $\pi_s^0(\mathcal{X}_{Kh}^{sl(K)}(K)) \rightarrow \tilde{H}_s^0(\mathcal{X}_{Kh}^{sl(K)}(K))$ is an isomorphism. \square

It can be checked that Corollary 5.4 applies to all topological knot types up to 11 crossings, and all 12-crossing knots except for $12n_{749}$. That is, for these knots, $Kh^{i,j}(K; \mathbb{Z}) = 0$ for all $i > 0$ and all $j \leq \overline{sl}(K)$ (recall \overline{sl} denotes the maximal self-linking number). Therefore, for any transverse representative of any of these knot types, Ψ will be determined by $[\psi_{0,1}]$. To date, we have not been able to use the cohomotopy Plamenevskaya invariant Ψ to distinguish transverse links with the same self-linking number.

We conclude by mentioning two strategies for using the cohomotopy invariant or the Khovanov homotopy type that could conceivably be useful. Suppose that K_1 and K_2 are transverse representatives for the same topological link type.

Idea 1. The spectrum $\mathcal{X}_{Kh}(K)$ induces stable cohomology operations like the Steenrod squares $Sq^k: \tilde{H}^i(\mathcal{X}_{Kh}^j(K); \mathbb{F}_2) \rightarrow \tilde{H}^{i+k}(\mathcal{X}_{Kh}^j(K); \mathbb{F}_2)$. Perhaps for some choice of k , we have $[\psi_{0,1}(K_1)] \in \text{im}(Sq^k)$ and $[\psi_{0,1}(K_2)] \notin \text{im}(Sq^k)$. Or, perhaps for some choice of k , $[\psi_{0,1}(K_1)] \in \text{ker}(Sq^k)$ and $[\psi_{0,1}(K_2)] \notin \text{ker}(Sq^k)$. Since the Reidemeister isomorphisms commute with the action of Sq^k , either of these phenomena would distinguish K_1 and K_2 .

In general, computing the Steenrod squares for an arbitrary CW complex (with exponentially many cells) is not so easy. In [LSc], it is explained how to compute the operation Sq^2 , and this would be a reasonable place to start exploring Idea 1.

Idea 2. Let $\text{Cone}(\Psi)$ denote the mapping cone of Ψ . Then the stable homotopy type of $\text{Cone}(\Psi(K))$ is a transverse invariant. So, perhaps for some choices of K_1 and K_2 , the mapping cones of $\Psi(K_i)$ distinguish the K_i .

For example, one could try to use the action of Steenrod squares to distinguish $\text{Cone}(\Psi(K_1))$ and $\text{Cone}(\Psi(K_2))$. In particular, it should be reasonably straightforward to extend techniques from [LSc] to compute the operation Sq^2 on $\text{Cone}(\Psi(K))$.

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E-mail address: lipshitz@math.columbia.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA, CHAPEL HILL, NC 27599

E-mail address: ng@math.duke.edu

DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY, DURHAM, NC 27708

E-mail address: sucharit@math.princeton.edu

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544