

AN L_∞ STRUCTURE FOR LEGENDRIAN CONTACT HOMOLOGY

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ABSTRACT. For any Legendrian knot or link in \mathbb{R}^3 , we construct an L_∞ algebra that can be viewed as an extension of the Chekanov–Eliashberg differential graded algebra. The L_∞ structure incorporates information from rational Symplectic Field Theory and can be formulated combinatorially. One consequence is the construction of a Poisson bracket on commutative Legendrian contact homology, and we show that the resulting Poisson algebra is an invariant of Legendrian links under isotopy.

1. INTRODUCTION

1.1. **Main results.** The modern study of the topology of 3-dimensional contact manifolds is inextricably tied to the study of Legendrian links, which are knots or links in the 3-manifold that are everywhere tangent to the contact structure. Associated to a Legendrian link is an algebraic invariant known as Legendrian contact homology (LCH), which dates back to work of Chekanov [Che02] and Eliashberg–Hofer [Eli98]. LCH, which is the homology of a complex now commonly called the Chekanov–Eliashberg differential graded algebra (DGA), is a key tool for studying both Legendrian links and the topology of associated contact and symplectic manifolds.

In this paper, we will restrict our attention to the basic setting of Legendrian links in \mathbb{R}^3 equipped with the standard contact structure $\ker(dz - y dx)$. From its original definition in [Che02], the study of Legendrian contact homology in \mathbb{R}^3 has developed into a rich subject. In just the \mathbb{R}^3 setting, one can use LCH to distinguish non-isotopic Legendrian links and analyze Lagrangian cobordisms between them; there are surgery formulas expressing invariants of symplectic manifolds constructed by critical handle attachment in terms of LCH; and LCH has connections to the theory of constructible sheaves (see e.g. [STZ17, NRS⁺20]) and cluster varieties (see e.g. [CGG⁺22, GSW20]). We refer the reader to [EN22] for a fairly recent survey of results concerning LCH in standard contact \mathbb{R}^3 .

The purpose of this paper is to extend the Chekanov–Eliashberg DGA for Legendrian links in \mathbb{R}^3 to a larger algebraic structure, namely the structure of an L_∞ algebra (homotopy Lie algebra). If we write (\mathcal{A}, ∂) for the DGA, then the L_∞ structure consists of a sequence of graded-symmetric multilinear maps $\ell_k : \mathcal{A}^{\otimes k} \rightarrow \mathcal{A}$, $k \geq 1$, satisfying a sequence of L_∞ relations. The first L_∞ operation ℓ_1 is the differential ∂ ; the second operation ℓ_2 is a homotopy

Lie bracket, with ℓ_1 being a derivation with respect to ℓ_2 and ℓ_2 satisfying the Jacobi identity up to correction terms involving ℓ_1 and ℓ_3 . In particular, ℓ_2 induces a Lie bracket on the homology $H_*(\mathcal{A}, \ell_1)$.

We now state our main results more precisely. We say that a *pointed Legendrian link* is an oriented Legendrian link with a collection of base points, at least one on each component. Let Λ be a pointed Legendrian link with s base points and n *Reeb chords*, which are integral curves of the Reeb vector field ($\partial/\partial z$ for standard contact \mathbb{R}^3) with endpoints on Λ . We associate to Λ the graded polynomial ring

$$\mathcal{A}^{\text{comm}} = \mathbb{Z}[q_1, \dots, q_n, t_1^{\pm 1}, \dots, t_s^{\pm 1}]$$

where the q_j 's represent Reeb chords, the t_i 's represent base points, and the grading is induced by a Maslov potential on Λ . This is the algebra underlying a differential graded algebra $(\mathcal{A}^{\text{comm}}, \partial)$, which is the commutative version of the Chekanov–Eliashberg DGA of Λ . Here the differential ∂ counts holomorphic disks in the symplectization $\mathbb{R} \times \mathbb{R}^3$ with boundary on the Lagrangian $\mathbb{R} \times \Lambda$, asymptotic to a single Reeb chord of Λ at $+\infty$ and some collection of Reeb chords at $-\infty$.

We note that the standard definition of the Chekanov–Eliashberg DGA involves a tensor algebra \mathcal{A} generated by the same generators as $\mathcal{A}^{\text{comm}}$: $q_1, \dots, q_n, t_1^{\pm 1}, \dots, t_s^{\pm 1}$. However, in the usual DGA, the q_j and t_i variables do not commute with each other; $\mathcal{A}^{\text{comm}}$ is the quotient of \mathcal{A} by the two-sided ideal generated by graded commutators. In order for the L_∞ relations to hold, we will be forced to use $\mathcal{A}^{\text{comm}}$ rather than \mathcal{A} ; see however Section 3.5 for a variant L_∞ structure in the case of single-component Legendrian knots that does retain some of the noncommutativity of \mathcal{A} .

The homology of the usual Chekanov–Eliashberg DGA (\mathcal{A}, ∂) is an invariant of the pointed Legendrian link Λ under Legendrian isotopy, and is called the Legendrian contact homology of Λ . The same is true for the commutative DGA once we tensor by a field \mathbb{k} of characteristic 0: it follows from [ENS02, Theorem 3.14] that the homology of $(\mathcal{A}^{\text{comm}} \otimes \mathbb{k}, \partial)$ is a Legendrian-isotopy invariant of Λ . We will refer to $H_*(\mathcal{A}^{\text{comm}} \otimes \mathbb{k}, \partial)$ as the commutative Legendrian contact homology of Λ and denote it by $LCH_*^{\text{comm}}(\Lambda)$.

The main construction of this paper is an L_∞ algebra structure on $\mathcal{A}^{\text{comm}}$. In fact it follows from the construction of the ℓ_k maps that each ℓ_k satisfies the Leibniz rule with respect to standard (associative) multiplication on $\mathcal{A}^{\text{comm}}$; an L_∞ algebra satisfying Leibniz is usually called a *homotopy Poisson algebra*.

Theorem 1.1 (see Definition 3.3 and Proposition 3.6). *Let Λ be any pointed Legendrian link in \mathbb{R}^3 with commutative Chekanov–Eliashberg DGA $(\mathcal{A}^{\text{comm}}, \partial)$. There are maps $\ell_k : (\mathcal{A}^{\text{comm}})^{\otimes k} \rightarrow \mathcal{A}^{\text{comm}}$ for $k \geq 1$, with $\ell_1 = \partial$, such that $(\mathcal{A}^{\text{comm}}, \{\ell_k\})$ is a homotopy Poisson algebra.*

The operation ℓ_k counts holomorphic disks in $(\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \Lambda)$ with k positive ends, as we discuss further in Section 1.2. As an immediate consequence of Theorem 1.1, ℓ_2 induces a Poisson bracket on the commutative LCH of Λ :

Corollary 1.2 (see Corollary 3.7). *For any pointed Legendrian link Λ , $(LCH_*^{\text{comm}}(\Lambda), \ell_2)$ is a Poisson algebra.*

In this paper, we will also study, but not completely resolve, the question of invariance of the L_∞ algebra under Legendrian isotopy. We conjecture that our construction is invariant:

Conjecture 1.3 (see Conjecture 5.1). *The homotopy Poisson algebra $(\mathcal{A}^{\text{comm}}, \{\ell_k\})$ of a pointed Legendrian link Λ is invariant under Legendrian isotopy of Λ , up to L_∞ equivalence.*

Here an L_∞ equivalence is a L_∞ morphism between L_∞ algebras, given by a collection of maps $\{f_k\}_{k=1}^\infty$, inducing an isomorphism on ℓ_1 homology; see Section 5.1 for details. We will partially resolve this conjecture, constructing the maps f_1, f_2 and thus establishing invariance up through ℓ_2 .

Proposition 1.4 (see Proposition 5.4). *The homotopy Poisson algebra $(\mathcal{A}^{\text{comm}}, \{\ell_k\})$ of Λ is invariant under Legendrian isotopy of Λ , up through the ℓ_2 operation.*

Corollary 1.5 (see Proposition 5.4). *The Poisson algebra $(LCH_*^{\text{comm}}(\Lambda), \ell_2)$ is invariant under Legendrian isotopy of Λ .*

Our construction of the L_∞ structure, like Legendrian contact homology itself in \mathbb{R}^3 , can be described in completely combinatorial (non-analytic) terms. As a consequence, one can readily calculate the L_∞ structure for any given Legendrian link. One particularly interesting family of Legendrian links is the collection of (-1) -closures of admissible positive braids (see Section 3.4), including rainbow closures of positive braids. For this class of Legendrians, the L_∞ algebra is strict in the sense that $\ell_k = 0$ for $k \geq 3$, and it follows that the commutative Chekanov–Eliashberg DGA is a DG Poisson algebra. In particular, for an admissible positive braid with N strands and k crossings, the ℓ_2 operation on $\mathcal{A}^{\text{comm}}$ induces a Poisson bracket on the polynomial ring $\mathbb{C}[q_1, \dots, q_k, t_1^{\pm 1}, \dots, t_N^{\pm 1}]$. In the specific case where Λ is a Legendrian $(2, 2n)$ torus link given by the (-1) -closure of the 2-braid σ_1^{2n+2} , this Poisson bracket agrees with the Flaschka–Newell bracket; see Proposition 3.11. We believe that the general family of Poisson brackets associated to arbitrary admissible positive braids may be new.

In upcoming work [CGN⁺], the L_∞ structure constructed in this paper will be used to construct a symplectic form on the augmentation variety of a large family of Legendrian links. Here the augmentation variety is a Legendrian-isotopy invariant constructed from the Chekanov–Eliashberg DGA—roughly speaking, it is given by $\text{Spec } LCH_0^{\text{comm}}(\Lambda)$ —and the ℓ_2 operation dualizes to a closed differential 2-form on the augmentation variety. The fact that this 2-form is nondegenerate follows from Sabloff duality

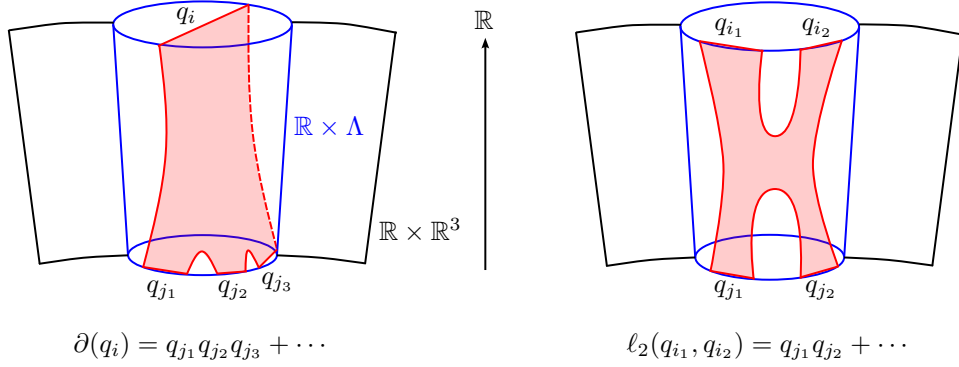


FIGURE 1. Left: a holomorphic disk contributing to the differential ∂ in the (commutative) Chekanov–Eliashberg DGA $(\mathcal{A}^{\text{comm}}, \partial)$ of a Legendrian Λ ; here $a_i, a_{j_1}, a_{j_2}, a_{j_3}$ are Reeb chords of Λ . Right: a holomorphic disk contributing to the ℓ_2 operation on $\mathcal{A}^{\text{comm}}$.

[EES09]; conversely, it will follow that the L_∞ structures in this paper are nondegenerate in a suitable sense.

In another direction, we conjecture that the L_∞ structure is functorial under exact Lagrangian cobordism:

Conjecture 1.6. *Let Λ^\pm be pointed Legendrian links, and suppose that there is an exact Lagrangian cobordism from Λ^- to Λ^+ . Then there is an L_∞ morphism from the L_∞ algebra $((\mathcal{A}^+)^{\text{comm}}, \{\ell_k^+\})$ of Λ^+ to the L_∞ algebra $((\mathcal{A}^-)^{\text{comm}}, \{\ell_k^-\})$ of Λ^- extending the usual cobordism map of Chekanov–Eliashberg DGAs $(\mathcal{A}^+)^{\text{comm}} \rightarrow (\mathcal{A}^-)^{\text{comm}}$.*

We hope to return to this conjecture in future work.

1.2. Geometric motivation. We now discuss the construction of the L_∞ structure in slightly more detail. The ℓ_k operations that constitute our L_∞ structure are defined using rational Symplectic Field Theory (SFT) [EGH00]. Where LCH counts rigid holomorphic disks in the symplectization $\mathbb{R} \times \mathbb{R}^3$ with boundary on $\mathbb{R} \times \Lambda$ with a single positive end at a Reeb chord (and any number of negative ends), rational SFT counts rigid holomorphic disks with any number of positive ends; the holomorphic disks with k positive ends contribute to ℓ_k for each $k \geq 1$. See Figure 1 for an illustration of terms in $\ell_1 = \partial$ and ℓ_2 .

The version of rational SFT that we consider in this paper is a relative version of rational SFT for closed contact manifolds V , which counts punctured holomorphic spheres in the symplectization $\mathbb{R} \times V$ whose punctures are mapped to Reeb orbits at $\pm\infty$ in the \mathbb{R} direction. In the closed case, it is well-known that one can use rational SFT to produce an L_∞ structure on the polynomial algebra generated by Reeb orbits: see [Sie19, MZ20]. More concretely, $\ell_k(q_{j_1}, \dots, q_{j_k})$ is a sum, over all rigid punctured holomorphic

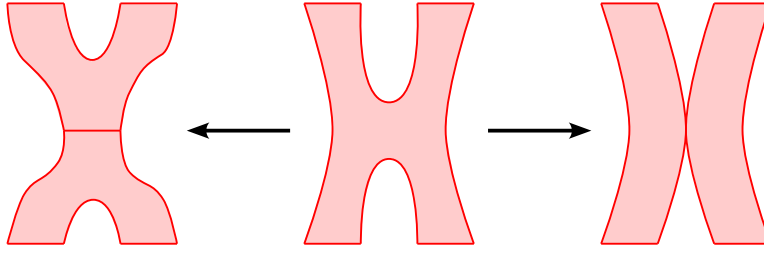


FIGURE 2. Possible degenerations of a 1-dimensional moduli space of holomorphic disks (center). Left: a two-story building given by gluing together a positive end of one rigid disk to a negative end of another rigid disk. Right: two rigid disks joined at a point on their boundaries.

spheres with positive ends at the Reeb orbits q_{j_1}, \dots, q_{j_k} , of the product of the negative ends of each sphere.

In the closed case, the claim that the ℓ_k operations satisfy the L_∞ relations is derived from the fact that 1-dimensional moduli spaces of punctured holomorphic spheres can be compactified by adding two-story buildings consisting of two rigid punctured holomorphic spheres with a positive end of one sphere glued to a negative end of the other. It then follows, as in standard Floer-theoretic $\partial^2 = 0$ arguments, that the endpoints of any 1-dimensional moduli space contribute canceling terms to the L_∞ relations. The standard way to express the fact about two-story buildings is an equation of the form

$$\{h, h\} = 0.$$

Here h is a Hamiltonian in the variables q_j and dual variables p_j given by summing the product of the positive ends (recorded as p_j 's) and the negative ends (recorded as q_j 's) of all rigid punctured holomorphic spheres, and the SFT bracket $\{\cdot, \cdot\}$ is defined by $\{p_{j_1}, q_{j_2}\} = -\{q_{j_2}, p_{j_1}\} = \delta_{j_1 j_2}$: geometrically this glues a punctured sphere with a negative end at some Reeb orbit to a punctured sphere with a positive end at the same Reeb orbit. See [Lat22] for further discussion of this perspective on the L_∞ structure in the closed setting.

Compared to the closed case, the relative case of rational SFT has a significant complication: 1-dimensional moduli spaces of punctured holomorphic disks can degenerate to two-story buildings, but there are also “boundary degenerations” (essentially boundary bubbling) consisting of two holomorphic disks joined by a common point on their boundary. See Figure 2 for an illustration of each type of degeneration. This is a well-known issue, and for Legendrians in \mathbb{R}^3 there are at least two (related) approaches to work around this difficulty. One approach, due to Ekholm [Ekh08], is to restrict to multi-component Legendrian links and count only holomorphic disks whose boundary components lie on all different components of the Legendrian; this eliminates boundary bubbling.

Another approach, and the one that underlies the present paper, is given in previous work of the author [Ng10]. This approach fleshes out a strategy laid out by Cieliebak, Latschev, and Mohnke [CL09], using ideas from string topology to treat the boundary-degeneration problem. As in the closed case, one can define a Hamiltonian h associated to a Legendrian Λ , but now the equation $\{h, h\} = 0$ is replaced by

$$\frac{1}{2}\{h, h\} = \delta h,$$

where δ is an operation called the string coproduct. In [Ng10], this equation (the “quantum master equation”) is used to produce a curved DGA associated to any Legendrian knot (the “LSFT algebra”), which is invariant in a suitable sense under Legendrian isotopy.

Analogously to the closed case, one can try to define an L_∞ structure on the commutative Chekanov–Eliashberg DGA $\mathcal{A}^{\text{comm}}$ of a Legendrian Λ by defining $\ell_k(q_{i_1}, \dots, q_{i_k})$ to be a sum over all rigid holomorphic disks with positive ends at the Reeb chords q_{i_1}, \dots, q_{i_k} of the product of the negative ends of the disk; see Figure 1 for illustrations for $k = 1$ and $k = 2$. For $k = 1$ this recovers the usual differential ∂ on $\mathcal{A}^{\text{comm}}$. However, the presence of δh in the quantum master equation means that these ℓ_k ’s do not satisfy the L_∞ relations, unlike in the closed case.

We will solve this problem in this paper by defining ℓ_k as described above for all $k \neq 2$, but adding a correction term ℓ_2^{str} to ℓ_2 . This correction term, which is of the form $\ell_2^{\text{str}}(q_{j_1}, q_{j_2}) = \alpha_{j_1, j_2} q_{j_1} q_{j_2}$ for some $\alpha_{j_1, j_2} \in \frac{1}{2}\mathbb{Z}$, is a string-topology contribution to ℓ_2 and depends on the placement of base points on Λ . By choosing these coefficients α_{j_1, j_2} appropriately, it is possible to cancel out the effect of the δh term in the quantum master equation and thus produce operations $\{\ell_k\}_{k=1}^\infty$ satisfying the L_∞ relations. As an added benefit, even though rational SFT was only defined for Legendrian knots and not multi-component Legendrian links in [Ng10], we will show that our construction of $\{\ell_k\}$ extends to links as well as knots, and that the L_∞ relations are satisfied in all cases.

We remark that even though defining the L_∞ structure combinatorially is relatively straightforward—see Section 3—verifying the L_∞ relations is technically rather intricate, and this verification forms the heart of this paper. One specific notable technical issue is that because ℓ_2 needs to be graded symmetric, (anti)symmetrization forces us to define string contributions ℓ_2^{str} to ℓ_2 that have coefficients in $\frac{1}{2}\mathbb{Z}$ rather than in \mathbb{Z} . This means that our L_∞ algebras are forced to be defined in fields of characteristic $\neq 2$. In turn, this forces us to keep careful track of signs (or, geometrically, orientations of moduli spaces) at all times. A fair amount of the difficulty in verifying the L_∞ relations comes from working through the resulting profusion of signs.

In the special case where Λ is a single-component Legendrian knot, the string corrections to ℓ_2 simplify significantly, and the full ℓ_2 operation can be expressed in terms of the differential on the LSFT algebra, which was

constructed in [Ng10] to encode rational SFT. Indeed, for Legendrian knots, the L_∞ structure defined in this paper is equivalent to a version of the LSFT algebra, the *commutative complex* defined in [Ng10, section 2.4], in the sense that one can read the commutative complex from the L_∞ structure and vice versa; see Proposition 3.13. Thus we can view the L_∞ structure in this case as a repackaging of a previously-defined Legendrian invariant. However, we note two points: first, it is unclear that invariance of the commutative complex as shown in [Ng10] implies invariance of the L_∞ structure, or vice versa; second, our construction of the L_∞ structure is entirely new for Legendrian links, or indeed for Legendrian knots with multiple base points.

1.3. Outline of paper. In Section 2, we describe the ingredients that go into defining the L_∞ operations, including broken closed strings, the Hamiltonian, the string coproduct, and the quantum master equation, and establish some of their fundamental properties. This is an upgrade of material from [Ng10] from the setting of Legendrian knots to the setting of pointed Legendrian links. In particular, we include a discussion of grading for the Chekanov–Eliashberg DGA in the presence of multiple base points that we believe is new and may be of independent interest.

We present the construction of the L_∞ structure, with examples, in Section 3. We then establish the L_∞ relations in Section 4 and prove invariance under Legendrian isotopy in Section 5.

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2. RATIONAL SYMPLECTIC FIELD THEORY FOR LEGENDRIAN LINKS

In this section, we present background material that goes into defining rational SFT for 1-dimensional Legendrians, following [Ng10]; the constructions described in this section will then be used to define the L_∞ structure in Section 3. We begin in Sections 2.1 and 2.2 with a description of the algebraic framework; this is followed in Sections 2.3 through 2.5 by a discussion of capping paths and the more general notion of broken closed strings. We introduce the Hamiltonian for rational SFT in Section 2.6 and the algebraic operations of the SFT bracket and string coproduct in Sections 2.7 through 2.9. We conclude by discussing the quantum master equation in Section 2.10.

Many of the constructions in this section were originally presented in [Ng10], and we will make frequent reference to that paper, but we have tried to keep our exposition as self-contained as possible. We will also extend the constructions from [Ng10], which were defined specifically for only Legendrian knots with a single base point, to the more general case of Legendrian links with arbitrarily many base points. This adds a significant level of complication to the arguments as compared to [Ng10]: for instance, the string coproduct is no longer a derivation with respect to the SFT bracket, cf. Proposition 2.30.

Throughout this section and the paper, we will be considering \mathbb{R}^3 with standard coordinates x, y, z and the standard contact structure $\ker(dz - y dx)$. Smoothly embedded links in \mathbb{R}^3 are Legendrian if they are everywhere tangent to the plane field given by the contact structure. One reference for relevant background material about Legendrian links in standard contact \mathbb{R}^3 is the survey [EN22].

2.1. Algebras and modules associated to Legendrian SFT. Here we review the basic algebraic structures from rational Legendrian Symplectic Field Theory [Ng10], with some modifications for our purposes. These are the building blocks that we will use to define the L_∞ structure.

Definition 2.1. A *pointed Legendrian link* in \mathbb{R}^3 is an oriented Legendrian link equipped with a collection of base points, with each connected component of Λ containing at least one base point. A pointed Legendrian link is *diagram-generic* if the only singularities of the immersed curve $\Pi_{xy}(\Lambda)$ are transverse double points and no base point is mapped by Π_{xy} to one of these double points.

Let Λ be a pointed Legendrian link in \mathbb{R}^3 . We label the base points on Λ by $\bullet_1, \dots, \bullet_s$, and the crossings of $\Pi_{xy}(\Lambda)$, which are the Reeb chords of Λ , by a_1, \dots, a_n . For bookkeeping of paths on Λ , we introduce a partner to each base point \bullet_i : a point \star_i immediately preceding \bullet_i with respect to the orientation of Λ . To distinguish \star_i from \bullet_i , we will refer to it as a “marked point” rather than a base point.

We associate variables to the base points and Reeb chords as follows:

- to each marked point \star_i , an invertible variable t_i ;
- to each Reeb chord a_j , two variables q_j and p_j .

Throughout the paper we will write

$$\mathcal{S} := \{q_1, p_1, \dots, q_n, p_n, t_1^{\pm 1}, \dots, t_s^{\pm 1}\}$$

for the collection of these variables. We will define a \mathbb{Z} -valued grading on each variable in \mathcal{S} in Section 2.4; for now we assume these gradings are given, and proceed to construct some algebras and \mathbb{Z} -modules out of these variables.

Let

$$\mathcal{A} = \mathbb{Z}\langle q_1, \dots, q_n, t_1^\pm, \dots, t_s^\pm \rangle$$

be the noncommutative graded algebra generated by the q 's and t 's: as a \mathbb{Z} -module, this is freely generated by words in the q 's and t 's (i.e., words in the elements of \mathcal{S} besides p_1, \dots, p_n). This is the algebra underlying the usual Chekanov–Eliashberg DGA of Λ (more precisely, the “fully noncommutative” version of the C-E DGA).

Next define

$$\mathcal{A}_{\text{SFT}}^0 = \mathbb{Z}\langle q_1, \dots, q_n, p_1, \dots, p_n, t_1^\pm, \dots, t_s^\pm \rangle :$$

as a \mathbb{Z} -module, this is freely generated by words in the elements of \mathcal{S} . This has a filtration $\mathcal{A}_{\text{SFT}}^0 = \mathcal{F}^0 \mathcal{A}_{\text{SFT}}^0 \supset \mathcal{F}^1 \mathcal{A}_{\text{SFT}}^0 \supset \mathcal{F}^2 \mathcal{A}_{\text{SFT}}^0 \supset \dots$, where $\mathcal{F}^k \mathcal{A}_{\text{SFT}}^0$ is the \mathbb{Z} -submodule freely generated by words with at least k p 's. We can use this filtration to define the completion

$$\mathcal{A}_{\text{SFT}} = \widehat{\mathcal{A}_{\text{SFT}}^0}$$

whose elements are formal sums of the form $\sum_{k=0}^{\infty} z_k$ with $z_k \in \mathcal{F}^k \mathcal{A}_{\text{SFT}}^0$. The algebra \mathcal{A}_{SFT} is the “LSFT algebra” from [Ng10]. Note that \mathcal{A}_{SFT} inherits the filtration from $\mathcal{A}_{\text{SFT}}^0$, and that $\mathcal{A} \cong \mathcal{A}_{\text{SFT}} / \mathcal{F}^1 \mathcal{A}_{\text{SFT}}$.

We can take two successive quotients of each of \mathcal{A} and \mathcal{A}_{SFT} . The first is the cyclic quotient, obtained by quotienting by the \mathbb{Z} -submodule generated by commutators $[x, y] = xy - (-1)^{|x||y|}yx$. This produces the graded \mathbb{Z} -modules (not \mathbb{Z} -algebras)

$$\mathcal{A}^{\text{cyc}}, \mathcal{A}_{\text{SFT}}^{\text{cyc}} :$$

each of these is generated as a \mathbb{Z} -module by cyclic words in the q 's and t 's (and p 's for $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$). Given a word $v_1 \cdots v_k \in \mathcal{A}_{\text{SFT}}$ with $v_i \in \mathcal{S}$ for all i , any cyclic permutation $v_{j+1} \cdots v_k v_1 \cdots v_j$ of the word represents the same element in $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$, up to a sign:

$$v_{j+1} \cdots v_k v_1 \cdots v_j = (-1)^{(|v_1| + \dots + |v_j|)(|v_{j+1}| + \dots + |v_k|)} v_1 \cdots v_k \in \mathcal{A}_{\text{SFT}}^{\text{cyc}}.$$

The second quotient of \mathcal{A} and \mathcal{A}_{SFT} is the commutative quotient, obtained by quotienting by the subalgebra generated by commutators. This produces the graded \mathbb{Z} -algebras

$$\begin{aligned} \mathcal{A}^{\text{comm}} &= \mathbb{Z}[q_1, \dots, q_n, t_1^{\pm 1}, \dots, t_s^{\pm 1}] \\ \mathcal{A}_{\text{SFT}}^{\text{comm}} &= \mathbb{Z}[q_1, \dots, q_n, p_1, \dots, p_n, t_1^{\pm 1}, \dots, t_s^{\pm 1}]. \end{aligned}$$

These are graded polynomial rings with the usual commutation relations $yx = (-1)^{|x||y|}xy$.

2.2. Running example. For the constructions in the remainder of this section, it may helpful to have an illustrative running example of a pointed Legendrian link. We will use the Legendrian positive Hopf link shown in Figure 3. This is a two-component link such that each component has rotation number 0, with three base points distributed between the two components.

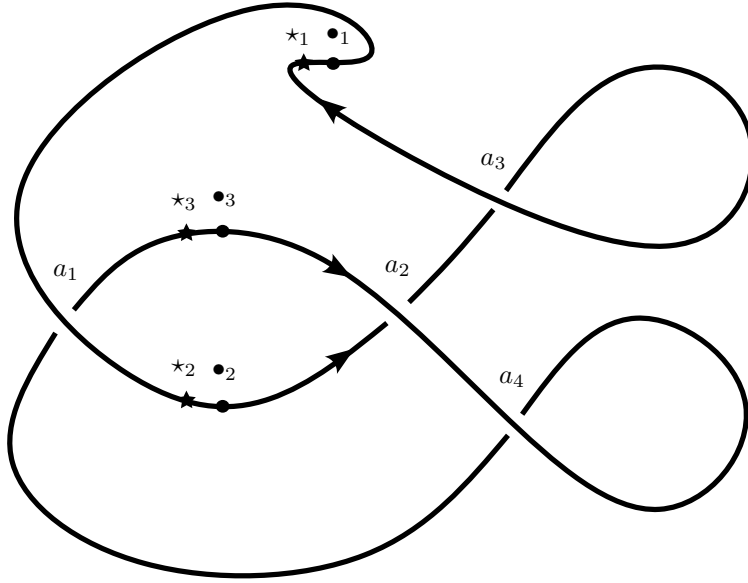


FIGURE 3. Hopf link in the xy projection. Reeb chords are labeled a_1, a_2, a_3, a_4 . The three base points \bullet_i and three marked points \star_i are as marked. The zigzag near \bullet_1 ensures that the tangent vectors at all of the base points are parallel, for grading purposes; see Remark 2.4.

There are four Reeb chords a_1, a_2, a_3, a_4 . The Chekanov–Eliashberg algebra of this link is

$$\mathcal{A} = \mathbb{Z}\langle q_1, q_2, q_3, q_4, t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1} \rangle$$

and the LSFT algebra \mathcal{A}_{SFT} is the completion of

$$\mathbb{Z}\langle q_1, q_2, q_3, q_4, p_1, p_2, p_3, p_4, t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1} \rangle.$$

2.3. Capping paths. We now again consider a general pointed Legendrian link Λ . Our next step is to construct a collection of capping paths, which are paths on Λ associated to each of the elements of \mathcal{S} . These paths are closely related to broken closed strings, to be discussed in Section 2.5, and will also be used to define gradings in Section 2.4.

The capping paths are defined using the positions of the base points $\bullet_1, \dots, \bullet_s$. Recall that we have a marked point \star_i immediately preceding each base point \bullet_i as we follow the orientation of Λ . Removing $\{\star_1, \dots, \star_s\}$ from Λ disconnects Λ into a disjoint union of s oriented segments, each containing one of the marked points \bullet_i (at the beginning of the segment). We define two functions

$$e_+, e_- : \mathcal{S} \rightarrow \{1, \dots, s\}$$

as follows:

$$\bullet e_+(t_i) = e_-(t_i^{-1}) = i;$$

- $e_-(t_i) = e_+(t_i^{-1})$ is the number of the marked point immediately preceding \bullet_i as we follow the orientation of Λ : that is, if we traverse Λ beginning at \bullet_i and traveling against the orientation of Λ , then $\bullet_{e_-(t_i)}$ is the first base point we encounter after \bullet_i ;
- $e_+(p_j) = e_-(q_j)$ is the number of the marked point on the same segment as the positive endpoint a_j^+ of the Reeb chord a_j , when the points \star_1, \dots, \star_s are removed from Λ ;
- $e_-(p_j) = e_+(q_j)$ is the number of the marked point on the same segment as the negative endpoint a_j^- of the Reeb chord a_j , when the points \star_1, \dots, \star_s are removed from Λ .

We can now define capping paths γ_v for each $v \in \{q_1, p_1, \dots, q_n, p_n, t_1^\pm, \dots, t_s^\pm\}$. In all cases, γ_v will be a path beginning at $\bullet_{e_-(v)}$ and ending at $\bullet_{e_+(v)}$. For $v = t_i$, we define γ_{t_i} to be the oriented segment of Λ beginning at $\bullet_{e_-(t_i)}$ and ending at $\bullet_{e_+(t_i)} = \bullet_i$; for $v = t_i^{-1}$, we define $\gamma_{t_i^{-1}}$ to be the reverse of γ_{t_i} .

For $v = q_j$ and $v = p_j$, the capping path γ_v has a discontinuity at the Reeb chord a_j and is constructed from two ‘‘half capping paths’’. Specifically, define γ_j^\pm to be the embedded paths in Λ , following the orientation of Λ , from $\bullet_{e_\pm(p_j)} = \bullet_{e_\mp(q_j)}$ to a_j^\pm . We then set

$$\begin{aligned}\gamma_{p_j} &= \gamma_j^- \cdot (-\gamma_j^+) \\ \gamma_{q_j} &= \gamma_j^+ \cdot (-\gamma_j^-),\end{aligned}$$

where \cdot denotes path concatenation. That is, γ_{p_j} is the path γ_j^- followed by the reverse of the path γ_j^+ , and similarly for γ_{q_j} . We note that the path γ_{p_j} has been constructed to begin at $\bullet_{e_-(p_j)}$, have a discontinuity from a_j^- to a_j^+ , and end at $\bullet_{e_+(p_j)}$. Similarly, γ_{q_j} , which is the same path in reverse, begins at $\bullet_{e_-(q_j)}$, has a discontinuity from a_j^+ to a_j^- , and ends at $\bullet_{e_+(q_j)}$.

Example 2.2. For the Hopf link from Figure 3, the capping paths for q_1 and t_1 are shown in Figure 4. We have $e_-(q_1) = 1$, $e_+(q_1) = 3$, and $\gamma_{q_1} = \gamma_1^+ \cdot (-\gamma_1^-)$ begins at \bullet_1 and ends at \bullet_3 , with a discontinuity going down the Reeb chord at a_1 ; and $e_-(t_1) = 2$, $e_+(t_1) = 1$, and γ_{t_1} begins at \bullet_2 and ends at \bullet_1 .

2.4. Gradings. Here we discuss how to use the capping paths defined in Section 2.3 to give \mathbb{Z} gradings to $t_i^{\pm 1}, q_j, p_j$, the elements of \mathcal{S} . This will induce gradings on the various algebraic structures $\mathcal{A}, \mathcal{A}_{\text{SFT}}, \mathcal{A}_{\text{SFT}}^{\text{cyc}}, \mathcal{A}_{\text{SFT}}^{\text{comm}}$ associated to a Legendrian link Λ in Section 2.1.

In combinatorial form, this story is well-known in the case where Λ has a single component with a single base point: see e.g. the papers [ENS02, Ng10], which build on the original construction of a $\mathbb{Z}/(2 \text{rot}(\Lambda)\mathbb{Z})$ grading by Chekanov [Che02]. When Λ has multiple components with a single base point on each, the grading is often defined in terms of the front projection of Λ , see e.g. [Ng03, NRS⁺20], implicitly using a correspondence between Reeb chords of Λ and crossings and right cusps of the front of Λ .

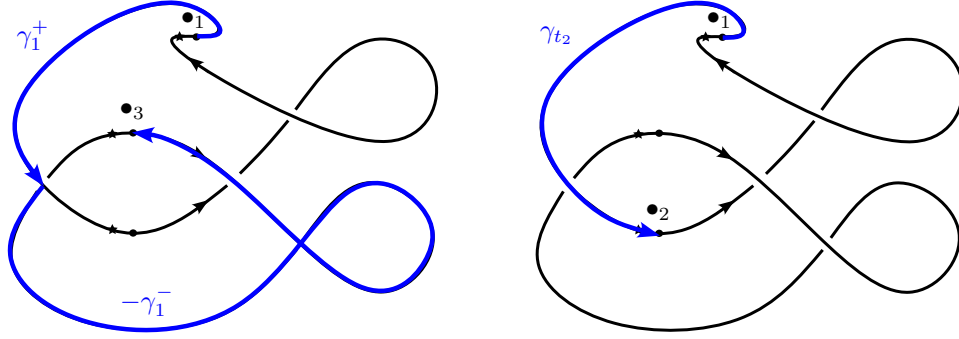


FIGURE 4. Capping paths γ_{q_1} (the concatenation of γ_1^+ and $-\gamma_1^-$) and γ_{t_2} , for the Hopf link from Figure 3.

Here we will present the grading in terms of the xy projection rather than the front projection, which is more natural geometrically. This may be of independent interest since to this author's knowledge, a combinatorial description of gradings in the multi-component case in the xy projection has not previously appeared in the literature, even though it is surely well known to experts. Furthermore, we extend the construction of gradings to the case where there are multiple base points on single components of Λ ; this is a setting that naturally arises in considering DGA maps induced by exact Lagrangian cobordisms, see e.g. [Pan17, CN22].

Let Λ be a pointed Legendrian link with base points $\bullet_1, \dots, \bullet_s$. In any case when Λ has more than one base point, the grading depends on an auxiliary piece of information, a Maslov potential. The oriented unit tangent vector in \mathbb{R}^2 to $\Pi_{xy}(\Lambda)$ at each base point is an element of S^1 , which we identify with \mathbb{R}/\mathbb{Z} in the usual way: to be concrete, the vector obtained from the positive x direction $\partial/\partial x$ by a counterclockwise θ rotation is associated with $\frac{\theta}{2\pi} \in \mathbb{R}/\mathbb{Z}$.

Definition 2.3. A *Maslov potential* on Λ is a map $m : \{\bullet_1, \dots, \bullet_s\} \rightarrow \mathbb{R}$ such that $m(\bullet_i)$ is a lift of the oriented unit tangent vector to $\Pi_{xy}(\Lambda)$ at $\Pi_{xy}(\bullet_i)$ from \mathbb{R}/\mathbb{Z} to \mathbb{R} .

The gradings will be defined in terms of differences of Maslov potentials, and so the collection of gradings for Λ will be an affine space based on \mathbb{Z}^{s-1} .

Remark 2.4. In practice, a convenient way to encode a Maslov potential in a link diagram is to perturb the xy projection of Λ by a planar isotopy in a neighborhood of each base point \bullet_i , so that the tangent vector at \bullet_i rotates in \mathbb{R} from $m(\bullet_i)$ at the beginning of the isotopy to 0 at the end. The end result is a link diagram for $\Pi_{xy}(\Lambda)$ where the tangent direction at each base point is $\partial/\partial x$. In this new link diagram, there is an obvious Maslov potential given by $m(\bullet_i) = 0$ for all i . All of the gradings that we will define below are unchanged by the isotopy. Consequently, in all examples in this paper,

we will choose link diagrams where the tangent direction at each base point is $\partial/\partial x$, where the Maslov potential is understood to be identically 0. See Figure 3 for an illustration.

For an immersed oriented curve γ in \mathbb{R}^2 , define $r(\gamma) \in \mathbb{R}$ to be the number of counterclockwise revolutions in S^1 made by the unit tangent vector to γ as we traverse γ ; note that in general this will not be an integer. Recall that in Section 2.3, we constructed capping paths γ_{t_i} associated to \bullet_i and half capping paths γ_j^\pm associated to the Reeb chord a_j ; each of these, when composed with the projection $\Pi_{xy} : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, yields an immersed oriented curve in \mathbb{R}^2 . We will abuse notation and write $r(\gamma_{t_i})$ and $r(\gamma_j^\pm)$ for $r(\Pi_{xy} \circ \gamma_{t_i})$ and $r(\Pi_{xy} \circ \gamma_j^\pm)$.

Definition 2.5. For $v \in \mathcal{S}$, we define:

$$|v| = \left\lfloor 2 \left(-r(\gamma_v) + m(\bullet_{e_+(v)}) - m(\bullet_{e_-(v)}) \right) \right\rfloor.$$

Here if $v = p_j$ or $v = q_j$, then the full capping path γ_v has a discontinuity, and by $r(\gamma_v)$ we mean the sum of r over each of the two half capping paths:

$$\begin{aligned} r(\gamma_{p_j}) &= r(\gamma_j^-) - r(\gamma_j^+) \\ r(\gamma_{q_j}) &= r(\gamma_j^+) - r(\gamma_j^-). \end{aligned}$$

Example 2.6. For the running example from Section 2.2, we see by inspection of Figure 4 that $r(\gamma_1^+) \approx \frac{7}{8}$ and $-r(\gamma_1^-) = r(-\gamma_1^-) \approx -\frac{1}{8}$, and so $|q_1| = -2$. The full grading in this example is:

$$\begin{array}{cccc} |q_1| = -2 & |p_1| = 1 & |q_2| = 0 & |p_2| = -1 \\ |q_3| = 1 & |p_3| = -2 & |q_4| = 1 & |p_4| = -2 \\ |t_1| = 2 & |t_2| = -2 & |t_3| = 0. & \end{array}$$

Proposition 2.7. (1) For any base point \bullet_i , $|t_i|$ is even and $|t_i^{-1}| = -|t_i|$. If Λ_0 is a component of Λ and $\bullet_{i_1}, \dots, \bullet_{i_m}$ are the base points on component Λ_0 , then

$$|t_{i_1}| + \dots + |t_{i_m}| = -2 \operatorname{rot}(\Lambda_0).$$

(2) For any Reeb chord a_j , we have

$$|q_j| + |p_j| = -1.$$

Furthermore, $|q_j|$ is even (resp. odd), and $|p_j|$ is odd (resp. even), if the crossing corresponding to a_j in the oriented link diagram given by the xy projection of Λ is positive (resp. negative).

- (3) If we change the Maslov potential m by adding $(n_1, \dots, n_s) \in \mathbb{Z}^s$ to m , then the gradings change as follows:

$$\begin{aligned}\Delta|t_i| &= 2(n_{e_+(t_i)} - n_{e_-(t_i)}) \\ \Delta|q_j| &= -\Delta|p_j| = 2(n_{e_+(q_j)} - n_{e_-(q_j)}).\end{aligned}$$

Proof. We first consider item (1). As we traverse γ_{t_i} , the tangent direction in \mathbb{R}^2 changes in $\mathbb{R}/\mathbb{Z} = S^1$, though not necessarily in a way that lifts continuously to \mathbb{R} , from $m(\bullet_{e_-(t_i)})$ to $m(\bullet_{e_+(t_i)})$. Thus

$$r(\gamma_{t_i}) \equiv m(\bullet_{e_+(t_i)}) - m(\bullet_{e_-(t_i)}) \pmod{\mathbb{Z}}$$

and the expression inside the floor function in Definition 2.5 is an even integer; it follows that $|t_i|$ is an even integer. Since $\gamma_{t_i^{-1}}$ is the reverse of γ_{t_i} , we also deduce from Definition 2.5 that $|t_i^{-1}| = -|t_i|$.

For the statement about the gradings of the base points on a component Λ_0 , suppose that we encounter $\bullet_{i_1}, \dots, \bullet_{i_m}$ in order as we traverse Λ_0 . Then $e_+(t_{i_j}) = i_j$, $e_-(t_{i_j}) = i_{j-1}$ (where $j-1$ is taken mod m), and

$$\sum_{j=1}^m |t_{i_j}| = \sum_j 2(-r(\gamma_{t_{i_j}}) + m(\bullet_{i_j}) - m(\bullet_{i_{j-1}})) = -2 \sum_j r(\gamma_{t_{i_j}}).$$

Since Λ_0 is the union of $\gamma_{t_{i_j}}$ over j , this last sum is $\text{rot}(\Lambda_0)$, as desired.

We next consider item (2). For $v = p_j$ and $v = q_j$, the expressions inside the parentheses in Definition 2.5 satisfy

$$-r(\gamma_{p_j}) + m(\bullet_{e_+(p_j)}) - m(\bullet_{e_-(p_j)}) = -\left(-r(\gamma_{q_j}) + m(\bullet_{e_+(q_j)}) - m(\bullet_{e_-(q_j)})\right)$$

and the two sides of this equality are both equal in \mathbb{R}/\mathbb{Z} to the difference between the tangent direction at a_j^- and the tangent direction at a_j^+ . Since we are assuming that all crossings of $\Pi_{xy}(\Lambda)$ are transverse, this quantity is neither an integer nor a half-integer. It follows that $|q_j| + |p_j| = -1$. To show that the parity of $|q_j|$ agrees with the sign of the crossing for a_j , note that the difference between the tangent directions at a_j^- and a_j^+ is between 0 and $1/2$ or between $1/2$ and $1 \pmod{\mathbb{Z}}$ depending on the sign of the crossing.

Finally, item (3) follows directly from the definition of the gradings. \square

Remark 2.8. When Λ has a single connected component with a single base point, the dependence on the Maslov potential m disappears and our gradings agree with the corresponding definitions in [Ng10, §2.3]. In particular, the path γ_j in [Ng10] is our $-\Pi_{xy} \circ \gamma_{q_j}$, and so our definition gives $|q_j| = \lfloor 2r(\gamma_j) \rfloor$, in agreement with [Ng10]. Also, in the examples considered in this paper, the Maslov potential will be understood to be identically 0 and so the contributions from $m(\cdot)$ disappear from the above formulas; see Remark 2.4.

2.5. Broken closed strings. In Section 2.3, we associated a path in Λ to each of t_i^\pm, q_j, p_j . This association can be extended to arbitrary words in these generators by concatenation, resulting in a geometric interpretation of generators of \mathcal{A}_{SFT} as a \mathbb{Z} -module. To make this precise, we next introduce the notion of a (based) broken closed string on Λ : the capping paths defined in Section 2.3 are examples of based broken closed strings.

Definition 2.9. Let Λ be a pointed Legendrian link. A *based broken closed string* on Λ is a path in the Legendrian link Λ which begins and ends at (possibly distinct) base points on Λ , and which is continuous except for the following allowed types of discontinuities:

- a discontinuity jumping from one endpoint of a Reeb chord of Λ to the other;
- a discontinuity jumping from one of the base points $\bullet_1, \dots, \bullet_s$ to another.

A *broken closed string* is a map from S^1 to Λ that is continuous except for the same types of discontinuities as a based broken closed string.

Note that there is an obvious broken closed string associated to any based broken closed string, given by gluing together the two endpoints: if the two endpoints of the based broken closed string are distinct, then this produces one more discontinuity in the associated broken closed string.

There is a correspondence between based broken closed strings (resp. broken closed strings) on Λ and words (resp. cyclic words) in the q_j, p_j, t_i^\pm . Define \mathcal{W} to be the collection of words in $q_1, p_1, \dots, q_n, p_n, t_1^\pm, \dots, t_s^\pm$, including the empty word. Also define \mathcal{W}^{cyc} to be the collection of cyclic words in the same generators; that is, words up to cyclic permutation. Note that \mathcal{W} and \mathcal{W}^{cyc} generate \mathcal{A}_{SFT} and $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$, respectively, as \mathbb{Z} -modules.

To any word $v_1 \cdots v_m \in \mathcal{W}$, where $v_1, \dots, v_m \in \mathcal{S}$, we can associate the based broken closed string

$$\gamma_{v_1} \cdot \gamma_{v_2} \cdots \gamma_{v_m}$$

given by the concatenation of the capping paths of v_1, \dots, v_m . This association is reversible. More precisely, there is a map

$$w : \{\text{based broken closed strings}\} \rightarrow \mathcal{W}$$

defined as follows: traverse the string and successively record t_i when the string crosses \star_i following the orientation of Λ , t_i^{-1} when the string crosses \star_i against the orientation of Λ , p_j when the string jumps from the bottom endpoint a_j^- of the Reeb chord a_j to the top endpoint a_j^+ , and q_j when the string jumps from a_j^+ to a_j^- . For example, if the based string passes through \star_1 positively, then jumps from a_1^- to a_1^+ , then passes through \star_1 negatively, then the associated word is $t_1 p_1 t_1^{-1}$. We now note that

$$w(\gamma_{v_1} \cdot \gamma_{v_2} \cdots \gamma_{v_m}) = v_1 v_2 \cdots v_m.$$

We claim that w induces a bijection between \mathcal{W} and homotopy classes of based broken closed strings. To define homotopy for broken closed strings, consider the quotient $\tilde{\Lambda}$ of Λ obtained by identifying all of the base points $\bullet_1, \dots, \bullet_s$ with each other to produce a single base point \bullet of $\tilde{\Lambda}$. Then a broken closed string (respectively based broken closed string) is a map from S^1 (resp. $[0, 1]$) to $\tilde{\Lambda}$ (resp. $(\tilde{\Lambda}, \bullet)$) whose only discontinuities are jumps from one endpoint of a Reeb chord to the other. We then define homotopy of (based) broken closed strings to mean a homotopy in $\tilde{\Lambda}$ that preserves all discontinuities at Reeb chords.

Since $\tilde{\Lambda}$ is a bouquet of circles and removing the marked points \star_1, \dots, \star_s results in a contractible space, the map w is injective: if two based broken closed strings map under w to the same word, then they are homotopic. We conclude the following.

Proposition 2.10. *The map*

$$w : \{\text{based broken closed strings up to homotopy}\} \rightarrow \mathcal{W}$$

is a bijection, as is the induced map

$$w : \{\text{broken closed strings up to homotopy}\} \rightarrow \mathcal{W}^{\text{cyc}}.$$

Remark 2.11. Consider a cyclic word $v_1 \cdots v_m$ in \mathcal{W}^{cyc} with $v_i \in \mathcal{S}$ for each i . We say that the cyclic word is *cyclically composable* if

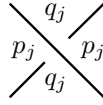
$$e_+(v_i) = e_-(v_{i+1})$$

for each $i = 1, \dots, m-1$, and $e_+(v_m) = e_-(v_1)$. Then a word in \mathcal{W}^{cyc} is cyclically composable if and only if the corresponding broken closed string has discontinuities only at Reeb chords and not at base points. One can form a graded \mathbb{Z} -module $\mathcal{A}^{\text{cyc, comp}}$ generated by composable cyclic words, where we replace the empty word in the generating set by a collection of r “idempotent” empty words. This composable module appears in the surgery formulas of [BEE12, BEE11].

2.6. The Hamiltonian. Given a Legendrian link Λ , rational Legendrian Symplectic Field Theory counts rigid holomorphic disks in the symplectization $\mathbb{R} \times \mathbb{R}^3$ with boundary on the Lagrangian $\mathbb{R} \times \Lambda$. In a familiar manner dating back to the work of Chekanov [Che02], these holomorphic disks are in one-to-one correspondence with certain immersed disks-with-corners in $(\mathbb{R}^2, \Pi_{xy}(\Lambda))$. We review how to use this correspondence to construct a Hamiltonian in $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$, following [Ng10].

Definition 2.12 (cf. [Ng10, Definition 2.18]). Given a diagram-generic Legendrian link Λ , let $\Delta(\Lambda)$ denote the set of all maps Δ from a disk D^2 equipped with any number of boundary marked points on the boundary to \mathbb{R}^2 , up to domain reparametrization, such that:

- Δ is continuous;

FIGURE 5. Decorating the quadrants at a Reeb chord a_j .

- Δ is an orientation-preserving immersion away from the boundary marked points;
- $\Delta(\partial D^2) \subset \Pi_{xy}(\Lambda)$;
- Δ sends each boundary marked point to a crossing of $\Pi_{xy}(\Lambda)$, and a neighborhood of each boundary marked point injectively to one of the four quadrants at that crossing.

Observe that the oriented boundary of any $\Delta \in \Delta(\Lambda)$ is (the projection to \mathbb{R}^2 of) a broken closed string of Λ , with discontinuities precisely at the images of the boundary marked points. Thus, following Section 2.5, to any $\Delta \in \Delta(\Lambda)$ we can associate a cyclic word

$$w(\Delta) \in \mathcal{W}^{\text{cyc}}.$$

Each of these cyclic words $w(\Delta)$ is cyclically composable in the terminology from Section 2.5, since the corresponding broken closed string $\partial\Delta$ has discontinuities only at Reeb chords and not at marked points.

An easy way to read the cyclic word $w(\Delta)$ from Δ is as follows. Decorate the four quadrants at any Reeb chord a_j by p_j, q_j as shown in Figure 5. Now traverse the boundary $\partial\Delta$, with the orientation induced by the orientation on Δ , and successively read off: at the image a_j of a boundary marked point, p_j or q_j depending on whether a neighborhood of this marked point is sent to a quadrant labeled by p_j or q_j ; and when $\partial\Delta$ passes through a marked point \bullet_i, t_i or t_i^{-1} depending on whether the orientations of $\partial\Delta$ and Λ agree at this point. The concatenation of these letters is the cyclic word $w(\Delta)$.

Each $w(\Delta)$ inherits a grading from the gradings in Section 2.4, and we have the following result, cf. [Ng10, Lemma 2.19].

Proposition 2.13. *For any immersed disk $\Delta \in \Delta(\Lambda)$, we have*

$$|w(\Delta)| = -2.$$

Proof. In the case where Λ has a single component and a single base point, this was proven in [Ng10, Lemma 2.19], and the proof here is very similar. Given $\Delta \in \Delta(\Lambda)$, write $w(\Delta) = v_1 v_2 \cdots v_k$, where each $v_\ell \in \mathcal{S}$. For each v_ℓ that is a q_j or p_j , Δ has a corner at the Reeb chord a_j ; let $\theta_\ell \in (0, \pi)$ denote the angle of the corner of Δ at a_j , and let $\eta_\ell = \pi - \theta_\ell \in (0, \pi)$ be the exterior angle of the curved polygon Δ at a_j . Then the angle formed by the tangent vectors to $\Pi_{xy}(\Lambda)$ at a_j is

$$r(\Pi_{xy} \circ \gamma_{v_\ell}) - m(\bullet_{e_+(v_\ell)}) + m(\bullet_{e_-(v_\ell)}) \equiv \frac{\theta_\ell}{2\pi} \equiv -\frac{\eta_\ell}{2\pi} \pmod{\frac{1}{2}\mathbb{Z}}$$

and so

$$\begin{aligned} |v_\ell| + \frac{\eta_\ell}{\pi} &= \frac{\eta_\ell}{\pi} + [-2(r(\Pi_{xy} \circ \gamma_{v_\ell}) - m(\bullet_{e_+(v_\ell)}) + m(\bullet_{e_-(v_\ell)}))] \\ &= -2(r(\Pi_{xy} \circ \gamma_{v_\ell}) - m(\bullet_{e_+(v_\ell)}) + m(\bullet_{e_-(v_\ell)})). \end{aligned}$$

Note that this equation also holds if $v_\ell = t_i^{\pm 1}$ if we set $\eta_\ell = 0$ in this case.

Now sum over ℓ : then the differences $-m(\bullet_{e_+(v_\ell)}) + m(\bullet_{e_-(v_\ell)})$ telescope since $v_1 v_2 \cdots v_k$ is cyclically composable, and we are left with

$$\sum_{\ell=1}^k |v_\ell| + \frac{\sum_{\ell=1}^k \eta_\ell}{\pi} = -2 \sum_{\ell=1}^k r(\Pi_{xy} \circ \gamma_{v_\ell}).$$

Observe that the oriented boundary $\partial\Delta$ of Δ is homotopic as a broken closed string to $\cup_{\ell=1}^k \gamma_{v_\ell}$; thus $\sum_{\ell=1}^k r(\Pi_{xy} \circ \gamma_{v_\ell}) = r(\partial\Delta)$. But since Δ is an immersed disk, the sum of $2\pi r(\partial\Delta)$ and the external angles η_ℓ is equal to 2π . Consequently, we have

$$|w(\Delta)| = \sum_{\ell=1}^k |v_\ell| = -2r(\partial\Delta) - \sum_{\ell=1}^k \frac{\eta_\ell}{\pi} = -2,$$

as desired. \square

We construct the Hamiltonian for Λ by summing the cyclic words $w(\Delta)$ over all immersed disks $\Delta \in \Delta(\Lambda)$ to obtain an element of $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$. To do this precisely, we need to associate signs to each of these cyclic words.

To this end, consider an immersed disk $\Delta \in \Delta(\Lambda)$. The disk Δ must have a corner at some Reeb chord of Λ since otherwise it has area 0 because each component of $\Pi_{xy}(\Lambda)$ bounds a region of signed area 0; see also Definition/Proposition 2.15 below. Thus we can write the cyclic word $w(\Delta)$ as $v_1 v_2 \cdots v_k$ with $v_i \in \mathcal{S}$ for all i and $v_k \in \{q_1, p_1, \dots, q_n, p_n\}$. For $i = 1, \dots, k$, define

$$\epsilon_i(\Delta) = \begin{cases} +1 & v_i = t_j^{\pm 1} \text{ for some } j \\ +1 & v_i = q_j \text{ or } v_i = p_j \text{ and the corner of } \Delta \text{ at } v_i \text{ is unshaded} \\ -1 & v_i = q_j \text{ or } v_i = p_j \text{ and the corner of } \Delta \text{ at } v_i \text{ is shaded,} \end{cases}$$

where “shaded” and “unshaded” refer to the shading of quadrants at the crossing a_j shown in Figure 6. We also define another sign $\epsilon'(\Delta; v_k)$ as follows: if the orientation of $\partial\Delta$ immediately after the corner v_k agrees with the orientation of Λ , then $\epsilon'(\Delta; v_k) = +1$; if it disagrees, then $\epsilon'(\Delta; v_k) = -1$. Finally, we define

$$\tilde{w}(\Delta) = (\epsilon_1(\Delta) \cdots \epsilon_k(\Delta) \epsilon'(\Delta; v_k)) v_1 v_2 \cdots v_k \in \mathcal{A}_{\text{SFT}}^{\text{cyc}}.$$

Lemma 2.14. *For any $\Delta \in \Delta(\Lambda)$, the element $\tilde{w}(\Delta) \in \mathcal{A}_{\text{SFT}}^{\text{cyc}}$ is independent of the choice of the representative word $v_1 v_2 \cdots v_k$ for $w(\Delta)$.*



FIGURE 6. Orientation signs. On the left, a positive crossing (q even, p odd); on the right, a negative crossing (q odd, p even). Each corner of an immersed disk contributes a $+$ sign if it occupies one of the unshaded quadrants or a $-$ sign if it occupies one of the shaded quadrants.

Proof. We need to check that if we replace $v_1 v_2 \cdots v_k$ by a cyclic permutation $v_{i+1} \cdots v_k v_1 \cdots v_i$, then the sign $(-1)^{(|v_1| + \cdots + |v_i|)(|v_{i+1}| + \cdots + |v_k|)}$ associated to this cyclic permutation is canceled by the product of the signs $\epsilon'(\Delta; v_k) \epsilon'(\Delta; v_i)$. This follows essentially immediately from the fact that $w(\Delta)$ has even degree from Proposition 2.13; see [Ng10, Lemma 3.10] for details. \square

Definition/Proposition 2.15. The Hamiltonian $h \in \mathcal{A}_{\text{SFT}}^{\text{cyc}}$ is defined by

$$h = \sum_{\Delta \in \Delta(\Lambda)} \tilde{w}(\Delta).$$

This is a well-defined element and satisfies $|h| = -2$ and $h \in \mathcal{F}^1 \mathcal{A}_{\text{SFT}}^{\text{cyc}}$.

Proof. The fact that $|h| = -2$ follows from Proposition 2.13. To see that $h \in \mathcal{F}^1 \mathcal{A}_{\text{SFT}}^{\text{cyc}}$, we note that for any $\Delta \in \Delta(\Lambda)$, the standard Stokes' Theorem argument (cf. [Ng10, Lemma 2.22]) shows that the area of Δ is equal to the sum of the heights of every Reeb chord appearing as a p in $w(\Delta)$, minus the sum of the heights of every Reeb chord appearing as a q in $w(\Delta)$. It follows that for each $\Delta \in \Delta(\Lambda)$, $w(\Delta)$ contains at least one p : $w(\Delta) \in \mathcal{F}^1 \mathcal{A}_{\text{SFT}}^{\text{cyc}}$. Furthermore, for any $k \geq 1$, only finitely many $\Delta \in \Delta(\Lambda)$ are such that $w(\Delta)$ has at most k p 's; it follows that $h \in \mathcal{F}^1 \mathcal{A}_{\text{SFT}}^{\text{cyc}}$. \square

Example 2.16. For the example from Section 2.2, we have

$$h = p_3 + t_1 t_2 p_3 + t_1 q_1 t_3 q_2 p_3 + p_4 + t_3^{-1} p_4 + q_2 t_2^{-1} q_1 p_4 + t_2 p_2 t_3^{-1} p_1.$$

When we define the L_∞ structure in Section 3, it will be helpful to subdivide h into parts according to the number of positive punctures of the immersed disks. For $k \geq 1$, define

$$\Delta_k(\Lambda) = \{\Delta \in \Delta(\Lambda) \mid w(\Delta) \text{ contains exactly } k \text{ } p\text{'s}\}$$

and

$$h_k = \sum_{\Delta \in \Delta_k(\Lambda)} \tilde{w}(\Delta).$$

Then h_k counts disks with k positive punctures, and $h = \sum_{k \geq 1} h_k$.

Remark 2.17 (sign conventions). The sign convention that we use in this paper for the Hamiltonian h , determined by the shading of corners in Figure 6, is different from the sign convention in [Ng10], which is given in [Ng10, Figure 4]. The difference comes from our desire in this paper to have h_1 induce the LCH differential ∂ on the Chekanov–Eliashberg DGA with its standard sign convention as described in Section 2.7 below; see also Remark 2.39. By contrast, the sign convention for h used in [Ng10] corresponds to an unusual choice of signs for the LCH differential: see [Ng10, Appendix A].

2.7. SFT bracket. We next introduce another ingredient necessary to define the L_∞ structure, the SFT bracket

$$\{\cdot, \cdot\} : \mathcal{A}_{\text{SFT}}^{\text{cyc}} \otimes \mathcal{A}_{\text{SFT}}^{\text{cyc}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{cyc}}.$$

Geometrically, the SFT bracket is defined as follows. Consider two cyclic words in $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$, represented as broken closed strings. If one of the cyclic words involves q_j and the other involves p_j for some j , then the corresponding broken closed strings both have a discontinuity at the Reeb chord a_j , traversed in opposite directions. We can glue together the broken closed strings at a_j to obtain another broken closed string that no longer has the discontinuity at a_j . Summing over all possible ways to glue the two broken closed strings at a Reeb chord yields the SFT bracket of the original cyclic words.

We now give the precise definition of the SFT bracket, with signs. First note that given two cyclic words $v_1 \cdots v_k$ and $u_1 \cdots u_\ell$ where $v_1, \dots, v_k, u_1, \dots, u_\ell \in \mathcal{S}$, we have that for each $i = 0, \dots, k-1$ and each $j = 1, \dots, \ell$, $v_1 \cdots v_k = \sigma_{v,i} v_{i+1} \cdots v_k v_1 \cdots v_i$ and $u_1 \cdots u_\ell = \sigma_{u,j-1} u_j \cdots u_\ell u_1 \cdots u_{j-1}$, where

$$\begin{aligned} \sigma_{v,i} &:= (-1)^{(|v_1| + \cdots + |v_i|)(|v_{i+1}| + \cdots + |v_k|)} \\ \sigma_{u,j-1} &:= (-1)^{(|u_1| + \cdots + |u_{j-1}|)(|u_j| + \cdots + |u_\ell|)}. \end{aligned}$$

Also, for $v, u \in \mathcal{S}$, define

$$\{v, u\} := \begin{cases} 1 & \text{if } v = p_j \text{ and } u = q_j \text{ for some } j \\ -1 & \text{if } v = q_j \text{ and } u = p_j \text{ for some } j \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.18. The *SFT bracket* $\{\cdot, \cdot\} : \mathcal{A}_{\text{SFT}}^{\text{cyc}} \otimes \mathcal{A}_{\text{SFT}}^{\text{cyc}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{cyc}}$ is the \mathbb{Z} -bilinear map defined on generators $v_1 \cdots v_k, u_1 \cdots u_\ell$ of $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$ as follows:

$$\begin{aligned} &\{v_1 \cdots v_k, u_1 \cdots u_\ell\} \\ &= \sum_{i=0}^{k-1} \sum_{j=1}^{\ell} \sigma_{v,i} \sigma_{u,j-1} \{v_i, u_j\} v_{i+1} \cdots v_k v_1 \cdots v_{i-1} u_{j+1} \cdots u_\ell u_1 \cdots u_{j-1}. \end{aligned}$$

Note that, because of the presence of the signs $\sigma(v, i)$ and $\sigma(u, j-1)$, the SFT bracket is well-defined independent of the choice of representative for the cyclic words $v_1 \cdots v_k$ and $u_1 \cdots u_\ell$.

Example 2.19. For the example from Section 2.2, we have

$$\{t_1q_1t_3q_2p_3, t_2p_2t_3^{-1}p_1\} = t_3q_2p_3t_1t_2p_2t_3^{-1} - p_3t_1q_1p_1t_2 = t_1t_2p_2q_2p_3 - t_1q_1p_1t_2p_3.$$

Note that $t_1q_1t_3q_2p_3$ and $t_2p_2t_3^{-1}p_1$ are both terms in the Hamiltonian h for this link, and the SFT bracket of these two terms involves the string cobracket δh of h ; cf. Example 2.34 below.

The SFT bracket that we have presented here is nearly identical to the construction in [Ng10, §3], with the small difference that we are allowing for more than one t variable. As in [Ng10], we have the following.

Proposition 2.20 ([Ng10, Proposition 3.4]). *The SFT bracket on $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$ has degree +1 and satisfies the following properties:*

- *symmetry:*

$$\{y, x\} = (-1)^{|x||y|+|x|+|y|}\{x, y\};$$

- *Leibniz:*

$$\begin{aligned}\{x, yz\} &= \{x, y\}z + (-1)^{(|x|+1)|y|}y\{x, z\} \\ \{xy, z\} &= x\{y, z\} + (-1)^{|y|(|z|+1)}\{x, z\}y;\end{aligned}$$

- *Jacobi:*

$$\{x, \{y, z\}\} + (-1)^{(|x|+1)(|y|+|z|)}\{y, \{z, x\}\} + (-1)^{(|z|+1)(|x|+|y|)}\{z, \{x, y\}\} = 0.$$

The SFT bracket in Definition 2.18, which is defined on $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$, descends to a map on $\mathcal{A}_{\text{SFT}}^{\text{comm}}$, which we also call the SFT bracket. On $\mathcal{A}_{\text{SFT}}^{\text{comm}}$, there is a simpler way to define the SFT bracket than on $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$. For clarity, we separate out this “definition”, even though in fact it follows from Definition 2.18.

Definition 2.21. The *SFT bracket* $\{\cdot, \cdot\} : \mathcal{A}_{\text{SFT}}^{\text{comm}} \otimes \mathcal{A}_{\text{SFT}}^{\text{comm}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{comm}}$ is the Poisson bracket defined on generators by:

$$\begin{aligned}\{p_i, q_j\} &= -\{q_j, p_i\} = \delta_{ij} \\ \{p_i, p_j\} &= \{q_i, q_j\} = \{t_i^{\pm 1}, p_j\} = \{t_i^{\pm 1}, q_j\} = \{t_i^{\pm 1}, t_j^{\pm 1}\} = 0.\end{aligned}$$

That is, $\{\cdot, \cdot\}$ is uniquely determined by its action on generators, along with symmetry and the Leibniz rule.

We conclude this subsection by reviewing the differential ∂ in the (commutative) Chekanov–Eliashberg DGA $(\mathcal{A}^{\text{comm}}, \partial)$ and its relation to the Hamiltonian and SFT bracket. Recall that $\Delta_1(\Lambda)$ is the collection of immersed disks with boundary on $\Pi_{xy}(\Lambda)$ and convex corners, exactly one of which is labeled by a p (a “positive corner”). For a given $p_j \in \mathcal{S}$, let $\Delta_1(\Lambda; p_j)$ denote the subset of $\Delta_1(\Lambda)$ consisting of disks whose positive corner is at p_j .

Given $\Delta \in \Delta_1(\Lambda; p_j)$, we can associate a word $w'(\Delta)$ by taking $w(\Delta)$ and removing p_j : to be precise, $w'(\Delta) = v_1 \cdots v_k$ where $v_1, \dots, v_k \in \{q_1, \dots, q_n, t_1^\pm, \dots, t_s^\pm\}$ are the labels of the corners and base points that are encountered when we traverse the boundary of Δ starting just after p_j and ending just before p_j .



FIGURE 7. Orientation signs for the differential in the Chekanov–Eliashberg DGA. On the left, a positive crossing (q even, p odd); on the right, a negative crossing (q odd, p even). Each corner of an immersed disk contributes a $+$ sign if it occupies one of the unshaded quadrants or a $-$ sign if it occupies one of the shaded quadrants.

We can also define a sign $\text{sgn}(\Delta) \in \{\pm 1\}$ to be the product of the “LCH orientation signs” associated to the corners of Δ , including p_j , as shown in Figure 7: that is, $\text{sgn}(\Delta)$ is (-1) raised to the number of corners of Δ that are shaded.

The differential ∂ on $\mathcal{A}^{\text{comm}}$ is now defined as follows:

$$\partial(q_j) = \sum_{\Delta \in \Delta_1(\Lambda; p_j)} \text{sgn}(\Delta) w'(\Delta).$$

Extend ∂ to all of $\mathcal{A}^{\text{comm}}$ by setting $\partial(t_i^{\pm 1}) = \partial(1) = 0$ and using the Leibniz rule.

Note that the orientation signs for LCH, as determined by the shadings in Figure 7, are different from the orientation signs for the Hamiltonian h , as shown in Figure 6. However, they have been chosen to be compatible in the following sense; see also Remark 2.17.

Proposition 2.22. *For any $x \in \mathcal{A}^{\text{comm}}$, we have*

$$\partial(x) = \{h_1, x\}.$$

Proof. Since both ∂ and $\{h_1, \cdot\}$ satisfy the Leibniz rule, it suffices to check $\partial(x) = \{h_1, x\}$ for $x \in \{q_1, \dots, q_n, t_1^{\pm}, \dots, t_s^{\pm}\}$. For $x = t_i^{\pm}$, both sides are 0. Now suppose that $x = q_j$. Then for each $\Delta \in \Delta_1(\Lambda; p_j)$, we can write $\tilde{w}(\Delta) = \text{sgn}'(\Delta) v_1 \cdots v_{k-1} p_j$, where $\text{sgn}'(\Delta)$ is the sign defined in Section 2.6: in the notation from that subsection, we have $\text{sgn}'(\Delta) = \epsilon_1(\Delta) \cdots \epsilon_k(\Delta) \epsilon'(\Delta; p_j)$ and $v_k = p_j$. Now note that the shadings in Figures 6 and 7 differ in two quadrants, namely the p quadrants where the orientation of the quadrant’s boundary (induced from the standard orientation on the quadrant) disagrees with the orientation of Λ after the corner. That is, the difference between the sign $\epsilon_k(\Delta)$ associated to the corner at p_j from Figure 6 and the corresponding sign from Figure 7 is precisely $\epsilon'(\Delta; p_j)$. It follows that $\text{sgn}'(\Delta) = \text{sgn}(\Delta)$ and

$$\{\text{sgn}'(\Delta) v_1 \cdots v_{k-1} p_j, q_j\} = \text{sgn}(\Delta) v_1 \cdots v_{k-1}.$$

Summing over all $\Delta \in \Delta_1(\Lambda; p_j)$ yields $\partial(q_j) = \{h_1, q_j\}$, as desired. \square

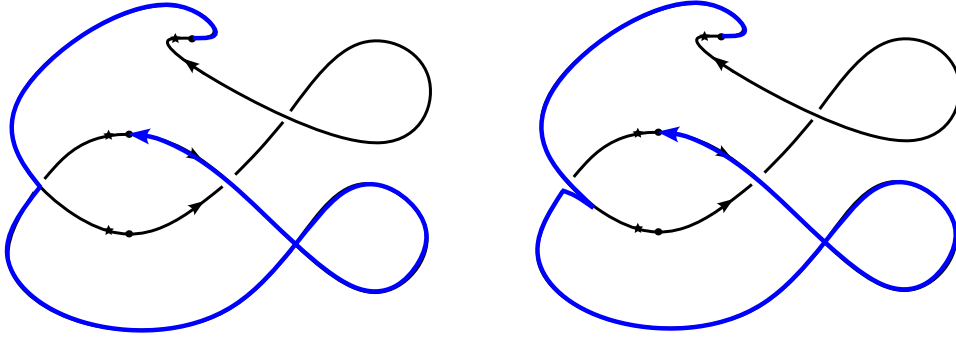


FIGURE 8. The capping path γ_{q_1} from Figure 4 (left) does not have a holomorphic corner at a_1 . It can be perturbed in a neighborhood of a_1 to produce a generic based broken closed string that does have a holomorphic corner at a_1 (right).

2.8. String coproduct. The final ingredient we need in order to define the L_∞ structure is the string coproduct. This is a map

$$\delta : \mathcal{A}_{\text{SFT}} \rightarrow \mathcal{A}_{\text{SFT}},$$

descending to $\delta : \mathcal{A}_{\text{SFT}}^{\text{cyc}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{cyc}}$ and $\delta : \mathcal{A}_{\text{SFT}}^{\text{comm}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{comm}}$, that inserts a (p, q) pair somewhere in the middle of any (based) broken closed string. Here we define δ and present some of its basic properties, following [Ng10, §3.2].

Definition 2.23. A (based) broken closed string γ on Λ is *generic* if whenever $\gamma(t)$ is at the endpoint of some Reeb chord, $\gamma'(t) \neq 0$; in particular, if γ has a Reeb chord discontinuity at t , then $\gamma'(t^-) := \lim_{\tau \rightarrow t^-} \gamma'(\tau)$ and $\gamma'(t^+) := \lim_{\tau \rightarrow t^+} \gamma'(\tau)$ are nonzero.

A generic (based) broken closed string γ *has holomorphic corners* if whenever γ has a discontinuity at a Reeb chord, the xy projection of γ turns left at the crossing corresponding to the Reeb chord.

Any (based) broken closed string is homotopic to a generic (based) broken closed string that has holomorphic corners: see Figure 8 for an illustration.

Now suppose that $\gamma : [0, 1] \rightarrow \Lambda$ is a generic based broken closed string with holomorphic corners. We say that $\tau \in (0, 1)$ is *interior Reeb* for γ if γ does not have a Reeb chord discontinuity at τ but $\gamma(\tau)$ lies at the endpoint of some Reeb chord of Λ . In this case, we can define a new based broken closed string $\delta(\gamma; \tau)$ that coincides with γ except that at time τ we insert a jump from $\gamma(\tau)$ to the other endpoint of the Reeb chord and back again.

As in Section 2.5, the based broken closed string γ corresponds to a word $w(\gamma) = v_1 \cdots v_m$. Suppose that τ is interior Reeb for γ and $\gamma(\tau)$ lies on the

segment of γ between v_i and v_{i+1} . Then

$$w(\delta(\gamma; \tau)) = \begin{cases} v_1 \cdots v_i p_j q_j v_{i+1} \cdots v_m & \text{if } \gamma(\tau) = a_j^-, \text{ the negative endpoint of Reeb chord } a_j \\ v_1 \cdots v_i q_j p_j v_{i+1} \cdots v_m & \text{if } \gamma(\tau) = a_j^+, \text{ the positive endpoint of Reeb chord } a_j. \end{cases}$$

Furthermore, we can associate a sign to $\delta(\gamma; \tau)$ as follows. Let $\epsilon_1 = +1$ if $\gamma(\tau) = a_j^-$ and $\epsilon_1 = -1$ if $\gamma(\tau) = a_j^+$. Let $\epsilon_2 = \pm 1$ depending on whether the orientation of γ at τ agrees or disagrees with the orientation of Λ . Finally, define

$$\epsilon(\gamma; \tau) = (-1)^{|v_1| + \cdots + |v_i|} \epsilon_1 \epsilon_2.$$

Definition 2.24. The *string coproduct* $\delta : \mathcal{A}_{\text{SFT}} \rightarrow \mathcal{A}_{\text{SFT}}$ is defined as follows. Suppose that $w \in \mathcal{W}$ is a word in \mathcal{S} and let γ be any generic based broken closed string with holomorphic corners for which $w(\gamma) = w$. Then

$$\delta(w) = \sum_{\tau \text{ interior Reeb for } \gamma} \epsilon(\gamma; \tau) w(\delta(\gamma; \tau)).$$

The definition of $\delta(w)$ is independent of the choice of γ ; see [Ng10, §3.2].

Example 2.25. Here we calculate $\delta(t_2)$, $\delta(q_1)$, and $\delta(p_1)$ for the example from Section 2.2. The path γ_{t_2} , as shown in Figure 4, passes through the positive endpoint a_1^+ of the Reeb chord a_1 ; it follows that $\delta(t_2) = -q_1 p_1 t_2$. To compute $\delta(q_1)$, note that the path γ_{q_1} in Figure 4 does not have holomorphic corners. Once we perturb it to have holomorphic corners as shown in the right hand diagram in Figure 8, we see that it passes through a_1^+ , a_4^- , a_4^+ , and a_2^+ in succession. This yields the four terms in

$$\delta(q_1) = -q_1 p_1 q_1 - q_1 p_4 q_4 + q_1 q_4 p_4 + q_1 q_2 p_2.$$

By contrast, to compute $\delta(p_1)$, we note that the path γ_{p_1} given by the reverse of γ_{q_1} from Figure 4 does have holomorphic corners and does not need to be perturbed. This leads to one fewer term in $\delta(p_1)$ than in $\delta(q_1)$:

$$\delta(p_1) = -q_2 p_2 p_1 - q_4 p_4 p_1 + p_4 q_4 p_1.$$

Proposition 2.26 ([Ng10, Proposition 3.8]). *The map $\delta : \mathcal{A}_{\text{SFT}} \rightarrow \mathcal{A}_{\text{SFT}}$ has degree -1 and satisfies the following properties:*

- $\delta(xy) = (\delta x)y + (-1)^{|x|} x(\delta y)$;
- $\delta^2 = 0$.

The map $\delta : \mathcal{A}_{\text{SFT}} \rightarrow \mathcal{A}_{\text{SFT}}$ descends to maps $\mathcal{A}_{\text{SFT}}^{\text{cyc}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{cyc}}$ and $\mathcal{A}_{\text{SFT}}^{\text{comm}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{comm}}$, which we will also write as δ . Note that $\delta^2 = 0$ also holds in these quotients, and that δ is also a derivation in $\mathcal{A}_{\text{SFT}}^{\text{comm}}$ with respect to associative multiplication.

2.9. Derivation property for δ . When we construct the L_∞ structure on $\mathcal{A}^{\text{comm}}$ in Section 3, a key ingredient in the proof of the L_∞ relations is that the string coproduct δ behaves nicely with respect to the SFT bracket.

Specifically, when Λ has a single base point, δ is a derivation with respect to $\{\cdot, \cdot\}$:

Proposition 2.27 ([Ng10, Proposition 3.8]). *If Λ is a Legendrian knot with a single base point, then for any $x, y \in \mathcal{A}_{\text{SFT}}^{\text{cyc}}$,*

$$\delta\{x, y\} = \{\delta x, y\} - (-1)^{|x|}\{x, \delta y\}.$$

In the general case where Λ has multiple components and base points, δ is not a derivation with respect to $\{\cdot, \cdot\}$, even if we pass to the commutative quotient $\mathcal{A}_{\text{SFT}}^{\text{comm}}$. However, δ does still satisfy a modified version of the derivation property on $\mathcal{A}_{\text{SFT}}^{\text{comm}}$. This subsection is devoted to stating and proving this property. To state it, we introduce some more notation that will also play a key role in our construction of the L_∞ structure.

We first note that while \mathcal{A}_{SFT} is generated as a \mathbb{Z} module by elements of \mathcal{W} , which are words in the alphabet $\mathcal{S} = \{q_1, \dots, q_n, p_1, \dots, p_n, t_1^\pm, \dots, t_s^\pm\}$, $\mathcal{A}_{\text{SFT}}^{\text{comm}}$ is instead generated as a \mathbb{Z} -module by “commutative words”, words in the alphabet \mathcal{S} up to arbitrary permutation. (Note that two words that are related by permutation are only equal up to the usual sign, e.g., $x_2x_1 = (-1)^{|x_1||x_2|}x_1x_2$.) Write $\mathcal{W}^{\text{comm}}$ as the collection of commutative words in \mathcal{S} , including the empty word.

Definition 2.28. Given a pointed Legendrian link Λ with s base points, define maps

$$\mathbf{v}, \mathbf{w}: \mathcal{W}^{\text{comm}} \rightarrow \mathbb{Z}^s$$

as follows. Let $\mathbf{e}_1, \dots, \mathbf{e}_s$ be the standard basis of \mathbb{Z}^s . For any $v \in \mathcal{S} = \{q_1, \dots, q_n, p_1, \dots, p_n, t_1^\pm, \dots, t_s^\pm\}$, set

$$\mathbf{v}(v) = \mathbf{e}_{e_+(v)} - \mathbf{e}_{e_-(v)} \quad \mathbf{w}(v) = \begin{cases} \mathbf{e}_i & v = t_i \\ -\mathbf{e}_i & v = t_i^{-1} \\ 0 & v \in \{q_1, \dots, q_n, p_1, \dots, p_n\}. \end{cases}$$

Now extend \mathbf{v}, \mathbf{w} multiplicatively to $\mathcal{W}^{\text{comm}}$:

$$\mathbf{v}(xy) = \mathbf{v}(x) + \mathbf{v}(y) \quad \mathbf{w}(xy) = \mathbf{w}(x) + \mathbf{w}(y).$$

The maps \mathbf{v}, \mathbf{w} can be interpreted geometrically as follows. Represent a commutative word $x \in \mathcal{W}^{\text{comm}}$ by a word $\tilde{x} \in \mathcal{W}$ (in any way) and let γ be the based broken closed string associated to \tilde{x} as in Section 2.5. Then the j -th entry in $\mathbf{v}(x)$ is the number of end points of γ at \bullet_j minus the number of beginning points of γ at \bullet_j , and the j -th entry in $\mathbf{w}(x)$ is the number of times γ passes through \star_j , counted with sign.

Definition 2.29. Define a map $\beta: \mathcal{W}^{\text{comm}} \times \mathcal{W}^{\text{comm}} \rightarrow \frac{1}{2}\mathbb{Z}$ by

$$\beta(x, y) = \left(-\frac{1}{2}\mathbf{v}(x) + \mathbf{w}(x)\right) \cdot \mathbf{v}(y),$$

and define a map $\tilde{\beta}: \mathcal{A}_{\text{SFT}}^{\text{comm}} \otimes \mathcal{A}_{\text{SFT}}^{\text{comm}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{comm}}$ to be the bilinear map determined by

$$\tilde{\beta}(x, y) = \beta(x, y)xy$$

for $x, y \in \mathcal{W}^{\text{comm}}$.

We now have the following derivation property for the string cobracket δ .

Proposition 2.30. *For any $x, y \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$,*

$$\delta\{x, y\} = \{\delta x, y\} - (-1)^{|x|}\{x, \delta y\} - (-1)^{|x|}\tilde{\beta}(x, y) - (-1)^{|x|(|y|+1)}\tilde{\beta}(y, x).$$

Note that the correction to the derivation property for δ , $-(-1)^{|x|}\tilde{\beta}(x, y) - (-1)^{|x|(|y|+1)}\tilde{\beta}(y, x)$, is a half-integer multiple of xy . In the special case where Λ is a knot with a single base point, $\mathbf{v} \equiv 0$ and thus this correction disappears, in agreement with Proposition 2.27 above.

Proof of Proposition 2.30. For $x, y \in \mathcal{W}^{\text{comm}}$, define

$$\begin{aligned} f(x, y) &:= \delta\{x, y\} - \{\delta x, y\} + (-1)^{|x|}\{x, \delta y\} \\ g(x, y) &:= -(-1)^{|x|}\tilde{\beta}(x, y) - (-1)^{|x|(|y|+1)}\tilde{\beta}(y, x). \end{aligned}$$

We wish to show that $f(x, y) = g(x, y)$ for all $x, y \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$. Note that $g(y, x) = (-1)^{|x||y|+|x|+|y|}g(x, y)$, while by the definition of $\tilde{\beta}$, $g(x, yz) = g(x, y)z + (-1)^{|x||y|}yg(x, z)$ for any $x, y, z \in \mathcal{W}^{\text{comm}}$. But f satisfies the same properties: $f(y, x) = (-1)^{|x||y|+|x|+|y|}f(x, y)$ from the symmetry property of $\{\cdot, \cdot\}$ in Proposition 2.20, while $f(x, yz) = f(x, y)z + (-1)^{|x||y|}yf(x, z)$ by Propositions 2.20 and 2.26. To show $f(x, y) = g(x, y)$ for all $x, y \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$, it thus suffices to check that $f(x, y) = g(x, y)$ for all $x, y \in \mathcal{S}$; the general result will follow for $x, y \in \mathcal{W}^{\text{comm}}$ by induction on wordlength, and then for $x, y \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$ by linearity.

We show that $f(x, y) = g(x, y)$ for $x, y \in \mathcal{S}$ by breaking into cases. Note that for $x, y \in \mathcal{S}$, $\delta\{x, y\} = 0$ and so $f(x, y) = -\{\delta x, y\} + (-1)^{|x|}\{x, \delta y\}$.

CASE 1: $x \in \{q_i, p_i\}$ and $y \in \{q_j, p_j\}$, where $i \neq j$.

In this case we have $\beta(x, y) = \beta(y, x) = -\frac{1}{2}\mathbf{v}(x) \cdot \mathbf{v}(y)$ and thus $g(x, y) = (-1)^{|x|}\mathbf{v}(x) \cdot \mathbf{v}(y)xy$.

Write x^\pm for the endpoints of the Reeb chord a_i , where x travels along the Reeb chord from x^- to x^+ : that is, $x^\pm = a_i^\pm$ if $x = p_i$ and $x^\pm = a_i^\mp$ if $x = q_i$. Similarly define y^\pm . Write the capping path γ_x as $\gamma_x^- \cdot \gamma_x^+$, where γ_x^- is the portion of γ_x up to x^- (i.e., γ_i^- if $x = p_i$, γ_i^+ if $x = q_i$) and γ_x^+ is the portion after x^+ ($-\gamma_i^+$ if $x = p_i$, $-\gamma_i^-$ if $x = q_i$). Similarly write $\gamma_y = \gamma_y^- \cdot \gamma_y^+$. Finally, write x^*, y^* for the ‘‘signed duals’’ to x, y : $x^* = -p_i$ if $x = q_i$ and $x^* = q_i$ if $x = p_i$, and similarly for y . Note that $\{x, x^*\} = -\{x^*, x\} = \{y, y^*\} = -\{y^*, y\} = 1$.

Now $\{\delta x, y\} = 0$ unless at least one of γ_x^\pm passes through an endpoint a_j^\pm of y . By the definition of δ , if γ_x^- passes through a_j^- (resp. a_j^+), then δx contains the term $+p_jq_jx$ (resp. $-q_jp_jx$). Stated another way, if γ_x^- passes through

y^+ (resp. y^-), then δx contains the term $-y^*yx$ (resp. yy^*x). The resulting contribution to $-\{\delta x, y\}$ is $-\{-y^*yx, y\} = (-1)^{|x|}\{xyy^*, y\} = -(-1)^{|x|}xy$ (resp. $-\{yy^*x, y\} = -(-1)^{|x|}\{xyy^*, y\} = (-1)^{|x|}xy$). But in both cases this exactly equals the contribution to $(-1)^{|x|}\mathbf{v}(x) \cdot \mathbf{v}(y)xy$ arising from the fact that γ_x^- shares an endpoint with γ_y^\pm .

In a similar way, we can show that in each instance where γ_x shares an endpoint with γ_y at a base point, we get equal contributions to $f(x, y)$ and $g(x, y)$:

	$f(x, y)$		$g(x, y)$
	$-\{\delta x, y\}$	$(-1)^{ x }\{x, \delta y\}$	$(-1)^{ x }\mathbf{v}(x) \cdot \mathbf{v}(y)xy$
γ_x^- contains y^+	$-\{-y^*yx, y\}$		$-(-1)^{ x }xy$
γ_x^- contains y^-	$-\{yy^*x, y\}$		$(-1)^{ x }xy$
γ_x^+ contains y^+	$-\{-(-1)^{ x }xyy^*, y\}$		$(-1)^{ x }xy$
γ_x^+ contains y^-	$-\{(-1)^{ x }xyy^*, y\}$		$-(-1)^{ x }xy$
γ_y^- contains x^+		$(-1)^{ x }\{x, -x^*xy\}$	$-(-1)^{ x }xy$
γ_y^- contains x^-		$(-1)^{ x }\{x, xx^*y\}$	$(-1)^{ x }xy$
γ_y^+ contains x^+		$(-1)^{ x }\{x, -(-1)^{ y }yx^*x\}$	$(-1)^{ x }xy$
γ_y^+ contains x^-		$(-1)^{ x }\{x, (-1)^{ y }yx^*x\}$	$-(-1)^{ x }xy$

In each case the expression in the $f(x, y)$ columns equals the expression in the $g(x, y)$ column. This shows that $f(x, y) = g(x, y)$.

CASE 2: $x, y \in \{q_i, p_i\}$.

This is similar to Case 1 but with additional contributions to $f(x, y)$ and $g(x, y)$. Specifically, if $|q_i|$ is even, then the capping path γ_{q_i} does not have holomorphic corners and must be perturbed, leading to an additional term $-q_i p_i q_i$ in δq_i . This adds $2q_i^2, -2q_i p_i, -2p_i q_i, 0$ to $f(q_i, q_i), f(q_i, p_i), f(p_i, q_i), f(p_i, p_i)$ respectively, in each case exactly matching the contribution to $g(x, y) = (-1)^{|x|}\mathbf{v}(x) \cdot \mathbf{v}(y)xy$ coming from the fact that γ_x and γ_y have the same endpoints, either in the same order if $x = y$ or in opposite order otherwise. (Note for $x = y = p_i$ that $p_i^2 = 0$.) Similarly, if $|q_i|$ is odd, then γ_{p_i} does not have holomorphic corners, leading to an additional term $+p_i q_i p_i$ in δp_i . This adds $0, 2q_i p_i, -2p_i q_i, 2p_i^2$ to $f(q_i, q_i), f(q_i, p_i), f(p_i, q_i), f(p_i, p_i)$ respectively, again matching the corresponding contribution to $g(x, y)$ in each case.

CASE 3: $x \in \{t_i, t_i^{-1}\}, y \in \{q_j, p_j\}$. By symmetry this also covers the case when x and y are swapped.

For $x = t_i^{\pm 1}$, we have $f(x, y) = -\{\delta t_i^{\pm 1}, y\}$. This is nonzero if either endpoint y^\pm of y lies on the capping path γ_{t_i} between $t_{e_-(t_i)}$ and t_i . If y^+ (resp. y^-) lies on γ_{t_i} , then δt_i contains a term $-y^*yt_i$ (resp. yy^*t_i), and this contributes $-t_i y$ (resp. $t_i y$) to $f(t_i, y)$ and $t_i^{-1}y$ (resp. $-t_i^{-1}y$) to $f(t_i^{-1}, y)$. On the other hand, we compute that

$$g(t_i^{\pm 1}, y) = ((\mathbf{v}(t_i^{\pm 1}) - \mathbf{w}(t_i^{\pm 1})) \cdot \mathbf{v}(y)) t_i^{\pm 1}y = \mp(\mathbf{e}_{e_-(t_i)} \cdot \mathbf{v}(y)) y t_i^{\pm 1}$$

and the presence of y^\pm on γ_{t_i} contributes ± 1 to $\mathbf{e}_{e_-(t)} \cdot \mathbf{v}(y)$. We conclude that $f(x, y) = g(x, y)$.

CASE 4: $x \in \{t_i, t_i^{-1}\}, y \in \{t_j, t_j^{-1}\}$.

In this case we have $f(x, y) = 0$ and $g(x, y) = -(\beta(x, y) + \beta(y, x))xy$. Define signs $\pm_1, \pm_2 \in \{\pm 1\}$ by $x = t_i^{\pm_1}, y = t_j^{\pm_2}$. Then

$$\begin{aligned} \beta(x, y) + \beta(y, x) &= (\mathbf{w}(t_i^{\pm_1}) - \mathbf{v}(t_i^{\pm_1})) \cdot (\mathbf{w}(t_j^{\pm_2}) - \mathbf{v}(t_j^{\pm_2})) - \mathbf{w}(t_i^{\pm_1}) \cdot \mathbf{w}(t_j^{\pm_2}) \\ &= \pm_1 \pm_2 (\mathbf{e}_{e_-(t_i)} \cdot \mathbf{e}_{e_-(t_j)} - \mathbf{e}_i \cdot \mathbf{e}_j). \end{aligned}$$

But both $\mathbf{e}_i \cdot \mathbf{e}_j$ and $\mathbf{e}_{e_-(t_i)} \cdot \mathbf{e}_{e_-(t_j)}$ are 0 unless $i = j$, in which case they are both 1. Thus $g(x, y) = 0$ and we are done. \square

To conclude this subsection, we collect some easy properties of $\tilde{\beta}$ that will be useful later. To set this up, we extend the definitions of $\mathbf{v}, \mathbf{w}, \beta$ slightly, as follows. If $x \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$ is a linear combination of commutative words w_i such that $\mathbf{v}(w_i) \in \mathbb{Z}^s$ is the same for all i , then we define $\mathbf{v}(x)$ to be equal to this same element of \mathbb{Z}^s . Similarly we can define $\mathbf{w}(x) \in \mathbb{Z}^s$ if $\mathbf{w}(w_i)$ is the same for all i , and $\beta(x, y) \in \frac{1}{2}\mathbb{Z}$ is defined as in Definition 2.29 whenever $\mathbf{v}(x), \mathbf{w}(x), \mathbf{v}(y)$ are well-defined.

Proposition 2.31. (1) *The Hamiltonian h satisfies $\mathbf{v}(h) = 0$.*

- (2) *If $y \in \mathcal{W}^{\text{comm}}$ satisfies $\mathbf{v}(y) = 0$, then $\tilde{\beta}(x, y) = 0$ for any $x \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$. It follows that the Hamiltonian h satisfies $\tilde{\beta}(x, h) = 0$ for any $x \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$.*
- (3) *For any $x_1, x_2 \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$ for which $\mathbf{v}(x_1), \mathbf{v}(x_2)$ are defined, $\mathbf{v}(\{x_1, x_2\}) = \mathbf{v}(x_1 x_2) = \mathbf{v}(x_1) + \mathbf{v}(x_2)$, and similarly for \mathbf{w} .*
- (4) *If $\mathbf{v}(x)$ is defined for $x \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$, then $\mathbf{v}(\delta x) = \mathbf{v}(x)$.*
- (5) *For any $x_1, x_2, x_3 \in \mathcal{W}^{\text{comm}}$, we have*

$$\begin{aligned} \beta(\{x_1, x_2\}, x_3) &= \beta(x_1 x_2, x_3) = \beta(x_1, x_3) + \beta(x_2, x_3) \\ \beta(x_1, \{x_2, x_3\}) &= \beta(x_1, x_2 x_3) = \beta(x_1, x_2) + \beta(x_1, x_3). \end{aligned}$$

- (6) *For any $x_1, x_2 \in \mathcal{A}^{\text{comm}}$ and $y \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$, we have*

$$\{y, \tilde{\beta}(x_2, x_1)\} + (-1)^{|x_1|(|x_2|+1)} \{\tilde{\beta}(y, x_1), x_2\} = \tilde{\beta}(\{y, x_2\}, x_1) + (-1)^{|x_2|(|y|+1)} \beta(x_2, x_1) x_2 \{y, x_1\}.$$

Proof. (1) holds because h is a linear combination of cyclically composable words, each of which satisfies $\mathbf{v} = 0$; (2) is an immediate consequence. (3) is clear by inspection of the definitions of \mathbf{v}, \mathbf{w} in Definition 2.28 and the SFT bracket in Definition 2.18, and (4) similarly follows from the definition of δ in Definition 2.24; (5) follows from (3).

To establish (6), we may assume that x_1, x_2, y are all words in $\mathcal{W}^{\text{comm}}$. Then by applying Proposition 2.20 and using the fact that $\{x_1, x_2\} = 0$ since

$x_1, x_2 \in \mathcal{A}^{\text{comm}}$, we compute that

$$\begin{aligned} \{y, \tilde{\beta}(x_2, x_1)\} &= \beta(x_2, x_1)\{y, x_2x_1\} = \beta(x_2, x_1)(\{y, x_2\}x_1 + (-1)^{(|y|+1)|x_2|}x_2\{y, x_1\}) \\ \{\tilde{\beta}(y, x_1), x_2\} &= \beta(y, x_1)\{yx_1, x_2\} = \beta(y, x_1)(-1)^{|x_1|(|x_2|+1)}\{y, x_2\}x_1 \\ \{\tilde{\beta}(y, x_2), x_1\} &= \beta(y, x_2)\{yx_2, x_1\} = \beta(y, x_2)(-1)^{|x_2|(|x_1|+1)}\{y, x_1\}x_2; \end{aligned}$$

combining these and using (5) yields (6). \square

2.10. The quantum master equation and differential. We conclude this section by discussing the relation between the Hamiltonian h and the SFT bracket and string cobracket. This allows us to define a “differential” d on $\mathcal{A}_{\text{SFT}}^{\text{comm}}$ that extends the standard differential ∂ in the (commutative) Chekanov–Eliashberg DGA.

The key relation is a formula called the “quantum master equation” in [Ng10]. We omit the proof here since it is identical to the proof given there: at heart, a standard Floer-theoretic argument decomposing the boundary of a 1-dimensional moduli space into products of two 0-dimensional moduli spaces.

Proposition 2.32 ([Ng10, Proposition 3.13]). *If we view the Hamiltonian h as an element of either $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$ or $\mathcal{A}_{\text{SFT}}^{\text{comm}}$, then we have:*

$$\delta h = \frac{1}{2}\{h, h\}.$$

Remark 2.33. In [Ng10, Proposition 3.13], the quantum master equation is presented as $\delta h + \frac{1}{2}\{h, h\} = 0$. However, our choice of sign convention for h in Section 2.6 differs from the one in [Ng10], and consequently the sign in Proposition 2.32 differs as well. See [Ng10, Appendix A] for a full discussion of this sign difference.

Example 2.34. Using our calculation of h in Example 2.16, we can verify Proposition 2.32 for the example from Section 2.2. For instance, the bracket $\{t_1q_1t_3q_2p_3, t_2p_2t_3^{-1}p_1\} = t_1t_2p_2q_2p_3 - t_1q_1p_1t_2p_3$ computed in Example 2.19 contributes to $\frac{1}{2}\{h, h\}$ and corresponds to the terms $t_1t_2(p_2q_2p_3)$ and $t_1(-q_1p_1t_2)p_3$ from $t_1t_2(\delta p_3)$ and $t_1(\delta t_2)p_3$, which appear in δh .

Definition 2.35. The *SFT differential* is the map $d: \mathcal{A}_{\text{SFT}}^{\text{comm}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{comm}}$ defined by:

$$d(x) = \{h, x\} - \delta x.$$

The map d on $\mathcal{A}_{\text{SFT}}^{\text{comm}}$ is induced from a map $d: \mathcal{A}_{\text{SFT}}^{\text{cyc}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{cyc}}$ with the same definition. In the single-component case, d is a differential:

Proposition 2.36 ([Ng10, Proposition 3.15]). *If Λ is a Legendrian knot with a single base point, then the map $d: \mathcal{A}_{\text{SFT}}^{\text{cyc}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{cyc}}$ satisfies $d^2 = 0$.*

In general, however, d does not satisfy $d^2 = 0$, even if we pass from $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$ to the quotient $\mathcal{A}_{\text{SFT}}^{\text{comm}}$. (We abuse language in referring to d as the “SFT

differential".) Instead, we have the following result, where $\tilde{\beta}$ was defined in Definition 2.29:

Proposition 2.37. *Let Λ be a pointed Legendrian link. Then the following equality holds for all $x \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$:*

$$d^2x = \tilde{\beta}(h, x).$$

Proof. By Propositions 2.20 and 2.32, we have

$$2\{h, \{h, x\}\} = -\{x, \{h, h\}\} = 2\{\delta h, x\}.$$

It follows that

$$\begin{aligned} d^2x &= \{h, \{h, x\}\} - \{h, \delta x\} - \delta\{h, x\} + \delta^2x \\ &= -\delta\{h, x\} + \{\delta h, x\} - \{h, \delta x\} \\ &= \tilde{\beta}(h, x) + \tilde{\beta}(x, h) \\ &= \tilde{\beta}(h, x) \end{aligned}$$

where we have used Propositions 2.26, 2.30, and 2.31 (2) in the second, third, and fourth lines, respectively. \square

For future use, we note that d satisfies essentially the same modified derivation property with respect to $\{\cdot, \cdot\}$ as does δ :

Proposition 2.38. *For any $x, y \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$,*

$$d\{x, y\} = \{dx, y\} - (-1)^{|x|}\{x, dy\} + (-1)^{|x|}\tilde{\beta}(x, y) + (-1)^{|x|(|y|+1)}\tilde{\beta}(y, x).$$

Proof. Immediate from Proposition 2.30 and the following consequence of the Jacobi identity:

$$\{h, \{x, y\}\} = \{\{h, x\}, y\} - (-1)^{|x|}\{x, \{h, y\}\}.$$

\square

Remark 2.39. It is clear from the construction of the differential d on $\mathcal{A}_{\text{SFT}}^{\text{comm}}$ that it extends the differential ∂ on the commutative Chekanov–Eliashberg DGA $(\mathcal{A}^{\text{comm}}, \partial)$. To be precise, $\mathcal{A}^{\text{comm}}$ is precisely the quotient $\mathcal{A}_{\text{SFT}}^{\text{comm}}/\mathcal{F}^1\mathcal{A}_{\text{SFT}}^{\text{comm}}$. Since the Hamiltonian h lies in $\mathcal{F}^1\mathcal{A}_{\text{SFT}}^{\text{comm}}$ by Definition/Proposition 2.15, it follows from Proposition 2.37 that the image of d^2 in $\mathcal{A}_{\text{SFT}}^{\text{comm}}$ lies in $\mathcal{F}^1\mathcal{A}_{\text{SFT}}^{\text{comm}}$. Thus d induces an honest differential on $\mathcal{A}^{\text{comm}} \cong \mathcal{A}_{\text{SFT}}^{\text{comm}}/\mathcal{F}^1\mathcal{A}_{\text{SFT}}^{\text{comm}}$. The only part of $dx = \{h, x\} + \delta x$ that does not raise filtration level is $\{h, x\}$, and we conclude from Proposition 2.22 that d induces the map ∂ on $\mathcal{A}^{\text{comm}}$.

3. THE L_∞ STRUCTURE

In this section, we construct the L_∞ structure associated to a pointed Legendrian link and present some examples. The proof that the L_∞ relations are satisfied, and thus that our construction actually constitutes an L_∞ algebra, is deferred until Section 4.

Throughout this section and for the rest of the paper, we fix a field \mathbb{k} of characteristic 0. Rather than using the \mathbb{Z} -algebra $\mathcal{A}^{\text{comm}}$, we will tensor with \mathbb{k} and use the \mathbb{k} -algebra $\mathcal{A}^{\text{comm}} \otimes \mathbb{k}$. However, to simplify notation, we will abuse notation and henceforth use $\mathcal{A}^{\text{comm}}$ to mean the \mathbb{k} -algebra $\mathcal{A}^{\text{comm}} \otimes \mathbb{k}$. We note that to define the L_∞ structure on $\mathcal{A}^{\text{comm}}$, as we will do, \mathbb{k} could also be any field of characteristic $\neq 2$; however, the invariance result in Section 5 relies on \mathbb{k} having characteristic 0.

3.1. L_∞ algebras. We begin by reviewing the definition of an L_∞ algebra, partly to state the sign conventions that we use in this paper.

Definition 3.1. An L_∞ algebra (or *strong homotopy Lie algebra*) is a \mathbb{Z} -graded vector space A over the field \mathbb{k} equipped with operations $\ell_k : A^{\otimes k} \rightarrow A$ of degree $k - 2$ for each $k \geq 1$, satisfying:

- graded symmetry: for any permutation $\sigma \in S_k$,

$$\ell_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \chi(\sigma, x_1, \dots, x_k) \ell_k(x_1, \dots, x_k);$$

- strong homotopy Jacobi identities: for any $k \geq 1$,

$$\sum_{i=1}^k (-1)^{i(k-i)} \sum_{\sigma \in \text{Unshuff}(i, k-i)} \chi(\sigma, x_1, \dots, x_k) \ell_{k-i+1}(\ell_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(k)}) = 0.$$

Here $\chi(\sigma, x_1, \dots, x_k)$ is the alternating Koszul sign, defined to be $-(-1)^{|x_i||x_{i+1}|}$ for $\sigma = (i \ i+1)$ and extended multiplicatively on S_k , and $\text{Unshuff}(i, k-i)$ is the subset of S_k consisting of permutations σ with $\sigma(1) < \sigma(2) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(k)$.

For concreteness, the first three Jacobi identities are:

$$\begin{aligned} 0 &= \ell_1(\ell_1(x_1)) \\ 0 &= \ell_1(\ell_2(x_1, x_2)) - \ell_2(\ell_1(x_1), x_2) - (-1)^{|x_1|} \ell_2(x_1, \ell_1(x_2)) \\ 0 &= \ell_2(\ell_2(x_1, x_2), x_3) + (-1)^{|x_1|(|x_2|+|x_3|)} \ell_2(\ell_2(x_2, x_3), x_1) + (-1)^{|x_3|(|x_1|+|x_2|)} \ell_2(\ell_2(x_3, x_1), x_2) \\ &\quad + \ell_3(\ell_1(x_1), x_2, x_3) + (-1)^{|x_1|} \ell_3(x_1, \ell_1(x_2), x_3) + (-1)^{|x_1|+|x_2|} \ell_3(x_1, x_2, \ell_1(x_3)) \\ &\quad + \ell_1(\ell_3(x_1, x_2, x_3)). \end{aligned}$$

If $(A, \{\ell_k\})$ is an L_∞ algebra, then (A, ℓ_1) is a complex, with ℓ_1 being a differential of degree -1 , and the homology $H_*(A, \ell_1)$ is a Lie algebra with Lie bracket induced by ℓ_2 .

We say that an L_∞ algebra $(A, \{\ell_k\})$ is *strict* if $\ell_k = 0$ for $k \geq 3$. In this case, the usual Jacobi identity for ℓ_2 holds on the nose, and (A, ℓ_1, ℓ_2) is a DG Lie algebra.

Definition 3.2. A *homotopy Poisson algebra* is an L_∞ algebra A that is also an associative algebra A , such that the L_∞ operations ℓ_k satisfy the

Leibniz rule for all $k \geq 1$:

$$\ell_k(x_1, \dots, x_k x'_k) = \ell_k(x_1, \dots, x_k) x'_k + (-1)^{(|x_1| + \dots + |x_{k-1}| + k)|x_k|} x_k \ell_k(x_1, \dots, x'_k).$$

If A is a homotopy Poisson algebra, then (A, ℓ_1) is a DG algebra, and the homology $H_*(A, \ell_1)$ is an associative algebra. Furthermore, ℓ_2 induces a Poisson bracket on $H_*(A, \ell_1)$, and so $(H_*(A, \ell_1), \ell_2)$ is a Poisson algebra. If in addition A is strict, then A itself is also a Poisson algebra with Poisson bracket ℓ_2 .

3.2. Definition of the L_∞ structure for Legendrian links. We now construct the L_∞ algebra associated to a Legendrian link. Let Λ be a pointed Legendrian link equipped with a Maslov potential, and let \mathbb{k} be a field of characteristic 0. The vector space underlying the L_∞ algebra (A, ℓ_*) in the notation of Section 3.1) is $\mathcal{A}^{\text{comm}}$, which we recall from the introduction to Section 3 is now the graded polynomial ring over \mathbb{k} generated by $q_1, \dots, q_n, t_1^{\pm 1}, \dots, t_s^{\pm 1}$ (but not p_1, \dots, p_n) in the notation of Section 2.1.

To define the L_∞ operations on $\mathcal{A}^{\text{comm}}$, we use the machinery built up in Section 2. Let $h \in \mathcal{A}_{\text{SFT}}^{\text{cyc}}$ be the Hamiltonian for Λ . Recall from Definition/Proposition 2.15 that $h \in \mathcal{F}^1 \mathcal{A}_{\text{SFT}}^{\text{cyc}}$: that is, every term in h contains at least one p . We can further write

$$h = h_1 + h_2 + h_3 + \dots$$

where h_k consists of all terms in h containing exactly k p 's; in the language of SFT, h_k counts holomorphic disks with exactly k positive punctures.

Definition 3.3. For $k \geq 1$, define $\ell_k : (\mathcal{A}^{\text{comm}})^{\otimes k} \rightarrow \mathcal{A}^{\text{comm}}$ as follows:

$$\begin{aligned} \ell_1(x_1) &= \{h_1, x_1\} \\ \ell_2(x_1, x_2) &= (-1)^{|x_1|} \{\{h_2, x_1\}, x_2\} - \frac{1}{2} (-1)^{|x_1|} \{\delta x_1, x_2\} - \frac{1}{2} \{x_1, \delta x_2\} \\ &\quad - \frac{1}{2} \tilde{\beta}(x_1, x_2) + \frac{1}{2} (-1)^{|x_1||x_2|} \tilde{\beta}(x_2, x_1) \end{aligned}$$

and for $k \geq 3$,

$$\ell_k(x_1, \dots, x_k) = (-1)^{|x_{k-1}| + |x_{k-3}| + \dots} \{ \dots \{ \{ \{ h_k, x_1 \}, x_2 \}, x_3 \}, \dots, x_k \}.$$

See Definition 2.29 for the definition of $\tilde{\beta}$ appearing in ℓ_2 ; for computational purposes, it may be helpful to note that when $x_1, x_2 \in \mathcal{W}^{\text{comm}}$,

$$-\frac{1}{2} \tilde{\beta}(x_1, x_2) + \frac{1}{2} (-1)^{|x_1||x_2|} \tilde{\beta}(x_2, x_1) = \frac{1}{2} (\mathbf{v}(x_1) \cdot \mathbf{w}(x_2) - \mathbf{w}(x_1) \cdot \mathbf{v}(x_2)) x_1 x_2.$$

Also note that each ℓ_k does indeed map into $\mathcal{A}^{\text{comm}}$ and not just into the larger $\mathcal{A}_{\text{SFT}}^{\text{comm}}$: this is because the SFT bracket reduces the number of p 's by 1, δ increases the number of p 's by 1, and each x_i contains no p 's.

The first operation ℓ_1 is precisely the Chekanov–Eliashberg differential ∂ on $\mathcal{A}^{\text{comm}}$; see Section 2.10. Thus the first L_∞ relation simply states the well-known fact that $\partial^2 = 0$.

Proposition 3.4. *The operations ℓ_k constructed in Definition 3.3 satisfy graded symmetry and the Leibniz rule.*

Proof. Note that if $x \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$ and $y, z \in \mathcal{A}^{\text{comm}}$, then by properties of $\{\cdot, \cdot\}$ (Proposition 2.20),

$$\begin{aligned} \{\{x, y\}, z\} &= (-1)^{(|x|+|y|)(|z|+1)+1} \{z, \{x, y\}\} \\ &= \{x, \{y, z\}\} + (-1)^{(|x|+1)(|y|+|z|)} \{y, \{z, x\}\} \\ &= (-1)^{(|x|+1)(|y|+|z|)} \{y, \{z, x\}\} \\ &= (-1)^{(|x|+1)(|y|+1)} \{\{x, z\}, y\}, \end{aligned}$$

where the third equality follows from the fact that $\{y, z\} = 0$ since neither y nor z contains any p 's. Thus we have

$$(-1)^{|x_2|} \{\{h_2, x_2\}, x_1\} = (-1)^{|x_1||x_2|+|x_1|+1} \{\{h_2, x_1\}, x_2\}.$$

The fact that $\ell_2(x_2, x_1) = (-1)^{|x_1||x_2|+1} \ell_2(x_1, x_2)$ is now easy to check from this.

The proof that ℓ_k is graded-symmetric for $k \geq 3$ is similar. For $j = 1, \dots, k-1$, transposing x_j and x_{j+1} in the definition of ℓ_k has the effect of replacing $\{\{y, x_j\}, x_{j+1}\}$ by $(-1)^{|x_j|+|x_{j+1}|} \{\{y, x_{j+1}\}, x_j\}$, where $y = \{\dots \{h_k, x_1\}, \dots, x_{j-1}\}$. Now $\{\{y, x_{j+1}\}, x_j\} = (-1)^{(|x_j|+1)(|x_{j+1}|+1)} \{\{y, x_j\}, x_{j+1}\}$ and thus transposing x_j and x_{j+1} changes $\ell_k(x_1, \dots, x_k)$ by multiplication by $(-1)^{|x_j||x_{j+1}|+1}$, as desired.

It remains to show that for $k \geq 1$, ℓ_k satisfies the Leibniz rule as stated in Definition 3.2. For $k \neq 2$, this is immediate from the fact that $\{\cdot, \cdot\}$ satisfies the Leibniz rule (Proposition 2.20). For $k = 2$ we additionally compute that

$$\{x_1, \delta(x_2 x'_2)\} = \{x_1, (\delta x_2) x'_2\} + (-1)^{|x_2|} \{x_1, x_2 (\delta x'_2)\} = \{x_1, \delta x_2\} x'_2 + (-1)^{|x_1||x_2|} x_2 \{x_1, \delta x'_2\}$$

by Propositions 2.20 and 2.26 and the fact that $\{x_1, x_2\} = \{x_1, x'_2\} = 0$, and that

$$\begin{aligned} \tilde{\beta}(x_1, x_2 x'_2) &= \tilde{\beta}(x_1, x_2) x'_2 + (-1)^{|x_1||x_2|} x_2 \tilde{\beta}(x_1, x'_2) \\ \tilde{\beta}(x_2 x'_2, x_1) &= (-1)^{|x_1||x'_2|} \tilde{\beta}(x_2, x_1) x'_2 + x_2 \tilde{\beta}(x'_2, x_1) \end{aligned}$$

from Proposition 2.31 (5). Combining these, we conclude that ℓ_2 satisfies the Leibniz rule. \square

Remark 3.5. The need for ℓ_2 to be graded-symmetric is what forces the presence of the $\frac{1}{2}$ fractions in Definition 3.3. Alternatively, one could multiply every ℓ_k by 2 to obtain an L_∞ structure whose operations have integer coefficients, for instance if one wished to work over a field of characteristic 2. However, ℓ_1 would no longer coincide with the usual Chekanov–Eliashberg differential ∂ but would instead equal 2∂ .

As an additional reason not to work over a field of characteristic 2, we note that the quasi-isomorphism type of $(\mathcal{A}^{\text{comm}}, \partial)$ is only guaranteed to be invariant under Legendrian isotopy if we work over a field of characteristic 0: see [ENS02, section 3.4].

We can now state our main result.

Proposition 3.6. *For any pointed Legendrian link, $(\mathcal{A}^{\text{comm}}, \{\ell_k\}_{k=1}^\infty)$ is a homotopy Poisson algebra.*

To prove Proposition 3.6, it remains to check that the ℓ_k operations satisfy the L_∞ relations. We defer this to Section 4.

An immediate corollary of Proposition 3.6 is that the commutative Legendrian contact homology of a Legendrian link inherits a Poisson bracket from ℓ_2 .

Corollary 3.7. *For any pointed Legendrian link, $(H_*(\mathcal{A}^{\text{comm}}, \partial), \ell_2)$ is a Poisson algebra.*

3.3. A formula for the string portion of ℓ_2 . The definition of the ℓ_k operations in Definition 3.3 is relatively straightforward except for ℓ_2 . Here we take a brief digression and present an equivalent formula for ℓ_2 that is convenient for computations. Write

$$\ell_2(x_1, x_2) = (-1)^{|x_1|} \{\{h_2, x_1\}, x_2\} + \ell_2^{\text{str}}(x_1, x_2)$$

where we define

$$\ell_2^{\text{str}}(x_1, x_2) := \frac{1}{2} \left((-1)^{|x_1|+1} \{\delta x_1, x_2\} - \{x_1, \delta x_2\} - \tilde{\beta}(x_1, x_2) + (-1)^{|x_1||x_2|} \tilde{\beta}(x_2, x_1) \right).$$

Note that $\ell_2^{\text{str}}(x_1, x_2)$ is the part of $\ell_2(x_1, x_2)$ that does not involve any disks in the Hamiltonian but does involve the string coproduct δ ; we call ℓ_2^{str} the *string portion* of ℓ_2 . For any x_1, x_2 , $\ell_2^{\text{str}}(x_1, x_2)$ is some half-integer multiple of $x_1 x_2$. Here we present a simple formula for this half-integer.

To that end, we set some notation. Given two distinct points z_1, z_2 on a pointed Lagrangian link Λ , a *segment* between z_1 and z_2 is an embedded path in Λ whose endpoints are z_1 and z_2 ; there are exactly 2 or 0 segments between z_1 and z_2 depending on whether the points lie on the same component of Λ or not. A segment between z_1 and z_2 is *unbroken* if its interior does not contain any base points of Λ .

Write $\mathcal{S}_0 = \{q_1, \dots, q_n, t_1, \dots, t_s\}$ for the collection of q -variables associated to the Reeb chords of Λ , along with the base points of Λ , and note that the elements of \mathcal{S}_0 along with the inverses $t_1^{-1}, \dots, t_s^{-1}$ generate $\mathcal{A}^{\text{comm}}$. We associate *endpoints* to each $x \in \mathcal{S}_0$ as follows: for $x = q_j$, the endpoints of x are a_j^+ and a_j^- , the top and bottom endpoints of the Reeb chord a_j ; for $x = t_i$, we say that the point \bullet_i itself is the unique endpoint of t_i .

For distinct $x_1, x_2 \in \mathcal{S}_0$, write $\mathcal{P}(x_1, x_2)$ for the collection of unbroken segments between an endpoint of x_1 and an endpoint of x_2 ; there are at most 4 elements in $\mathcal{P}(x_1, x_2)$. Given a path $\gamma \in \mathcal{P}(x_1, x_2)$, orient γ following the orientation of Λ , and define $\sigma(\gamma)$ to be +1 if γ is oriented from x_1 to x_2

and -1 if it is oriented from x_2 to x_1 . Now define $n(\gamma) \in \{\pm 1\}$ as follows:

$$n(\gamma) = \begin{cases} -\sigma_1\sigma_2\sigma(\gamma) & \text{if } (x_1, x_2) = (q_{j_1}, q_{j_2}) \text{ and the endpoints of } \gamma \text{ are } a_{j_1}^{\sigma_1}, a_{j_2}^{\sigma_2} \\ -\sigma & \text{if } (x_1, x_2) = (q_j, t_i) \text{ and the endpoints of } \gamma \text{ are } a_j^\sigma, \bullet_i \\ \sigma & \text{if } (x_1, x_2) = (t_i, q_j) \text{ and the endpoints of } \gamma \text{ are } \bullet_i, a_j^\sigma \\ \sigma(\gamma) & \text{if } (x_1, x_2) = (t_{i_1}, t_{i_2}). \end{cases}$$

We now have the following result, which completely determines ℓ_2^{str} since it is easy to check that $\ell_2^{\text{str}}(t_i^{-1}, \cdot) = -t_i^{-2}\ell_2^{\text{str}}(t_i, \cdot)$.

Proposition 3.8. *Let $x_1, x_2 \in \mathcal{S}_0$. If $x_1 = x_2$, then $\ell_2^{\text{str}}(x_1, x_2) = 0$; if $x_1 \neq x_2$, then*

$$\ell_2^{\text{str}}(x_1, x_2) = \frac{1}{2} \left(\sum_{\gamma \in \mathcal{P}(x_1, x_2)} n(\gamma) \right) x_1 x_2.$$

Proof. From the definition of ℓ_2^{str} , we have $\ell_2^{\text{str}}(x_2, x_1) = (-1)^{|x_1||x_2|+1}\ell_2^{\text{str}}(x_1, x_2)$. Thus if $x_1 = x_2$, then $\ell_2^{\text{str}}(x_1, x_1) = 0$: either $|x_1|$ is even, in which case $\ell_2^{\text{str}}(x_1, x_1) = 0$ from the graded symmetry of ℓ_2^{str} , or $|x_1|$ is odd, in which case $\ell_2^{\text{str}}(x_1, x_1)$ is a multiple of x_1^2 , which is 0 in $\mathcal{A}^{\text{comm}}$.

Now suppose $x_1 \neq x_2$. By the symmetry of ℓ_2^{str} , it suffices to consider 3 cases: $(x_1, x_2) = (q_{j_1}, q_{j_2})$; $(x_1, x_2) = (q_j, t_i)$; $(x_1, x_2) = (t_{i_1}, t_{i_2})$.

First suppose $(x_1, x_2) = (q_{j_1}, q_{j_2})$. Then

$$(1) \quad 2\ell_2^{\text{str}}(q_{j_1}, q_{j_2}) = (-1)^{|q_{j_1}|+1} \{ \delta q_{j_1}, q_{j_2} \} - \{ q_{j_1}, \delta q_{j_2} \}.$$

Contributions to the right hand side of (1) correspond exactly to instances in which the capping path for one of q_{j_1}, q_{j_2} passes through an endpoint of the other Reeb chord. By the construction of our capping paths, these instances are in precise correspondence to unbroken segments between $q_{j_1}^\pm$ and $q_{j_2}^\pm$.

It now suffices to check that for such a segment γ , the contribution to the right hand side of (1) is exactly $n(\gamma)x_1x_2$. This is a straightforward case-by-case check, which we will illustrate in two cases. If γ is a path from $a_{j_1}^+$ to $a_{j_2}^+$, then the front half $\gamma_{j_2}^+$ of the capping path $\gamma_{q_{j_2}} = \gamma_{j_2}^+ \cdot (-\gamma_{j_2}^-)$ passes through $a_{j_1}^+$, yielding a term $-q_{j_1}p_{j_1}q_{j_2}$ in δq_{j_2} , and the contribution to (1) is $-\{q_{j_1}, -q_{j_1}p_{j_1}q_{j_2}\} = -q_{j_1}q_{j_2} = n(\gamma)q_{j_1}q_{j_2}$. If γ is a path from $a_{j_1}^+$ to $a_{j_2}^-$, then the back half $-\gamma_{j_2}^-$ of $\gamma_{q_{j_2}}$ passes through $a_{j_1}^+$, yielding a term $(-1)^{|q_{j_2}|}q_{j_2}q_{j_1}p_{j_1} = q_{j_1}p_{j_1}q_{j_2}$ in δq_{j_2} , and the contribution to (1) is $+q_{j_1}q_{j_2} = n(\gamma)q_{j_1}q_{j_2}$. We leave the other 6 cases for the possible endpoints of γ to the reader.

Next suppose $(x_1, x_2) = (q_j, t_i)$, in which case

$$2\ell_2^{\text{str}}(q_j, t_i) = -\{q_j, \delta t_i\} + (\mathbf{v}(q_j) \cdot \mathbf{w}(t_i))q_j t_i.$$

Now unbroken segments between a_j^\pm and \bullet_i come in four types. A segment γ from a_j^+ to \bullet_i corresponds to γ_{t_i} passing through a_j^+ ; this gives a term $-q_j p_j t_i$

in δt_i and thus contributes $-q_j t_i = n(\gamma) q_j t_i$ to $2\ell_2^{\text{str}}(q_j, t_i)$. A segment γ from \bullet_i to a_j^\dagger , on the other hand, does not contribute to $-\{q_j, \delta t_i\}$ but does contribute -1 to $\mathbf{v}(q_j) \cdot \mathbf{w}(t_i)$, resulting in a contribution of $-q_j t_i = n(\gamma) q_j t_i$. We leave the other 2 cases to the reader.

Finally, suppose $(x_1, x_2) = (t_{i_1}, t_{i_2})$, in which case

$$2\ell_2^{\text{str}}(t_{i_1}, t_{i_2}) = (\mathbf{v}(t_{i_1}) \cdot \mathbf{w}(t_{i_2}) - \mathbf{w}(t_{i_1}) \cdot \mathbf{v}(t_{i_2})) t_{i_1} t_{i_2}.$$

An unbroken segment γ from \bullet_{i_1} to \bullet_{i_2} contributes -1 to $\mathbf{w}(t_{i_1}) \cdot \mathbf{v}(t_{i_2})$ and thus $+t_{i_1} t_{i_2} = n(\gamma) t_{i_1} t_{i_2}$ to $2\ell_2^{\text{str}}(t_{i_1}, t_{i_2})$, while a segment γ from \bullet_{i_2} to \bullet_{i_1} contributes -1 to $\mathbf{v}(t_{i_1}) \cdot \mathbf{w}(t_{i_2})$ and thus $-t_{i_1} t_{i_2} = n(\gamma) t_{i_1} t_{i_2}$ to $2\ell_2^{\text{str}}(t_{i_1}, t_{i_2})$. \square

3.4. (-1) -closures of admissible positive braids. To illustrate the construction of the L_∞ algebra from Section 3.2, here we consider a family of Legendrian links, (-1) -closures of admissible positive braids. These lie in a neighborhood of a standard Legendrian unknot in \mathbb{R}^3 and are braided around this unknot. This family includes “rainbow closures” of positive braids.

We first review some notation from [CN22, §2.2]. A positive braid is *admissible* if drawing the braid from left to right results in a diagram in $[0, 1] \times \mathbb{R}$ that represents the xy projection of a Legendrian link in $(J^1(S^1) = S^1 \times \mathbb{R} \times \mathbb{R}, \ker(dz - y dx))$, where we view $S^1 = [0, 1]/(0 \sim 1)$. Not all positive braids are admissible; for our purposes, we note that by [CN22, Proposition 2.7], any positive braid containing a positive half-twist Δ is admissible.

It is shown in [CN22, Proposition 2.6] that given any admissible positive braid β , there is a well-defined Legendrian link $\Lambda(\beta)$ in \mathbb{R}^3 which is topologically isotopic to the closure of $\beta\Delta^{-2}$, where Δ represents a half-twist in the braid group. This link $\Lambda(\beta)$ is called the (-1) -closure of β and is illustrated in Figure 9. In particular, the (-1) -closure of a braid of the form $\beta\Delta^2$, where β is any positive braid, is a Legendrian link often called the rainbow closure of β .

Suppose that β is an admissible positive braid with N strands, and that the length of β as a braid word is k . Then the (-1) -closure $\Lambda(\beta)$ as shown in Figure 9 has $k + N^2$ Reeb chords: the k crossings in β , which we label a_1, \dots, a_n , and the N^2 crossings where the link loops around itself, which we label \tilde{a}_{ij} for $1 \leq i, j \leq N$. We note that if we view $\Lambda(\beta)$ as a satellite of the standard Legendrian unknot, then the \tilde{a}_{ij} crossings of $\Lambda(\beta)$ correspond to the single Reeb chord of the standard unknot.

If we place one base point on each strand of β as shown in Figure 9, then we can write a generating set for the commutative algebra $\mathcal{A}^{\text{comm}}$ as

$$\mathcal{S} = \{q_1, \dots, q_k, \tilde{q}_{11}, \dots, \tilde{q}_{NN}, p_1, \dots, p_k, \tilde{p}_{11}, \dots, \tilde{p}_{NN}, t_1^{\pm 1}, \dots, t_N^{\pm 1}\}.$$

If we give $\Lambda(\beta)$ the Maslov potential determined by the given placement of base points, then it is straightforward to check that we have the following

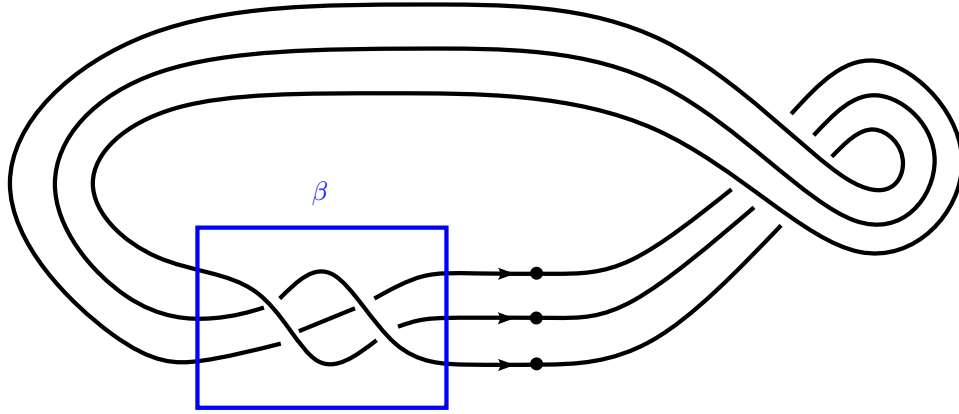


FIGURE 9. The (-1) -closure $\Lambda(\beta)$ of an admissible positive braid β , in the xy projection.

gradings:

$$\begin{aligned} |q_i| &= 0 & |p_i| &= -1 \\ |\tilde{q}_{ij}| &= 1 & |\tilde{p}_{ij}| &= -2 \\ |t_i^{\pm 1}| &= 0. \end{aligned}$$

Proposition 3.9. *For $\Lambda(\beta)$, the homotopy Poisson algebra $(\mathcal{A}^{\text{comm}}, \{\ell_k\})$ is strict: $\ell_k = 0$ for $k \geq 3$.*

Proof. We claim that any immersed disk in $\Delta(\Lambda(\beta))$ has at most two positive corners, whence $h_k = 0$ for $k \geq 3$. Note that $\Lambda(\beta)$ lies in a tubular neighborhood of a Legendrian unknot Λ_0 whose xy projection has a single crossing, and $\Pi_{xy}(\Lambda(\beta))$ lies in a tubular neighborhood of $\Pi_{xy}(\Lambda_0)$. Any disk $\Delta \in \Delta(\Lambda(\beta))$ either maps into this tubular neighborhood, in which case we call Δ “thin”, or it does not, in which case we call it “thick”. If Δ is thin, then by inspection, Δ has exactly two positive corners: these are the two points where $\partial\Delta$ changes direction along $\Pi_{xy}(\Lambda_0)$. If Δ is thick, then in fact Δ can only have one positive corner. Indeed, in the limit where we shrink the tubular neighborhood so that $\Pi_{xy}(\Lambda(\beta))$ approaches a multiple cover of $\Pi_{xy}(\Lambda_0)$, Δ approaches an (immersed) disk in $\Delta(\Lambda_0)$. But there are exactly two disks in $\Delta(\Lambda_0)$ and both have exactly one positive corner; thus Δ must have one positive corner as well. \square

It follows from Proposition 3.9 that for the Legendrian link $\Lambda(\beta)$, $\mathcal{A}^{\text{comm}}$ is a DG Poisson algebra with Poisson bracket ℓ_2 . In particular, the degree-0 subalgebra of $\mathcal{A}^{\text{comm}}$ generated by $q_1, \dots, q_k, t_1^{\pm 1}, \dots, t_N^{\pm 1}$ is a Poisson algebra. We thus have the following corollary of Proposition 3.9.

Corollary 3.10. *Given any admissible positive braid β with k crossings and N strands, the L_∞ structure for the Legendrian link $\Lambda(\beta)$ induces a Poisson*

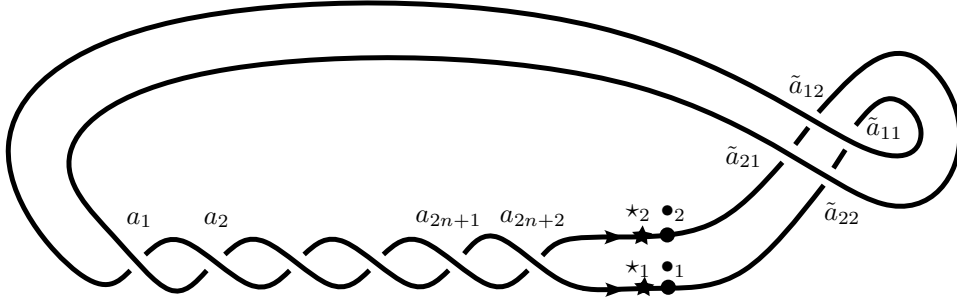


FIGURE 10. The $(2, 2n)$ torus link $\Lambda(\sigma_1^{2n+2})$, in the xy projection.

bracket on

$$\mathbb{k}[q_1, \dots, q_k, t_1^{\pm 1}, \dots, t_N^{\pm 1}],$$

the coordinate ring of $\mathbb{k}^k \times (\mathbb{k}^*)^N$.

As an example, we compute the Poisson bracket from Corollary 3.10 for the 2-strand braid $\beta = \sigma_1^{2n+2}$, $n \geq 1$. The (-1) -closure of this braid is a 2-component Legendrian link which is topologically a $(2, 2n)$ torus link. What we will find is that this Poisson bracket is essentially the same as the Flaschka–Newell Poisson bracket on $\mathbb{k}[s_1, \dots, s_{2n+2}, \lambda^{\pm 1}]$, which is defined by

$$\begin{aligned} \{s_i, s_j\}_{\text{FN}} &= \delta_{i,j-1} - \frac{\delta_{i,1}\delta_{j,2n+2}}{\lambda^2} + (-1)^{i-j+1} s_i s_j, & i < j \\ \{s_i, \lambda\}_{\text{FN}} &= (-1)^i s_i \lambda; \end{aligned}$$

see e.g. [BT22, equation (1.2)].

Proposition 3.11. *The Poisson bracket on*

$$\mathbb{k}[q_1, \dots, q_{2n+2}, t_1^{\pm 1}, t_2^{\pm 1}]$$

induced by the L_∞ structure on the $(2, 2n)$ torus link $\Lambda(\sigma_1^{2n+2})$ is:

$$\begin{aligned} \{q_i, q_j\} &= (-1)^{i+j+1} q_i q_j & j \geq i + 1 \\ \{q_i, q_{i+1}\} &= q_i q_{i+1} + 1 & 1 \leq i \leq 2n + 1 \\ \{q_1, q_{2n+2}\} &= q_1 q_{2n+2} - t_1 t_2^{-1} \\ \{q_i, t_1\} &= (-1)^{i+1} q_i t_1 & 1 \leq i \leq 2n + 2 \\ \{q_i, t_2\} &= (-1)^i q_i t_2 & 1 \leq i \leq 2n + 2 \\ \{t_1, t_2\} &= 0. \end{aligned}$$

It follows that the map $q_i \mapsto s_i$, $t_1 \mapsto \lambda^{-1}$, $t_2 \mapsto \lambda$ sends our $\{\cdot, \cdot\}$ to the Flaschka–Newell bracket $\{\cdot, \cdot\}_{\text{FN}}$.

Proof. We need to compute ℓ_2 on $q_1, \dots, q_{2n+2}, t_1, t_2$. To do this, we first compute the relevant portion of the Hamiltonian h_2 , namely the terms that

involve only q_j, p_j, t_i and not $\tilde{q}_{ij}, \tilde{p}_{ij}$. By inspection of Figure 10, we have:

$$h_2 = -p_1p_2 - p_2p_3 - \cdots - p_{2n+1}p_{2n+2} - t_2^{-1}p_{2n+2}t_1p_1 + \cdots$$

where the terms correspond to embedded bigons with corners at a_j and a_{j+1} for $j = 1, \dots, 2n+1$, except for the final term, which corresponds to an immersed bigon with corners at a_1 and a_{2n+2} and wraps around the loop region. The result now follows by using this formula for h_2 along with Proposition 3.8 to compute $\ell_2(x, y) = \{\{h_2, x\}, y\} + \ell_2^{\text{str}}(x, y)$ for $x, y \in \{q_1, \dots, q_{2n+2}, t_1, t_2\}$. \square

3.5. The L_∞ structure for Legendrian knots, and a variant. Here we consider the case where Λ is a single-component Legendrian knot with a single base point, rather than an arbitrary pointed Legendrian link. In this case the definition of the L_∞ structure from Section 3.2 simplifies considerably.

Definition 3.12. Let Λ be a Legendrian knot with a single base point. For $k \geq 1$, define $\ell_k : (\mathcal{A}^{\text{comm}})^{\otimes k} \rightarrow \mathcal{A}^{\text{comm}}$ as follows:

$$\begin{aligned} \ell_1(x_1) &= \{h_1, x_1\} \\ \ell_2(x_1, x_2) &= (-1)^{|x_1|} \{\{h_2, x_1\}, x_2\} - (-1)^{|x_1|} \{\delta x_1, x_2\} \end{aligned}$$

and for $k \geq 3$,

$$\ell_k(x_1, \dots, x_k) = (-1)^{|x_{k-1}| + |x_{k-3}| + \cdots} \{ \cdots \{ \{ \{ h_k, x_1 \}, x_2 \}, x_3 \}, \cdots, x_k \}.$$

Equivalently, if we define $d : \mathcal{A}_{\text{SFT}}^{\text{comm}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{comm}}$ by $d = \{h, \cdot\} - \delta$ as in Definition 2.35, and write $\Pi : \mathcal{A}_{\text{SFT}}^{\text{comm}} \rightarrow \mathcal{A}^{\text{comm}}$ for the projection map annihilating all words containing p 's, then we can define ℓ_k for all $k \geq 1$ by:

$$\ell_k(x_1, \dots, x_k) = (-1)^{|x_{k-1}| + |x_{k-3}| + \cdots} \Pi \{ \cdots \{ \{ dx_1, x_2 \}, x_3 \}, \cdots, x_k \}.$$

For Legendrian knots, this definition of the L_∞ structure agrees with the more general Definition 3.3; the simplification of ℓ_2 in this case is due to the facts that the $\tilde{\beta}$ terms disappear when there is a single base point, and the string coproduct δ is now an honest derivation with respect to the SFT bracket $\{\cdot, \cdot\}$.

The unified definition of ℓ_k involving d in Definition 3.12 implies the following result.

Proposition 3.13. *Let Λ be a Legendrian knot with a single base point. The L_∞ structure on $\mathcal{A}^{\text{comm}}$ determines the differential $d : \mathcal{A}^{\text{comm}} \rightarrow \mathcal{A}^{\text{comm}}$.*

Proof. Write the generators of $\mathcal{A}^{\text{comm}}$ as $q_1, \dots, q_n, t_1^\pm, \dots, t_s^\pm$ as usual. Since d satisfies the Leibniz rule, and $d(t_i) = 0$ for all i , it suffices to show that the L_∞ structure determines $d(q_j)$ for all j . We can write $d(q_j)$ as a polynomial (or power series) in p_1, \dots, p_n with coefficients in $q_1, \dots, q_n, t_1^\pm, \dots, t_s^\pm$:

$$d(q_j) = \sum_{k \geq 0} \sum_{j_1 \leq \cdots \leq j_k} f_{j_1, \dots, j_k} p_{j_1} \cdots p_{j_k}$$

with $f_{j_1, \dots, j_k} \in \mathcal{A}^{\text{comm}}$. Then for any j_1, \dots, j_k , f_{j_1, \dots, j_k} is determined by $\{\dots \{dq_j, q_{j_1}\}, \dots, q_{j_k}\}$ and thus by $\ell_{k+1}(q_j, q_{j_1}, \dots, q_{j_k})$. \square

In [Ng10], the complex $(\mathcal{A}^{\text{comm}}, d)$ is called the *commutative complex* associated to the LSFT algebra of the Legendrian knot Λ ; this is the commutative version of the rational SFT invariant that is the subject of that paper. Proposition 3.13 states that the L_∞ structure developed in the present paper encodes the commutative rational SFT invariant from [Ng10].

Remark 3.14. It is shown in [Ng10] that the commutative complex, up to filtered chain homotopy equivalence, is an invariant of the Legendrian knot under Legendrian isotopy. In Section 5, we will prove a restricted version of an invariance result for the L_∞ structure, and we conjecture that the L_∞ structure is in fact invariant up to L_∞ equivalence. However, it is not clear that this invariance conjecture would directly imply invariance for the commutative complex: the procedure described above to pass from the L_∞ structure to the commutative complex involves information about the L_∞ structure besides its equivalence class, namely the distinguished generators q_1, \dots, q_n of $\mathcal{A}^{\text{comm}}$.

We next note that when Λ is a Legendrian knot, we can lift the above L_∞ structure from $\mathcal{A}^{\text{comm}}$ to \mathcal{A}^{cyc} : the definition of $\ell_k : (\mathcal{A}^{\text{cyc}})^{\otimes k} \rightarrow \mathcal{A}^{\text{cyc}}$ is exactly the same as in Definition 3.12 above. Note that \mathcal{A}^{cyc} is not an associative algebra, and so the L_∞ structure in this case will not be a homotopy Poisson structure.

Proposition 3.15. *For any Legendrian knot with a single base point, $(\mathcal{A}^{\text{cyc}}, \{\ell_k\}_{k=1}^\infty)$ is an L_∞ algebra.*

Proposition 3.15 will be proved in Section 4.

The usual Chekanov–Eliashberg differential ∂ on the noncommutative algebra \mathcal{A} descends to a differential on \mathcal{A}^{cyc} . Proposition 3.15 immediately implies that the homology of \mathcal{A}^{cyc} with this differential inherits a Lie bracket from ℓ_2 .

Corollary 3.16. *For any Legendrian knot with a single base point, $(H_*(\mathcal{A}^{\text{cyc}}, \partial), \ell_2)$ is a Lie algebra.*

Remark 3.17. For a Legendrian knot Λ , the homology $H_*(\mathcal{A}^{\text{cyc}}, \partial)$ is the cylindrical contact homology of the contact manifold $\mathbb{R}^3(\Lambda)$ obtained from \mathbb{R}^3 by Legendrian surgery on Λ [BEE12]. In this context, the L_∞ operations appear to count rational holomorphic curves in the symplectization of $\mathbb{R}^3(\Lambda)$ with multiple positive punctures and a single negative puncture. In particular, the Lie bracket induced by ℓ_2 counts curves with two positive ends and one negative end.

There is a version of this story for Legendrian links as well. In this case, one can consider $\mathcal{A}^{\text{cyc, comp}}$, the module of composable cyclic words; see Remark 2.11. Then the homology $(\mathcal{A}^{\text{cyc, comp}}, d)$ is the cylindrical contact homology of the contact manifold $\mathbb{R}^3(\Lambda)$ obtained by Legendrian surgery on

each component of Λ [BEE12]. It appears to be the case that $(\mathcal{A}^{\text{cyc,comp}}, \{\ell_k\}_{k=1}^\infty)$ is again an L_∞ algebra, and the L_∞ operations again count rational curves with multiple positive and one negative puncture. However, we do not pursue this direction further in this paper.

4. PROOF OF THE L_∞ RELATIONS

This section is primarily devoted to proving Proposition 3.6 by verifying that the L_∞ operations that we defined for Legendrian links in Section 3.2 do indeed satisfy the L_∞ relations. We begin by laying the groundwork for the proof of Proposition 3.6, and then present the proof itself. We end the section with the proof of Proposition 3.15, which states that the variant L_∞ structure for Legendrian knots as constructed in Section 3.5 also satisfies the L_∞ relations.

For $k \geq 1$, we extend the maps ℓ_k to maps $m_k : (\mathcal{A}^{\text{comm}})^{\otimes k} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{comm}}$ defined as follows. Recall that $d : \mathcal{A}_{\text{SFT}}^{\text{comm}} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{comm}}$ is defined by $d(x) = \{h, x\} + \delta x$. For any $k \geq 1$ with $k \neq 2$, define

$$m_k(x_1, \dots, x_k) = (-1)^{|x_{k-1}| + |x_{k-3}| + \dots} \{ \{ \{ dx_1, x_2 \}, x_3 \}, \dots, x_k \};$$

note in particular that $m_1(x_1) = dx_1 = \{h, x_1\} - \delta x_1$. For $k = 2$, define

$$m_2(x_1, x_2) = \frac{1}{2} \left((-1)^{|x_1|} \{ dx_1, x_2 \} + \{ x_1, dx_2 \} - \tilde{\beta}(x_1, x_2) + (-1)^{|x_1||x_2|} \tilde{\beta}(x_2, x_1) \right).$$

For ease of notation, we will henceforth suppress the k index in m_k .

Let

$$\Pi : \mathcal{A}_{\text{SFT}}^{\text{comm}} \rightarrow \mathcal{A}^{\text{comm}}$$

denote the projection map, which annihilates all words that contain p 's and preserves all words that do not. Then ℓ_k and m are related by this projection:

$$\ell_k(x_1, \dots, x_k) = \Pi m(x_1, \dots, x_k)$$

for all $k \geq 1$.

Lemma 4.1. *The operation m satisfies the following properties:*

- (1) For any $k \geq 1$ and any permutation $\sigma \in S_k$,

$$m(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \chi(\sigma, x_1, \dots, x_k) m(x_1, \dots, x_k).$$

- (2) We have

$$m(x_1, x_2) = (-1)^{|x_1|} \{ m(x_1), x_2 \} + (-1)^{|x_1||x_2|} \tilde{\beta}(x_2, x_1) = \{ x_1, m(x_2) \} - \tilde{\beta}(x_1, x_2),$$

while for any $k \geq 3$,

$$m(x_1, \dots, x_{k-1}, x_k) = (-1)^{|x_1| + \dots + |x_{k-1}|} \{ m(x_1, \dots, x_{k-1}), x_k \}.$$

- (3) For any $k \geq 1$, if x_1, \dots, x_{k+1} are words in $\mathcal{W}^{\text{comm}}$, then

$$\tilde{\beta}(x_{k+1}, m(x_1, \dots, x_k)) = \sum_{i=1}^k \beta(x_{k+1}, x_i) x_{k+1} m(x_1, \dots, x_k).$$

Proof. (1) is proved in exactly the same way as the corresponding result for ℓ_k in Proposition 3.4. For (2), note that since $\{x_1, x_2\} = 0$, it follows from Proposition 2.38 that

$$0 = d\{x_1, x_2\} = \{mx_1, x_2\} - (-1)^{|x_1|}\{x_1, mx_2\} + (-1)^{|x_1|}\tilde{\beta}(x_1, x_2) + (-1)^{|x_1|(|x_2|+1)}\tilde{\beta}(x_2, x_1).$$

Combining this with the definition of $m(x_1, x_2)$ immediately gives (2).

For (3), we use Proposition 2.31(1,3,4) to see that $\mathbf{v}(\{h, x_1\}) = \mathbf{v}(\delta x_1) = \mathbf{v}(x_1)$ and thus $\mathbf{v}(dx_1) = \mathbf{v}(x_1)$. By Proposition 2.31(3) again, it follows that $\mathbf{v}(m(x_1, \dots, x_k)) = \sum_{i=1}^k \mathbf{v}(x_i)$. Thus $\beta(x_{k+1}, m(x_1, \dots, x_k)) = \sum_{i=1}^k \beta(x_{k+1}, x_i)$, and (3) follows. \square

To set up the next lemma, we introduce some notation. Fix $k \geq 2$ and $x_1, \dots, x_k \in \mathcal{A}^{\text{comm}}$. For a nonempty subset $S \subset [k] = \{1, \dots, k\}$, write $S = \{i_1, \dots, i_\ell\}$ with $i_1 < \dots < i_\ell$, define

$$\begin{aligned} m(x_S) &= m(x_{i_1}, \dots, x_{i_\ell}) \\ |x_S| &= \sum_{j=1}^{\ell} |x_{i_j}| \\ v_S &= \sum_{j=1}^{\ell} v(x_{i_j}). \end{aligned}$$

For a partition of $[k] = \{1, \dots, k\}$ into a union of two disjoint nonempty subsets $[k] = A \sqcup B$, let σ_A denote the permutation of $[k]$ that sends $1, \dots, k$ successively to the elements of A in increasing order, followed by the elements of B in increasing order; analogously let σ_B denote the permutation sending $1, \dots, k$ to the elements of B in increasing order, followed by the elements of A in increasing order.

Lemma 4.2. *For any $k \geq 2$, we have:*

$$\begin{aligned} (-1)^k dm(x_1, \dots, x_k) &= \sum_{[k]=A \sqcup B} \chi(\sigma_A) (-1)^{(|x_A|+1)(k-|A|+1)} \{m(x_A), m(x_B)\} \\ &\quad - \sum_{i=1}^k \chi(\sigma_{[k] \setminus \{i\}}) \tilde{\beta}(m(x_{[k] \setminus \{i\}}), x_i), \end{aligned}$$

where the first sum is over all unordered ways to partition $[k]$ into two disjoint subsets.

We note that the summand in the first sum is well-defined because a straightforward computation shows that it is symmetric in A and B :

$$\chi(\sigma_A) (-1)^{(|x_A|+1)(k-|A|+1)} \{m(x_A), m(x_B)\} = \chi(\sigma_B) (-1)^{(|x_B|+1)(k-|B|+1)} \{m(x_B), m(x_A)\}.$$

Proof of Lemma 4.2. We induct on k . For $k = 2$, the desired identity is:

$$(2) \quad dm(x_1, x_2) = \{m(x_1), m(x_2)\} - \tilde{\beta}(m(x_1), x_2) + (-1)^{|x_1||x_2|} \tilde{\beta}(m(x_2), x_1).$$

Now applying d to both sides of the equality $m(x_1, x_2) = \{x_1, m(x_2)\} - \tilde{\beta}(x_1, x_2)$ from Lemma 4.1 and using Propositions 2.38 and 2.37 yields

$$\begin{aligned} dm(x_1, x_2) &= \{m(x_1), m(x_2)\} - (-1)^{|x_1|} \{x_1, d^2 x_2\} + (-1)^{|x_1|} \tilde{\beta}(x_1, dx_2) \\ &\quad + (-1)^{|x_1||x_2|} \tilde{\beta}(dx_2, x_1) - d\tilde{\beta}(x_1, x_2); \end{aligned}$$

thus to prove (2), it suffices to show that

$$(3) \quad -(-1)^{|x_1|} \{x_1, d^2 x_2\} + (-1)^{|x_1|} \tilde{\beta}(x_1, dx_2) - d\tilde{\beta}(x_1, x_2) = -\tilde{\beta}(dx_1, x_2).$$

But by Propositions 2.37 and 2.31(6), we have

$$\begin{aligned} -(-1)^{|x_1|} \{x_1, d^2 x_2\} &= -(-1)^{|x_1|} \{x_1, \tilde{\beta}(h, x_2)\} \\ &= -(-1)^{|x_2|(|x_1|+1)} \{\tilde{\beta}(h, x_2), x_1\} \\ &= -\tilde{\beta}(\{h, x_1\}, x_2) + \{h, \tilde{\beta}(x_1, x_2)\} - (-1)^{|x_1|} \beta(x_1, x_2) x_1 \{h, x_2\} \\ &= -\tilde{\beta}(dx_1, x_2) + d\tilde{\beta}(x_1, x_2) - (-1)^{|x_1|} \beta(x_1, x_2) x_1 dx_2 \\ &\quad + \left(\delta \tilde{\beta}(x_1, x_2) - \tilde{\beta}(\delta x_1, x_2) + (-1)^{|x_1|} \beta(x_1, x_2) x_1 \delta x_2 \right) \\ &= -\tilde{\beta}(dx_1, x_2) + d\tilde{\beta}(x_1, x_2) - (-1)^{|x_1|} \tilde{\beta}(x_1, dx_2) \\ &\quad + \left(\delta \tilde{\beta}(x_1, x_2) - \tilde{\beta}(\delta x_1, x_2) + (-1)^{|x_1|} \beta(x_1, x_2) x_1 \delta x_2 \right) \end{aligned}$$

where we have used Proposition 2.37 in the first equality, Proposition 2.31(6) in the third equality, and Lemma 4.1(3) in the fifth equality. Finally, since $\mathbf{v}(\delta x_1) = \mathbf{v}(x_1)$ and $\mathbf{w}(\delta x_1) = \mathbf{w}(x_1)$ by Proposition 2.31(3), we have $\beta(\delta x_1, x_2) = \beta(x_1, x_2)$, and so

$$\begin{aligned} &\delta \tilde{\beta}(x_1, x_2) - \tilde{\beta}(\delta x_1, x_2) - (-1)^{|x_1|} \beta(x_1, x_2) x_1 \delta x_2 \\ &= \beta(x_1, x_2) \left(\delta(x_1 x_2) - (\delta x_1) x_2 - (-1)^{|x_1|} x_1 (\delta x_2) \right) \\ &= 0 \end{aligned}$$

This proves (3) and completes the proof of the lemma for the base case $k = 2$.

For the inductive step, assume that the desired equality holds for $k \geq 2$; we establish it for $k + 1$. Indeed, by Proposition 2.38, Lemma 4.1, and the induction assumption, we have

$$\begin{aligned} &(-1)^{k+1} dm(x_1, \dots, x_{k+1}) \\ &= (-1)^{k+1+|x_{[k]}|} d\{m(x_1, \dots, x_k), x_{k+1}\} \\ &= (-1)^{k+1+|x_{[k]}|} \{dm(x_1, \dots, x_k), x_{k+1}\} + \{m(x_{[k]}), dx_{k+1}\} \\ &\quad - \tilde{\beta}(m(x_{[k]}), x_{k+1}) - (-1)^{(k+|x_{[k]}|)|x_{k+1}|} \tilde{\beta}(x_{k+1}, m(x_{[k]})) \\ &= C + D + \{m(x_{[k]}), dx_{k+1}\} \\ &\quad - \tilde{\beta}(m(x_{[k]}), x_{k+1}) - (-1)^{(k+|x_{[k]}|)|x_{k+1}|} \tilde{\beta}(x_{k+1}, m(x_{[k]})) \end{aligned}$$

where

$$C = \sum_{[k]=A \sqcup B} \chi(\sigma_A)(-1)^{1+|x_{[k]}|+(|x_A|+1)(k-|A|+1)} \{\{m(x_A), m(x_B)\}, x_{k+1}\}$$

$$D = (-1)^{|x_{[k]}|} \sum_{i=1}^k \chi(\sigma_{[k] \setminus \{i\}}) \{\tilde{\beta}(m(x_{[k] \setminus \{i\}}), x_i), x_{k+1}\}.$$

Now by the Jacobi identity, we compute that the summand in C is equal to

$$\chi(\sigma_A)(-1)^{1+|x_{[k]}|+(|x_A|+1)(k-|A|+1)} \{m(x_A), \{m(x_B), x_{k+1}\}\}$$

$$+ \chi(\sigma_B)(-1)^{1+|x_{[k]}|+(|x_B|+1)(k-|B|+1)} \{m(x_B), \{m(x_A), x_{k+1}\}\}.$$

By Lemma 4.1, these two terms are equal to $\chi(\sigma_A)(-1)^{(|x_A|+1)(k-|A|)} \{m(x_A), m(x_{B \cup \{k+1\}})\}$ and $\chi(\sigma_B)(-1)^{(|x_B|+1)(k-|B|)} \{m(x_B), m(x_{A \cup \{k+1\}})\}$ respectively, unless either of A or B (say B) is a singleton $\{i\}$, in which case we need to add $\chi(\sigma_{[k] \setminus \{i\}})(-1)^{|x_{[k] \setminus \{i\}}|+|x_i||x_{k+1}|} \{m(x_{[k] \setminus \{i\}}), \tilde{\beta}(x_{k+1}, x_i)\}$ to the first term. It follows that

$$C = E + \sum_{[k]=A \sqcup B} \chi(\sigma_A)(-1)^{(|x_A|+1)(k-|A|)} \{m(x_A), m(x_{B \cup \{k+1\}})\}$$

$$+ \sum_{[k]=A \sqcup B} \chi(\sigma_B)(-1)^{(|x_B|+1)(k-|B|)} \{m(x_B), m(x_{A \cup \{k+1\}})\}$$

$$= E + \sum_{[k+1]=A \sqcup B} \chi(\sigma_A)(-1)^{(|x_A|+1)(k-|A|)} \{m(x_A), m(x_B)\} - \{m(x_{[k]}), dx_{k+1}\}$$

where

$$E = \sum_{i=1}^k \chi(\sigma_{[k] \setminus \{i\}})(-1)^{|x_{[k] \setminus \{i\}}|+|x_i||x_{k+1}|} \{m(x_{[k] \setminus \{i\}}), \tilde{\beta}(x_{k+1}, x_i)\}.$$

From the above expression for C , to complete the induction step it suffices to prove:

(4)

$$D+E = (-1)^{(k+|x_{[k]}|)|x_{k+1}|} \tilde{\beta}(x_{k+1}, m(x_{[k]})) - \sum_{i=1}^k \chi(\sigma_{[k+1] \setminus \{i\}}) \tilde{\beta}(m(x_{[k+1] \setminus \{i\}}), x_i).$$

We will establish (4) first for $k \geq 3$, and then for $k = 2$; the $k = 2$ case involves some additional correction terms that we address at that point. For now, assume $k \geq 3$. By Proposition 2.31 (6), we have

(5)

$$\{m(x_{[k] \setminus \{i\}}), \tilde{\beta}(x_{k+1}, x_i)\} + (-1)^{|x_i|(|x_{k+1}|+1)} \{\tilde{\beta}(m(x_{[k] \setminus \{i\}}), x_i), x_{k+1}\}$$

$$= \tilde{\beta}(\{m(x_{[k] \setminus \{i\}}), x_{k+1}\}, x_i) + (-1)^{|x_{k+1}|(|m(x_{[k] \setminus \{i\}})|+1)} \beta(x_{k+1}, x_i) x_{k+1} \{m(x_{[k] \setminus \{i\}}), x_i\}$$

$$= (-1)^{|x_{[k] \setminus \{i\}}|} \tilde{\beta}(m(x_{[k+1] \setminus \{i\}}), x_i) + \chi(\sigma_{[k] \setminus \{i\}})(-1)^{|x_{k+1}|(|x_{[k] \setminus \{i\}}|+k)+|x_{[k] \setminus \{i\}}|} \beta(x_{k+1}, x_i) x_{k+1} m(x_{[k]});$$

multiplying both sides by $\chi(\sigma_{[k]\setminus\{i\}})(-1)^{|x_{[k]\setminus\{i\}}|+|x_i||x_{k+1}|}$ and summing over i gives

$$\begin{aligned} D + E &= \sum_{i=1}^k \left(\chi(\sigma_{[k]\setminus\{i\}})(-1)^{|x_i||x_{k+1}|} \tilde{\beta}(m(x_{[k+1]\setminus\{i\}}), x_i) + \beta(x_{k+1}, x_i) m(x_{[k]}) x_{k+1} \right) \\ &= (-1)^{(k+|x_{[k]})|x_{k+1}|} \tilde{\beta}(x_{k+1}, m(x_{[k]})) - \sum_{i=1}^k \chi(\sigma_{[k+1]\setminus\{i\}}) \tilde{\beta}(m(x_{[k+1]\setminus\{i\}}), x_i) \end{aligned}$$

where in the second step we have used Lemma 4.1 (3). This completes the proof of (4) when $k \geq 3$.

When $k = 2$, since $\{m(x_1), x_2\}$ is equal not to $(-1)^{|x_1|} m(x_1, x_2)$ but rather to $(-1)^{|x_1|} m(x_1, x_2)$ plus the correction term $-(-1)^{|x_1|(|x_2|+1)} \tilde{\beta}(x_2, x_1)$, we need to add correction terms to the final expression in (5). The total contribution of these correction terms summed over $i = 1, 2$ is:

$$\begin{aligned} F &:= -(-1)^{|x_1||x_2|+|x_1||x_3|+|x_2||x_3|} \tilde{\beta}(\tilde{\beta}(x_3, x_2), x_1) - (-1)^{(|x_1|+|x_2|)|x_3|} \beta(x_3, x_1) x_3 \tilde{\beta}(x_1, x_2) \\ &\quad - (-1)^{(|x_1|+|x_2|)|x_3|+1} \tilde{\beta}(\tilde{\beta}(x_3, x_1), x_2) - (-1)^{|x_1||x_2|+|x_1||x_3|+|x_2||x_3|+1} \beta(x_3, x_2) x_3 \tilde{\beta}(x_2, x_1). \end{aligned}$$

To establish (4) when $k = 2$, it suffices to show $F = 0$. Now since

$$\begin{aligned} \tilde{\beta}(\tilde{\beta}(x_3, x_2), x_1) &= \beta(\tilde{\beta}(x_3, x_2), x_1) \tilde{\beta}(x_3, x_2) x_1 \\ &= (\beta(x_3, x_1) + \beta(x_2, x_1)) \beta(x_3, x_2) x_3 x_2 x_1 \\ &= (-1)^{|x_1||x_2|+|x_1||x_3|+|x_2||x_3|} (\beta(x_3, x_1) + \beta(x_2, x_1)) \beta(x_3, x_2) x_1 x_2 x_3 \end{aligned}$$

with a similar expression for $\tilde{\beta}(\tilde{\beta}(x_3, x_1), x_2)$, we find that

$$\begin{aligned} F &= -(\beta(x_3, x_1) + \beta(x_2, x_1)) \beta(x_3, x_2) x_1 x_2 x_3 - \beta(x_3, x_1) \beta(x_1, x_2) x_1 x_2 x_3 \\ &\quad + (\beta(x_3, x_2) + \beta(x_1, x_2)) \beta(x_3, x_1) x_1 x_2 x_3 + \beta(x_3, x_2) \beta(x_2, x_1) x_1 x_2 x_3 \\ &= 0, \end{aligned}$$

and we are done. \square

Proof of Proposition 3.6. Recall that ℓ_k and m are related by $\ell_k = \Pi \circ m$ where Π is the projection map $\mathcal{A}_{\text{SFT}}^{\text{comm}} \rightarrow \mathcal{A}^{\text{comm}}$. We note that Π satisfies the following properties:

- for any $x \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$, $\Pi d\Pi(x) = \Pi d(x)$;
- for any $x_1, x_2 \in \mathcal{A}_{\text{SFT}}^{\text{comm}}$, $\Pi\{x_1, x_2\} = \Pi\{\Pi x_1, x_2\} + \Pi\{x_1, \Pi x_2\}$.

We now turn to the proof of the Jacobi identities. The first Jacob identity is $\ell_1^2 = 0$, which follows directly from $\partial^2 = 0$ in the usual Chekanov–Eliashberg DGA for Λ . The second Jacobi identity is

$$\ell_1(\ell_2(x_1, x_2)) = \ell_2(\ell_1(x_1), x_2) + (-1)^{|x_1|} \ell_2(x_1, \ell_1(x_2)).$$

To prove this, we apply Π to both sides of the $k = 2$ case of Lemma 4.2, as presented in equation (2):

$$\begin{aligned}
\ell_1 \ell_2(x_1, x_2) &= \Pi d \Pi m(x_1, x_2) \\
&= \Pi dm(x_1, x_2) \\
&= \Pi\{\Pi m(x_1), m(x_2)\} + \Pi\{m(x_1), \Pi m(x_2)\} \\
&\quad - \Pi \tilde{\beta}(m(x_1), x_2) + (-1)^{|x_1||x_2|} \Pi \tilde{\beta}(m(x_2), x_1) \\
&= \Pi\left(\{\ell_1(x_1), m(x_2)\} - \tilde{\beta}(\ell_1(x_1), x_2)\right) \\
&\quad + \Pi\left(\{m(x_1), \ell_1(x_2)\} + (-1)^{|x_1||x_2|} \tilde{\beta}(\ell_1(x_2), x_1)\right) \\
&= \Pi m(\ell_1(x_1), x_2) + (-1)^{|x_1|} \Pi m(x_1, \ell_1(x_2)) \\
&= \ell_2(\ell_1(x_1), x_2) + (-1)^{|x_1|} \ell_2(x_1, \ell_1(x_2)).
\end{aligned}$$

It remains to prove the k -th Jacobi identity for $k \geq 3$, which we rewrite as:

$$(6) \quad 0 = \ell_1(\ell_k(x_1, \dots, x_k)) + \sum_{[k]=A \sqcup B} (-1)^{|A||B|} (\chi(\sigma_A) \ell_{|B|+1}(\ell_{|A|}(x_A), x_B) + \chi(\sigma_B) \ell_{|A|+1}(\ell_{|B|}(x_B), x_A)),$$

with the sum being over unordered nonempty sets A, B as before. Here we use analogous notation to above: if $A = \{i_1, \dots, i_p\}$ and $B = \{j_1, \dots, j_{k-p}\}$ where the elements of A and B are in increasing order, then $\ell_p(x_A) := \ell_p(x_{i_1}, \dots, x_{i_p})$, $\ell_{k-p+1}(\ell_p(x_A), x_B) := \ell_{k-p+1}(\ell_p(x_A), x_{j_1}, \dots, x_{j_{k-p}})$, and so forth.

Since $k \geq 3$, we can assume that in the unordered partition $[k] = A \sqcup B$, we always have $|B| > 1$. Then we have

$$\begin{aligned}
\Pi\{\Pi m(x_A), m(x_B)\} &= (-1)^{(|x_A|+|A|+1)(|x_B|+|B|+1)+1} \Pi\{m(x_B), \ell_{|A|}(x_A)\} \\
&= (-1)^{(|x_A|+|A|+1)(|x_B|+|B|+1)+1+|x_B|} \Pi m(x_B, \ell_{|A|}(x_A)) \\
&= (-1)^{(|x_A|+|A|)(|B|+1)} \ell_{|B|+1}(\ell_{|A|}(x_A), x_B).
\end{aligned}$$

Similarly, if $|A| > 1$ then

$$\begin{aligned}
\Pi\{m(x_A), \Pi m(x_B)\} &= (-1)^{|x_A|} \Pi m(x_A, \ell_{|B|}(x_B)) \\
&= (-1)^{|x_A|(|x_B|+|B|+1)+|A|} \ell_{|A|+1}(\ell_{|B|}(x_B), x_A) \\
&= \chi(\sigma_A) \chi(\sigma_B) (-1)^{(|x_A|+|A|)(|B|+1)} \ell_{|A|+1}(\ell_{|B|}(x_B), x_A),
\end{aligned}$$

while if $|A| = 1$, say $A = \{i\}$, then there is a correction term:

$$\begin{aligned}
\Pi\{m(x_A), \Pi m(x_B)\} &= \Pi\{m(x_i), \ell_{k-1}(x_B)\} \\
&= (-1)^{|x_i|} \Pi\left(m(x_i, \ell_{k-1}(x_B)) - (-1)^{|x_i||\ell_{k-1}(x_B)|} \tilde{\beta}(\ell_{k-1}(x_B), x_i)\right) \\
&= \chi(\sigma_A) \chi(\sigma_B) (-1)^{(|x_A|+|A|)(|B|+1)} \ell_{|A|+1}(\ell_{|B|}(x_B), x_A) \\
&\quad + \chi(\sigma_{\{i\}}) \chi(\sigma_{[k]\setminus\{i\}}) (-1)^{k|x_i|+k} \tilde{\beta}(\ell_{k-1}(x_{[k]\setminus\{i\}}), x_i).
\end{aligned}$$

Now apply Π to both sides of the equation in the statement of Lemma 4.2. The left hand side is $(-1)^k \Pi dm(x_1, \dots, x_k) = (-1)^k \ell_1(\ell_k(x_1, \dots, x_k))$. Using the fact that $\Pi\{m(x_A), m(x_B)\} = \Pi\{\Pi m(x_A), x_B\} + \Pi\{m(x_A), \Pi x_B\}$, we find that the first sum on the right hand side is:

$$\sum_{[k]=A \sqcup B} (-1)^{|A||B|+k+1} (\chi(\sigma_A) \ell_{|B|+1}(\ell_{|A|}(x_A), x_B) + \chi(\sigma_B) \ell_{|A|+1}(\ell_{|B|}(x_B), x_A)) \\ + \sum_{i=1}^k \chi(\sigma_{[k] \setminus \{i\}}) \tilde{\beta}(\ell_{k-1}(x_{[k] \setminus \{i\}}), x_i).$$

This last sum exactly cancels the image under Π of the second sum on the right hand side of the statement of Lemma 4.2, and (6) follows for $k \geq 3$. \square

Proof of Proposition 3.15. A key to the above proof of the L_∞ relations for links is the statement (Proposition 2.30) that on $\mathcal{A}_{\text{SFT}}^{\text{comm}}$, δ is a derivation with respect to $\{\cdot, \cdot\}$ when we include some correction terms involving $\tilde{\beta}$. Now suppose that Λ is a knot with a single base point. Recall in this case that δ is actually a derivation with respect to $\{\cdot, \cdot\}$ on the module of cyclic words $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$, not just the commutative quotient $\mathcal{A}_{\text{SFT}}^{\text{comm}}$, and further that there are no $\tilde{\beta}$ correction terms: see Proposition 2.27.

This allows us to lift the argument in the above proof of Proposition 3.6 from $\mathcal{A}_{\text{SFT}}^{\text{comm}}$ to $\mathcal{A}_{\text{SFT}}^{\text{cyc}}$. We can define $m_k : (\mathcal{A}^{\text{cyc}})^{\otimes k} \rightarrow \mathcal{A}_{\text{SFT}}^{\text{cyc}}$ for $k \geq 1$ as in the previous proof:

$$m_k(x_1, \dots, x_k) = (-1)^{|x_{k-1}|+|x_{k-3}|+\dots} \{\dots \{\{\{dx_1, x_2\}, x_3\}, \dots, x_k\}.$$

Now however we do not need a separate definition for m_2 since the above definition for $m_2(x_1, x_2)$ is already graded-symmetric in x_1 and x_2 . We now follow the proof of Proposition 3.6 word for word, but with the significant simplification that we omit all terms involving β or $\tilde{\beta}$. The resulting proof verifies that the L_∞ operations on \mathcal{A}^{cyc} satisfy the L_∞ relations, as desired. \square

5. INVARIANCE

In this section, we prove a version of invariance for the L_∞ structure constructed in Section 3 under Legendrian isotopy. More precisely, we show that the L_∞ algebra $(\mathcal{A}^{\text{comm}}, \{\ell_k\}_{k=1}^\infty)$ is invariant up through the ℓ_2 operation. For ℓ_1 , this is contained in the usual invariance statement for the Chekanov–Eliashberg DGA (involving “stable tame isomorphism”); the new content is the invariance of ℓ_2 as well.

5.1. Statement of invariance. Given two L_∞ algebras $(A^+, \{\ell_k^+\})$ and $(A^-, \{\ell_k^-\})$, there is a standard notion of an L_∞ morphism from A^+ to A^- . This is a collection of graded-symmetric maps $\{f_k\}_{k=1}^\infty$ with $f_k : (A^+)^{\otimes k} \rightarrow$

A^- , satisfying a sequence of relations of the form:

$$(7) \quad \sum_{i=1}^k \sum_{\sigma \in \text{Unshuff}(i, k-i)} f_{k-i+1}(\ell_i^+(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(k)}) \\ = \sum_{k_1 + \dots + k_j = k} \sum_{\sigma \in \text{Unshuff}(k_1, \dots, k_j)} \ell_j^-(f_{k_1}(x_{\sigma(1)}, \dots, x_{\sigma(k_1)}), \dots, f_{k_j}(x_{\sigma(k-k_j+1)}, \dots, x_{\sigma(k)}))$$

for $k \geq 1$, where for readability we have omitted signs in the relations (7). (In our setting, where the L_∞ algebras also have associative multiplication and are homotopy Poisson algebras, we would also require that each f_k satisfies the Leibniz rule.) In particular, for $k = 1$ we have $f_1 \circ \ell_1^+ = \ell_1^- \circ f_1$, and so f_1 is a chain map $(A^+, \ell_1^+) \rightarrow (A^-, \ell_1^-)$. An L_∞ equivalence between A^+ and A^- is then an L_∞ morphism $\{f_k\}$ such that f_1 is a quasi-isomorphism, and two L_∞ algebras are said to be equivalent if there is an L_∞ equivalence between them.

We believe that the L_∞ algebra defined in this paper is invariant under this notion of equivalence, but do not prove this full result here.

Conjecture 5.1. *The homotopy Poisson algebra $(\mathcal{A}^{\text{comm}}, \{\ell_k\})$ associated to a pointed Legendrian link is an invariant of the Legendrian isotopy type of the pointed link, up to L_∞ equivalence.*

Instead, in this paper we establish a more limited statement of invariance, up through the ℓ_2 operation. That is, we construct maps f_1, f_2 and check the relation (7) for $k = 1, 2$. More precisely, we define the following.

Definition 5.2. Let $(A^+, \{\ell_k^+\})$ and $(A^-, \{\ell_k^-\})$ be two homotopy Poisson algebras. An L_2 morphism from $(A^+, \{\ell_k^+\})$ to $(A^-, \{\ell_k^-\})$ is a pair of maps $f_1 : A^+ \rightarrow A^-$ and $f_2 : A^+ \otimes A^+ \rightarrow A^-$ such that f_1 is an algebra map with $f_1 \circ \ell_1^+ = \ell_1^- \circ f_1$, and f_2 satisfies the following properties:

- graded symmetry: $f_2(x_2, x_1) = (-1)^{|x_1||x_2|+1} f_2(x_1, x_2)$;
- Leibniz rule:

$$f_2(x_1, x_2 x'_2) = f_2(x_1, x_2) f_1(x'_2) + (-1)^{(|x_1|+1)|x_2|} f_1(x_2) f_2(x_1, x'_2);$$

- L_2 morphism property:

$$f_1 \ell_2^+(x_1, x_2) = \ell_2^-(f_1 x_1, f_1 x_2) + f_2(\ell_1^+ x_1, x_2) + (-1)^{|x_1|} f_2(x_1, \ell_1^+ x_2) + \ell_1^- f_2(x_1, x_2).$$

An L_2 morphism such that f_1 is a quasi-isomorphism is an L_2 equivalence; two homotopy Poisson algebras are L_2 equivalent if there is an L_2 equivalence between them.

We note for future reference that graded symmetry and the Leibniz rule imply the following alternate version of Leibniz:

$$f_2(x_1 x'_1, x_2) = (-1)^{|x_1|} f_1(x_1) f_2(x'_1, x_2) + (-1)^{|x'_1||x_2|} f_2(x_1, x_2) f_1(x'_1).$$

Recall that the homology $H_*(A, \ell_1)$ of a homotopy Poisson algebra is a Poisson algebra, with multiplication induced from A and Poisson bracket induced by ℓ_2 . The following result is immediate from Definition 5.2.

Proposition 5.3. *If the homotopy Poisson algebras $(A^+, \{\ell_k^+\})$ and $(A^-, \{\ell_k^-\})$ are L_2 equivalent, then the Poisson algebras $(H_*(A^+, \ell_1^+), \ell_2^+)$ and $(H_*(A^-, \ell_1^-), \ell_2^-)$ are isomorphic.*

We can now state our invariance result; note in the following that “Legendrian isotopic” refers to an isotopy that sends base points to base points and Maslov potential to Maslov potential.

Proposition 5.4. *Suppose that Λ^+ and Λ^- are pointed Legendrian links in \mathbb{R}^3 that are Legendrian isotopic. Then the respective homotopy Poisson algebras $((\mathcal{A}^+)^{\text{comm}}, \{\ell_k^+\})$ and $((\mathcal{A}^-)^{\text{comm}}, \{\ell_k^-\})$ are L_2 equivalent. In particular, the commutative Legendrian contact homologies $(LCH_*^{\text{comm}}(\Lambda^+), \ell_2^+)$ and $(LCH_*^{\text{comm}}(\Lambda^-), \ell_2^-)$ are isomorphic as Poisson algebras.*

The remainder of Section 5 is devoted to the proof of Proposition 5.4. In order to do this, we follow Chekanov’s combinatorial proof that Legendrian contact homology is an isotopy invariant. In the xy projection, any Legendrian isotopy can be decomposed into planar isotopies along with a succession of elementary moves:

- moving a base point through the endpoint of a Reeb chord;
- a Reidemeister III move;
- a Reidemeister II move.

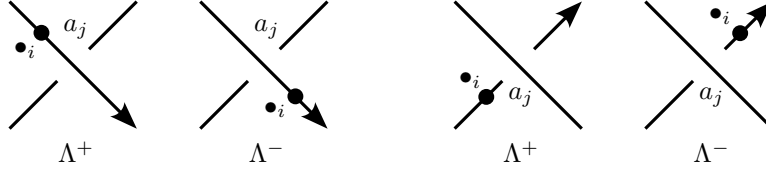
Recall that the xy projection of a Legendrian isotopy is a regular homotopy, and thus we do not need to consider a Reidemeister I move.

In the following subsections, we will construct an L_2 equivalence associated to each of the elementary moves. For each move, Chekanov [Che02] (see also [ENS02]) constructs a chain map between Chekanov–Eliashberg DGAs (“stable tame isomorphism”) that induces an isomorphism on the homology of the noncommutative DGA. This chain map is our f_1 ; when $\mathcal{A}^{\text{comm}}$ is the commutative DGA over a field \mathbb{k} of characteristic 0, f_1 induces an isomorphism on the homology of $(\mathcal{A}^{\text{comm}}, \partial)$ as well, see [ENS02, Theorem 3.14]. We then construct f_2 for each elementary move in such a way that (f_1, f_2) is an L_2 morphism (and hence an L_2 equivalence) in the sense of Definition 5.2. Proposition 5.4 then follows.

5.2. Invariance under base point move. Suppose that the pointed Legendrian links Λ^+ and Λ^- are identical except that the base point \bullet_i is moved through an endpoint of the Reeb chord a_j : see Figure 11.

We first consider the case where \bullet_i is moved through the positive endpoint a_j^+ of a_j , as shown in the left hand diagram in Figure 11. Define an algebra isomorphism

$$f_1 : (\mathcal{A}^+)^{\text{comm}} \rightarrow (\mathcal{A}^-)^{\text{comm}}$$

FIGURE 11. Moving a base point \bullet_i through a Reeb chord a_j .

by sending

$$f_1(q_j) = t_i^{-1}q_j$$

and letting f_1 act as the identity on all other generators of $(\mathcal{A}^+)^{\text{comm}}$ (including t_i).

To set notation, let h^\pm denote the Hamiltonians for Λ^\pm , and write $\mathcal{S}_0 = \{q_1, \dots, q_n, t_1, \dots, t_s\}$ for the set of generators for both $(\mathcal{A}^+)^{\text{comm}}_{\text{SFT}}$ and $(\mathcal{A}^-)^{\text{comm}}_{\text{SFT}}$. We now observe the following.

- Lemma 5.5.** (1) For any $x_1, x_2 \in (\mathcal{A}^+)^{\text{comm}}_{\text{SFT}}$, we have $\{f_1x_1, f_1x_2\} = f_1\{x_1, x_2\}$.
(2) $f_1(h^+) = h^-$.
(3) For any $x_1, x_2 \in \mathcal{S}$, we have $(\ell_2^-)^{\text{str}}(f_1x_1, f_1x_2) = f_1((\ell_2^+)^{\text{str}}(x_1, x_2))$, where $(\ell_2^\pm)^{\text{str}}$ are the versions for Λ^\pm of the operations ℓ_2^{str} defined in Section 3.3.

Proof. For (1), it suffices to check this when $x_1, x_2 \in \mathcal{S}$. In this case both sides are 0 unless $\{x_1, x_2\} = \{p_\ell, q_\ell\}$ for some ℓ . If $\ell \neq j$ then both sides are $\{x_1, x_2\}$, while if $\ell = j$ then the result follows from the fact that $\{f_1p_j, f_1q_j\} = \{t_ip_j, t_i^{-1}q_j\} = \{p_j, q_j\}$.

To see (2), note that there is an obvious one-to-one correspondence between disks contributing to the Hamiltonians h^\pm . An inspection of the four quadrants at the Reeb chord a_j shows that the boundaries of the disks that have a corner at a_j change between Λ^+ and Λ^- as follows: $t_iq_j \rightarrow q_j$; $q_j \rightarrow t_i^{-1}q_j$; $p_jt_i^{-1} \rightarrow p_j$; $p_j \rightarrow t_ip_j$. It follows immediately that $f_1(h^+) = h^-$.

For (3), recall from Proposition 3.8 that $\ell_2^{\text{str}}(x_1, x_2)$ is x_1x_2 times a signed count of the unbroken segments between x_1 and x_2 . If neither x_1 nor x_2 is q_j , then this collection of unbroken segments is the same for Λ^+ and Λ^- , and so $(\ell_2^-)^{\text{str}}(f_1x_1, f_1x_2) = (\ell_2^-)^{\text{str}}(x_1, x_2) = (\ell_2^+)^{\text{str}}(x_1, x_2) = f_1((\ell_2^+)^{\text{str}}(x_1, x_2))$. Since $(\ell_2^\pm)^{\text{str}}(q_j, q_j) = 0$, it only remains to check (3) when $x_1 = q_j$ and $x_2 \neq q_j$. In this case we have $t_if_1((\ell_2^+)^{\text{str}}(q_j, x_2)) = (\ell_2^+)^{\text{str}}(q_j, x_2)$ while

$$t_i(\ell_2^-)^{\text{str}}(f_1q_j, f_1x_2) = t_i(\ell_2^-)^{\text{str}}(t_i^{-1}q_j, x_2) = (\ell_2^-)^{\text{str}}(q_j, x_2) - q_jt_i^{-1}(\ell_2^-)^{\text{str}}(t_i, x_2).$$

Thus we need to show that

$$(8) \quad (\ell_2^-)^{\text{str}}(q_j, x_2) - (\ell_2^+)^{\text{str}}(q_j, x_2) = q_jt_i^{-1}(\ell_2^-)^{\text{str}}(t_i, x_2).$$

Now there is a one-to-one correspondence between unbroken segments between endpoints of a_j and of x_2 in Λ^+ and in Λ^- , with two exceptions:

(oriented) unbroken segments in Λ^+ from a_j^+ to an endpoint of x_2 , and unbroken segments in Λ^- from an endpoint of x_2 to a_j^+ have no counterpart on the other side. Thus $(\ell_2^-)^{\text{str}}(q_j, x_2) - (\ell_2^+)^{\text{str}}(q_j, x_2)$ is given by a signed count of these exceptional segments. But in these exceptional cases there are also companion unbroken segments between \bullet_i and the endpoint of x_2 : an unbroken segment in Λ^+ from \bullet_i to the endpoint of x_2 in the first case, and in Λ^- from the endpoint of x_2 to \bullet_i the latter. Thus each term on the left hand side of (8) has a corresponding term on the right hand side. Completing the proof now reduces to a case-by-case verification that the terms agree with signs, using Proposition 3.8. Suppose for example that $x_2 = q_{j_2}$ and there is an unbroken segment in Λ^+ from a_j^+ to $a_{j_2}^{\sigma_2}$: this contributes $\frac{1}{2}\sigma_2 q_j q_{j_2}$ to $(\ell_2^+)^{\text{str}}(q_j, x_2)$, while the companion segment in Λ^+ from \bullet_i to $a_{j_2}^{\sigma_2}$ contributes $\frac{1}{2}\sigma_2 q_j x_2$ to $q_j t_i^{-1} (\ell_2^-)^{\text{str}}(t_i, x_2)$. We leave the other cases (a segment in Λ^- from an endpoint of $x_2 = q_{j_2}$ to a_j^+ , and the cases where $x_2 = t_{i'}$ for some i') to the reader. \square

Proposition 5.6. *Suppose that Λ^\pm are related by moving a base point through the top endpoint of a Reeb chord. For any $k \geq 1$ and $x_1, \dots, x_k \in (\mathcal{A}^+)^{\text{comm}}$, we have*

$$f_1 \ell_k^+(x_1, \dots, x_k) = \ell_k^-(f_1 x_1, \dots, f_1 x_k).$$

Thus f_1 , as a map between the L_∞ algebras $((\mathcal{A}^+)^{\text{comm}}, \{\ell_k^+\})$ and $((\mathcal{A}^-)^{\text{comm}}, \{\ell_k^-\})$, is an L_∞ equivalence and in particular an L_2 equivalence.

Proof. For $k \neq 2$, we have

$$\begin{aligned} f_1 \ell_k^+(x_1, \dots, x_k) &= (-1)^{|x_{k-1}| + |x_{k-3}| + \dots} f_1 \{ \dots \{ \{ h_k^+, x_1 \}, x_2 \}, \dots, x_k \} \\ &= (-1)^{|x_{k-1}| + |x_{k-3}| + \dots} \{ \dots \{ \{ f_1 h_k^+, f_1 x_1 \}, f_1 x_2 \}, \dots, f_1 x_k \} \\ &= (-1)^{|x_{k-1}| + |x_{k-3}| + \dots} \{ \dots \{ \{ h_k^-, f_1 x_1 \}, f_1 x_2 \}, \dots, f_1 x_k \} \\ &= \ell_k^-(f_1 x_1, \dots, f_1 x_k) \end{aligned}$$

by Lemma 5.5(1,2). For $k = 2$, since $\ell_2(x_1, x_2) = (-1)^{|x_1|} \{ \{ h_2, x_1 \}, x_2 \} + \ell_2^{\text{str}}(x_1, x_2)$, we further need that $f_1((\ell_2^+)^{\text{str}}(x_1, x_2)) = (\ell_2^-)^{\text{str}}(f_1 x_1, f_1 x_2)$; but this is precisely Lemma 5.5 (3). \square

This completes the proof that the L_∞ structure is invariant under the move in the left hand diagram of Figure 11. For the move in the right hand diagram of Figure 11, an entirely analogous argument shows that $f_1 : (\mathcal{A}^+)^{\text{comm}} \rightarrow (\mathcal{A}^-)^{\text{comm}}$ defined by $f_1(q_j) = t_i q_j$ and $f_1(x) = x$ for all other generators is an L_∞ equivalence between the L_∞ algebras $((\mathcal{A}^+)^{\text{comm}}, \{\ell_k^+\})$ and $((\mathcal{A}^-)^{\text{comm}}, \{\ell_k^-\})$. We conclude that the L_∞ structure is invariant under any base point move.

5.3. Invariance under Reidemeister III move. Next we suppose that the pointed Legendrian links Λ^+ and Λ^- are related by a Reidemeister III move, as shown in Figure 12. By moving base points if necessary and

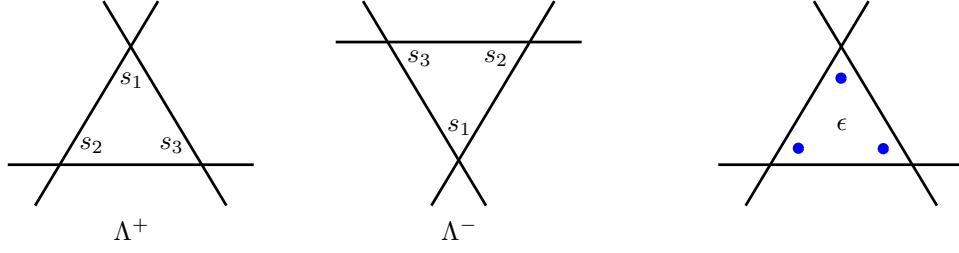


FIGURE 12. Reidemeister III move in the xy projection, with any possible choice of crossing information at the three crossings. In each of Λ^+ and Λ^- , the three depicted quadrants are labeled s_1, s_2, s_3 , where each s_j is either a p or q as in Figure 5. On the right, $\epsilon \in \{\pm 1\}$ is defined to be the product of the orientation signs of the three marked quadrants in Λ^+ .

applying the invariance result from Section 5.2, we can assume that the local Reidemeister III move does not involve any base points.

As usual, we append \pm superscripts to label constructions in Λ^\pm : so for instance ℓ_k^\pm are the L_∞ operations on $(\mathcal{A}^\pm)^{\text{comm}}$, and $h^\pm, \delta^\pm, d^\pm, \tilde{\beta}^\pm$ are the Hamiltonian, string coproduct, differential, and $\tilde{\beta}$ map (Definition 2.29) on $(\mathcal{A}^\pm)_{\text{SFT}}^{\text{comm}}$. There is an obvious one-to-one correspondence between base points and Reeb chords for Λ^+ and Λ^- , and we use the same labels for each, so that the \pm algebras are the same: we can write

$$\begin{aligned} (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}} &= (\mathcal{A}^-)_{\text{SFT}}^{\text{comm}} = \mathbb{Z}[q_1, \dots, q_n, t_1^{\pm 1}, \dots, t_s^{\pm 1}] \\ (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}} &= (\mathcal{A}^-)_{\text{SFT}}^{\text{comm}} = \mathbb{Z}[q_1, \dots, q_n, p_1, \dots, p_n, t_1^{\pm 1}, \dots, t_s^{\pm 1}]. \end{aligned}$$

In particular, we label the Reeb chords for Λ^\pm so that the three crossings involved in the Reidemeister III move correspond to Reeb chords a_1, a_2, a_3 ; we then define s_1, s_2, s_3 to be the associated p or q variables labeling the quadrants as shown in Figure 12, with $s_j \in \{p_j, q_j\}$.

To prove invariance for Reidemeister III, we will use the corresponding invariance proof for Legendrian SFT in [Ng10, §4.1]. First we introduce a bit of notation. We use asterisks to denote signed duals: $p_j^* = q_j$ and $q_j^* = -p_j$. This follows the notation from the proof of Proposition 2.30 above; please note that this notation is slightly different from the notation used in [Ng10], where q_j^* is $+p_j$ rather than $-p_j$. Also, let $\epsilon, \epsilon_1 \in \{\pm 1\}$ denote the following signs: ϵ is the product of the orientation signs of the three shaded quadrants in the diagram for Λ^+ shown in the right-hand diagram in Figure 12, and ϵ_1 records whether the orientation of Λ^+ agrees or disagrees with the depicted arrow.

We now define an algebra map $\phi : (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}} \rightarrow (\mathcal{A}^-)_{\text{SFT}}^{\text{comm}}$ by:

$$\begin{aligned}\phi(s_1) &= s_1 - \epsilon s_3^* s_2^* \\ \phi(s_2) &= s_2 - \epsilon s_1^* s_3^* \\ \phi(s_3) &= s_3 - \epsilon s_2^* s_1^*\end{aligned}$$

and ϕ acts as the identity on all other generators of $(\mathcal{A}^+)_{\text{SFT}}^{\text{comm}} = (\mathcal{A}^-)_{\text{SFT}}^{\text{comm}}$. Note that ϕ is a filtered map with respect to the p filtrations on $(\mathcal{A}^\pm)_{\text{SFT}}^{\text{comm}}$ since not all of s_1, s_2, s_3 can be p 's (otherwise the move is not topologically allowed). The map ϕ was previously defined in [Ng10, §4.1], with slightly different signs because of the above-mentioned convention change for duals, and satisfies the following key properties.

- Lemma 5.7.** (1) For any $x_1, x_2 \in (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}}$, we have $\phi(\{x_1, x_2\}) = \{\phi(x_1), \phi(x_2)\}$.
(2) We have $\phi \circ d^+ = d^- \circ \phi$.
(3) For any $x_1, x_2 \in (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}}$, we have $\phi(\tilde{\beta}^+(x_1, x_2)) = \tilde{\beta}^-(\phi(x_1), \phi(x_2))$.

Proof. Properties (1) and (2) are proven in [Ng10, Lemma 4.1] and [Ng10, Proposition 4.5], respectively; note as before that the sign conventions in [Ng10] are different from the ones here, but the proofs carry over to the new conventions. For (3), note that for any generator x of $(\mathcal{A}^+)_{\text{SFT}}^{\text{comm}}$, we have $\mathbf{v}^+(x) = \mathbf{v}^-(\phi(x))$ and $\mathbf{w}^+(x) = \mathbf{w}^-(\phi(x))$: this is clear unless $x = s_j$ for some $j = 1, 2, 3$, while for $x = s_j$, say $x = s_1$, it follows from the fact that $\mathbf{v}^-(s_3^* s_2^*) = \mathbf{v}^-(s_1)$ and $\mathbf{w}^-(s_3^* s_2^*) = \mathbf{w}^-(s_1)$. (The latter is trivial since both sides are 0; for the former, note that the capping paths γ_{s_1} and $\gamma_{s_3^*} \cdot \gamma_{s_2^*}$ agree outside of the triangle in the Reidemeister move.) Property (3) now follows directly from the definition of $\tilde{\beta}$ in Definition 2.29. \square

We now construct the maps f_1, f_2 that will constitute an L_2 equivalence between $((\mathcal{A}^+)_{\text{SFT}}^{\text{comm}}, \{\ell_k^+\})$ and $((\mathcal{A}^-)_{\text{SFT}}^{\text{comm}}, \{\ell_k^-\})$. Define $f_1 : (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}} \rightarrow (\mathcal{A}^-)_{\text{SFT}}^{\text{comm}}$ and $f_2 : (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}} \otimes (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}} \rightarrow (\mathcal{A}^-)_{\text{SFT}}^{\text{comm}}$ as follows:

$$\begin{aligned}f_1(x_1) &= \Pi\phi(x_1) \\ f_2(x_1, x_2) &= (-1)^{|x_1|+1} \Pi\{\Pi\phi(x_1), \phi(x_2)\},\end{aligned}$$

where we recall that $\Pi : (\mathcal{A}^\pm)_{\text{SFT}}^{\text{comm}} \rightarrow (\mathcal{A}^\pm)_{\text{SFT}}^{\text{comm}}$ is the projection map that sends any word containing p 's to 0.

Proposition 5.8. Suppose that Λ^\pm are related by a Reidemeister III move. Then the maps (f_1, f_2) are an L_2 equivalence between $((\mathcal{A}^+)_{\text{SFT}}^{\text{comm}}, \{\ell_k^+\})$ and $((\mathcal{A}^-)_{\text{SFT}}^{\text{comm}}, \{\ell_k^-\})$.

Proof. First note that by Lemma 5.7 (2) and the fact that ϕ and d^- are filtered maps, we have

$$f_1 \circ \ell_1^+ = \Pi\phi\Pi d^+ = \Pi\phi d^+ = \Pi d^- \phi = \Pi d^- \Pi\phi = \ell_1^- \circ f_1.$$

It thus suffices to show that f_2 satisfies the properties listed in Definition 5.2.

It will be useful to write

$$(\mathcal{A}^\pm)_{\text{SFT}}^{\text{comm}} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots$$

where \mathcal{A}_k is the \mathbb{Z} -module generated by words in $q_1, \dots, q_n, p_1, \dots, p_n, t_1^{\pm 1}, \dots, t_s^{\pm 1}$ containing exactly k p 's; then $\mathcal{A}_0 = (\mathcal{A}^\pm)_{\text{SFT}}^{\text{comm}}$, the map Π is simply projection to \mathcal{A}_0 , and the p filtration on $(\mathcal{A}^\pm)_{\text{SFT}}^{\text{comm}}$ is given by $\mathcal{F}^k(\mathcal{A}^\pm)_{\text{SFT}}^{\text{comm}} = \bigoplus_{m \geq k} \mathcal{A}_m$ for $k \geq 0$. Now decompose

$$\begin{aligned} \phi &= \phi_0 + \phi_1 + \cdots \\ d^+ &= d_0^+ + d_1^+ + \cdots \\ d^- &= d_0^- + d_1^- + \cdots \end{aligned}$$

where ϕ_m, d_m^+, d_m^- are the components of ϕ, d^+, d^- raising filtration level by m , i.e., mapping \mathcal{A}_k to \mathcal{A}_{m+k} for each k . Then we can write $\phi_0 = f_1$, $d_0^+ = \{h_1^+, \cdot\}$, $d_0^- = \{h_1^-, \cdot\}$, and

$$f_2(x_1, x_2) = (-1)^{|x_1|+1} \{\phi_0 x_1, \phi_1 x_2\}.$$

Now suppose $x_1, x_2 \in (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}}$. Then $\{x_1, x_2\} = 0$, and thus by Lemma 5.7 (1) we have $\{\phi(x_1), \phi(x_2)\} = \phi\{x_1, x_2\} = 0$. In particular we have

$$\begin{aligned} 0 &= \Pi\{\phi(x_1), \phi(x_2)\} = \{\phi_0 x_1, \phi_1 x_2\} + \{\phi_1 x_1, \phi_0 x_2\} \\ &= \{\phi_0 x_1, \phi_1 x_2\} + (-1)^{|x_1||x_2|+|x_1|+|x_2|} \{\phi_0 x_2, \phi_1 x_1\} \\ &= (-1)^{|x_1|+1} f_2(x_1, x_2) + (-1)^{|x_1||x_2|+|x_1|+1} f_2(x_2, x_1), \end{aligned}$$

and we conclude that f_2 is graded-symmetric. The fact that f_2 satisfies the Leibniz rule from Definition 5.2 follows from graded symmetry and the fact that $\phi_0 = f_1$ is an algebra map:

$$\begin{aligned} f_2(x_1, x_2 x'_2) &= (-1)^{|x_2 x'_2|(|x_1|+1)} \{\phi_0(x_2) \phi_0(x'_2), \phi_1(x_1)\} \\ &= (-1)^{|x_2 x'_2|(|x_1|+1)} \phi_0(x_2) \{\phi_0(x'_2), \phi_1(x_1)\} + (-1)^{|x_2|(|x_1|+1)} \{\phi_0(x_2), \phi_1(x_1)\} \phi_0(x'_2) \\ &= (-1)^{|x_2|(|x_1|+1)} f_1(x_2) f_2(x_1, x'_2) + f_2(x_1, x_2) f_1(x'_2). \end{aligned}$$

Finally we establish the L_2 relation. For $x_1, x_2 \in (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}}$, we calculate

$$\begin{aligned} f_1 \ell_2^+(x_1, x_2) &= \Pi \phi \Pi m_2^+(x_1, x_2) \\ &= \Pi \phi \{x_1, d^+ x_2\} - \Pi \phi \tilde{\beta}^+(x_1, x_2) \\ &= \Pi \{\phi x_1, d^- \phi x_2\} - \Pi \tilde{\beta}^-(\phi x_1, \phi x_2) \\ &= \{\phi_0 x_1, (d_1^- \phi_0 + d_0^- \phi_1) x_2\} + \{\phi_1 x_1, d_0^- \phi_0 x_2\} - \tilde{\beta}^-(\Pi \phi x_1, \Pi \phi x_2) \\ &= \Pi(\{\phi_0 x_1, d^- \phi_0 x_2\} - \tilde{\beta}^-(\phi_0 x_1, \phi_0 x_2)) + \{\phi_0 x_1, d_0^- \phi_1 x_2\} + \{\phi_1 x_1, d_0^- \phi_0 x_2\} \\ &= \Pi m_2^-(f_1 x_1, f_1 x_2) + \{\phi_0 x_1, d_0^- \phi_1 x_2\} + \{\phi_1 x_1, \phi_0 d_0^+ x_2\} \\ &= \ell_2^-(f_1 x_1, f_1 x_2) + \{\phi_0 x_1, d_0^- \phi_1 x_2\} + (-1)^{|x_1|} f_2(x_1, \ell_1^+ x_2). \end{aligned}$$

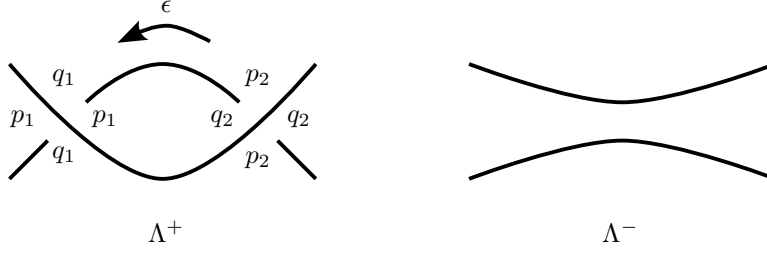


FIGURE 13. Reidemeister II move in the xy projection. The sign ϵ is ± 1 depending on whether the depicted arrow agrees or disagrees with the orientation of Λ^+ .

Furthermore, by Proposition 2.38 and the fact that Π annihilates $\tilde{\beta}(\phi_0 x_1, \phi_1 x_2)$ and $\tilde{\beta}(\phi_1 x_2, \phi_0 x_1)$ since both of these are multiples of $(\phi_0 x_1)(\phi_1 x_2) \in \mathcal{F}^1$, we have

$$\begin{aligned}
 \ell_1^- f_2(x_1, x_2) &= (-1)^{|x_1|+1} \Pi d^- \{ \phi_0 x_1, \phi_1 x_2 \} \\
 &= (-1)^{|x_1|+1} \Pi \{ d^- \phi_0 x_1, \phi_1 x_2 \} + \Pi \{ \phi_0 x_1, d^- \phi_1 x_2 \} \\
 &= (-1)^{|x_1|+1} \{ d_0^- \phi_0 x_1, \phi_1 x_2 \} + \{ \phi_0 x_1, d_0^- \phi_1 x_2 \} \\
 &= (-1)^{|x_1|+1} \{ \phi_0 d_0^+ x_1, \phi_1 x_2 \} + \{ \phi_0 x_1, d_0^- \phi_1 x_2 \} \\
 &= -f_2(\ell_1^+ x_1, x_2) + \{ \phi_0 x_1, d_0^- \phi_1 x_2 \}.
 \end{aligned}$$

Subtracting this from the formula for $f_1 \ell_2^+(x_1, x_2)$ gives the L_2 relation, as desired. \square

Remark 5.9. The proof of Proposition 5.8 works equally well to produce an L_2 morphism between the L_∞ algebras $(\mathcal{A}^\pm)^{\text{comm}}$ for any Legendrian links Λ_\pm , given the existence of a map $\phi : (\mathcal{A}^+)^{\text{SFT}} \rightarrow (\mathcal{A}^-)^{\text{SFT}}$ satisfying the properties in Lemma 5.7. For the base point move from Section 5.2, such a map also exists and is defined for the base point move in the left hand diagram of Figure 11 by $\phi(q_j) = t_i^{-1} q_j$, $\phi(p_j) = t_i p_j$, and ϕ acts as the identity on all other generators. (The same map with t_i replaced by t_i^{-1} works for the right hand diagram of Figure 11.) This gives an alternate proof of L_2 equivalence for the base point move. However, note that Proposition 5.6 proves the stronger result that a base point move gives an L_∞ equivalence.

5.4. Invariance under Reidemeister II move. We now suppose that Λ^+ and Λ^- are related by a Reidemeister II move, with the xy projection of Λ^+ having two more crossings a_1, a_2 as shown in Figure 13. By moving base points as before, we can assume that the local Reidemeister II move does not involve any base points. As usual we use \pm superscripts to denote operations on Λ^\pm .

In [Ng10, §4.2], it is proved that the curved DGA associated to rational Legendrian SFT is invariant under a Reidemeister II move. In the language of the present paper, part of this proof involves the construction of an algebra

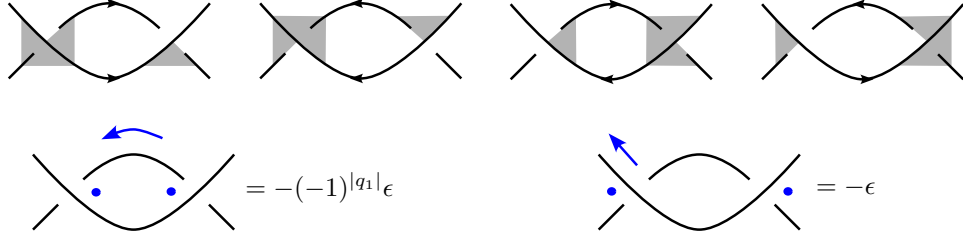


FIGURE 14. Top row: orientation signs for the four possible choices for strand orientations in Λ^+ ; shaded quadrants have $-$ orientation sign, unshaded quadrants $+$. For the first two diagrams, $|q_1|$ is even; for the last two, $|q_1|$ is odd. Bottom row: by inspection, in all four cases, the product of the orientation signs of the two marked quadrants, along with the sign of the arrow relative to the orientation of Λ^+ , is as shown.

map $\phi : (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}} \rightarrow (\mathcal{A}^-)_{\text{SFT}}^{\text{comm}}$ such that $\phi \circ d^+ = d^- \circ \phi$: in the language of [Ng10], this is $\lim_{n \rightarrow \infty} \psi^n$. If this map satisfied $\phi\{x_1, x_2\} = \{\phi x_1, \phi x_2\}$ for $x_1, x_2 \in (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}}$, then we could use the same proof as for Reidemeister III moves in Section 5.3 to construct an L_2 equivalence between $(\mathcal{A}^+)_{\text{SFT}}^{\text{comm}}$ and $(\mathcal{A}^-)_{\text{SFT}}^{\text{comm}}$; see Remark 5.9. Unfortunately the ϕ map for Reidemeister II does not usually commute with $\{\cdot, \cdot\}$. Instead we will use a different strategy to prove L_2 equivalence under Reidemeister II, constructing f_1 and f_2 by hand.

To define these maps, note that the bigon in Λ^+ with corners at p_1 and q_2 contributes $\epsilon q_2 p_1 = -(-1)^{|q_1|} p_1 q_2$ to the Hamiltonian h^+ : see Figure 14. Then $\ell_1^+(q_1) = \{h_1^+, q_1\} = \epsilon(q_2 - v)$ where we define

$$v := -\epsilon\{h_1^+ - q_2 p_1, q_1\};$$

geometrically v counts disks with a single positive corner at the Reeb chord a_1 besides the bigon. A standard argument implies that v does not involve either q_1 or q_2 : by Stokes, the (positive, bounded below) area of any disk contributing to v is the difference between the height of the Reeb chord a_1 and the sum of the heights of the Reeb chords for all negative corners of the disk, and the difference between the heights of the Reeb chords a_1 and a_2 can be arbitrarily small. Thus we have $v \in (\mathcal{A}^-)_{\text{SFT}}^{\text{comm}}$.

Define the algebra map $f_1 : (\mathcal{A}^+)_{\text{SFT}}^{\text{comm}} \rightarrow (\mathcal{A}^-)_{\text{SFT}}^{\text{comm}}$ by

$$f_1(q_1) = 0, \quad f_1(q_2) = v,$$

and f_1 is the identity on all other generators of $(\mathcal{A}^+)_{\text{SFT}}^{\text{comm}}$. As originally shown over $\mathbb{Z}/2$ by Chekanov [Che02] and extended to \mathbb{Z} coefficients in [ENS02], f_1 is a chain map between the commutative Chekanov–Eliashberg DGAs $((\mathcal{A}^+)_{\text{SFT}}^{\text{comm}}, \ell_1^+)$ and $((\mathcal{A}^-)_{\text{SFT}}^{\text{comm}}, \ell_1^-)$ inducing an isomorphism in homology. We record this fact in the following result.

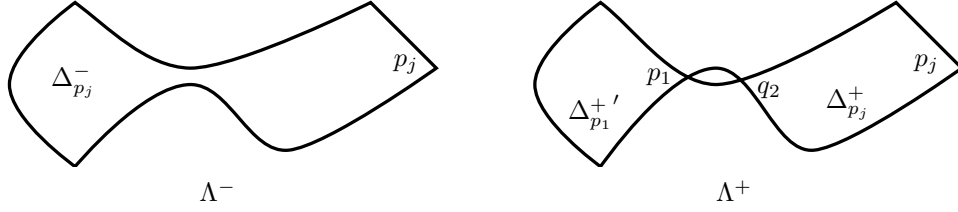


FIGURE 15. Following a disk in $\Delta_{p_j}^-$ from Λ^- to Λ^+ . On the right, the Reidemeister II move pinches the disk into two pieces connected by the $p_1 q_2$ bigon.

Proposition 5.10 ([Che02, ENS02]). *The map f_1 is a chain map between the commutative Chekanov–Eliashberg DGAs $((\mathcal{A}^+)^{\text{comm}}, \ell_1^+)$ and $((\mathcal{A}^-)^{\text{comm}}, \ell_1^-)$: $f_1 \circ \ell_1^+ = \ell_1^- \circ f_1$. Furthermore, f_1 is a quasi-isomorphism.*

Proposition 5.10 was originally shown over $\mathbb{Z}/2$ by Chekanov [Che02] in his proof that Legendrian contact homology is an isotopy invariant; the proof was extended to \mathbb{Z} coefficients in [ENS02]. We will recall the proof that f_1 is a chain map because it will be useful in what follows. For the proof that f_1 is a quasi-isomorphism, see [ENS02], specifically section 6.3 (the DGA is invariant up to stable tame isomorphism) and Theorem 3.14 (stable tame isomorphism is a quasi-isomorphism).

To set up the proof, we introduce some notation that we will use in the proof of the L_2 relation as well. For any j , write $\Delta_{p_j}^\pm$ for the subset of $\Delta(\Lambda^\pm)$ consisting of disks with a single positive corner at a_j (i.e., a corner at p_j) and any number of negative corners, that is, the disks contributing to $\ell_1^\pm(q_j)$. For $j \neq 1, 2$ and $m \geq 0$, write $\Delta_{p_j}^{+,m}$ for the subset of $\Delta_{p_j}^+$ consisting of disks with no negative corners at a_1 and m negative corners at a_2 (i.e., m corners at q_2 and no corners at p_1). Finally, the set $\Delta_{p_1}^+$ includes the bigon with corners at p_1 and q_2 ; we write $\Delta_{p_1}^{+'}$ for the complement of this bigon in $\Delta_{p_1}^+$. Note that no disk in $\Delta_{p_1}^{+'}$ has a corner at q_2 , and that v is the signed sum of terms corresponding to the disks in $\Delta_{p_1}^{+'}$.

Proof of Proposition 5.10. As mentioned above, we will show that f_1 is a chain map and refer the reader to [Che02, ENS02] for the proof that it is a quasi-isomorphism. It suffices to check $f_1 \circ \ell_1^+(s) = \ell_1^-(f_1(s))$ on generators s of $(\mathcal{A}^+)^{\text{comm}}$. For $s = t_i^{\pm 1}$, this is trivial since $\ell_1^\pm(t_i^{\pm 1}) = 0$; for $s = q_1$ and $s = q_2$, we have $f_1(\ell_1^+(q_1)) = f_1(\epsilon(q_2 - v)) = 0 = \ell_1^-(f_1(q_1))$ and

$$f_1(\ell_1^+(q_2)) = f_1(\ell_1^+(\epsilon \ell_1^+(q_1) + v)) = f_1(\ell_1^+(v)) = \ell_1^-(f_1(v)) = \ell_1^-(f_1(q_2)).$$

Now suppose that $s = q_j$ is a generator corresponding to a Reeb chord a_j different from a_1, a_2 ; we need to check $f_1(\ell_1^+(q_j)) = \ell_1^-(q_j)$. Consider a disk $\Delta \in \Delta_{p_j}^-$. As we pass through the Reidemeister II move from Λ^- to Λ^+ , Δ may survive as an immersed disk, or it may pinch into a union of disks if some portion of Δ passes through the neck of Λ^- . Suppose that Δ

passes through the neck of Λ^- m times. Then Δ will pinch into $m+1$ disks; see Figure 15 for an illustration for $m=1$. These $m+1$ disks in Λ^+ have corners collectively given by the corners of Δ along with m p_1, q_2 pairs (i.e. a positive corner at a_1 and a negative corner at a_2). One of the disks, call it Δ_0 , contains the original positive corner at a_j . Since the rest of the disks must contain a positive corner by Stokes, they must each have a positive corner at a_1 . It follows that each of these disks is in $\Delta_{p_1}^{+'}$, and that Δ_0 is in $\Delta_{p_j}^{+;m}$. Conversely, we can glue together a disk in $\Delta_{p_j}^{+;m}$ and m disks in $\Delta_{p_1}^{+'}$ to get a disk in $\Delta_{a_j}^-$. We conclude that there is a one-to-one correspondence

$$\Delta_{p_j}^- \longleftrightarrow \bigcup_{m \geq 0} \left(\Delta_{p_j}^{+;m} \times (\Delta_{p_1}^{+'})^m \right).$$

Suppose that Δ and $(\Delta_0, \Delta_1, \dots, \Delta_m)$ are mapped to each other under this correspondence. Let w denote the contribution of Δ to $\ell_1^-(q_j)$, and let w_0 and w_1, \dots, w_m denote the contributions of Δ_0 and $\Delta_1, \dots, \Delta_m$ to $\ell_1^+(q_j)$ and v , respectively. (Since $\ell_1^+(q_1) = \epsilon(q_2 - v)$, w_i is also $-\epsilon$ times the contribution of Δ_i to $\ell_1^+(q_1)$ for $i = 1, \dots, m$.) Then by the gluing process, w is precisely the result of replacing the m occurrences of q_2 in w_0 by w_1, \dots, w_m . Indeed, this is true even if we account for the signs of w, w_0, w_1, \dots, w_m : the cumulative difference in signs between the term in $\ell_1^-(q_j)$ and the terms in $\ell_1^+(q_j)$ and $\ell_1^+(q_1)$ is given by the product of the three signs in the bottom right diagram in Figure 14 for each pinch. But from Figure 14, this product is equal to $-\epsilon$ and exactly cancels the difference between the sign of w_i and the sign of its contribution to $\ell_1^+(q_1)$.

Restating the result of the previous paragraph in algebraic language, we have $f_1 \ell_1^+(q_j) = \ell_1^-(q_j)$, as desired. \square

Having defined the map $f_1 : (\mathcal{A}^+)^{\text{comm}} \rightarrow (\mathcal{A}^-)^{\text{comm}}$, we complete the construction of an L_2 morphism between $(\mathcal{A}^+)^{\text{comm}}$ and $(\mathcal{A}^-)^{\text{comm}}$ by defining a map $f_2 : (\mathcal{A}^+)^{\text{comm}} \otimes (\mathcal{A}^+)^{\text{comm}} \rightarrow (\mathcal{A}^-)^{\text{comm}}$. Write the Reeb chords of Λ^+ as a_1, a_2, \dots, a_n , where a_1, a_2 are the crossings involved in the Reidemeister II move. As usual, we will write h_1^+ and h_2^+ for the portions of the Hamiltonian of Λ^+ corresponding to disks with 1 and 2 positive punctures, respectively.

Define f_2 on the generators of $(\mathcal{A}^+)^{\text{comm}}$ as follows: for $x_1, x_2 \in \{q_1, \dots, q_n, t_1, \dots, t_s\}$, $f_2(x_1, x_2) = 0$ unless at least one of x_1, x_2 is equal to q_2 ; and

$$\begin{aligned} f_2(q_2, t_i) &= 0 \\ f_2(q_2, q_1) &= f_1 \left(\frac{\epsilon}{2} \ell_1^+(q_1, q_1) \right) \\ f_2(q_2, q_2) &= \begin{cases} \epsilon f_1 \ell_2^+(q_1, q_2 + v) & |q_1| \text{ even} \\ 0 & |q_1| \text{ odd} \end{cases} \\ f_2(q_2, q_j) &= f_1 \left(\epsilon \ell_2^+(q_1, q_j) - \frac{1}{2} \ell_2^+(q_1, q_1) \{p_2, \ell_1^+(q_j)\} \right). \end{aligned}$$

where the last equality is for all $j > 2$. Extend f_2 to a bilinear form on all of $(\mathcal{A}^+)^{\text{comm}}$ by graded symmetry and the Leibniz rule.

For future use, we note the following.

Lemma 5.11. *For any $x \in (\mathcal{A}^+)^{\text{comm}}$, we have*

$$f_2(q_1, x) = \frac{\epsilon(-1)^{|q_1|}}{2} f_1(\ell_2^+(q_1, q_1)\{p_2, x\}),$$

and for any $x \in (\mathcal{A}^+)^{\text{comm}}$ that does not involve q_2 , we have

$$f_2(q_2, x) = f_1\left(\epsilon\ell_2^+(q_1, x) - \frac{1}{2}\ell_2^+(q_1, q_1)\{p_2, \ell_1^+x\}\right).$$

Proof. By the definition of f_2 , the first equality holds for $x \in \{q_1, \dots, q_n, t_1, \dots, t_s\}$; note that both sides are trivially 0 unless $x = q_2$. Similarly, the second equality holds for $x \in \{q_1, q_3, \dots, q_n, t_1, \dots, t_s\}$. Now use the fact that both sides of both equalities are derivations with respect to x : we have

$$\begin{aligned} f_2(q_1, x_1x_2) &= f_2(q_1, x_1)f_1(x_2) + (-1)^{(|q_1|+1)|x_1|} f_1(x_1)f_2(q_1, x_2) \\ f_2(q_2, x_1x_2) &= f_2(q_2, x_2)f_1(x_2) + (-1)^{|q_1||x_1|} f_1(x_1)f_2(q_2, x_2) \end{aligned}$$

and it is straightforward to check that the right hand sides of the two equalities satisfy corresponding product rules in x . \square

We can now state the invariance result for the Reidemeister II move.

Proposition 5.12. *The maps (f_1, f_2) are an L_2 equivalence between $((\mathcal{A}^+)^{\text{comm}}, \{\ell_k^+\})$ and $((\mathcal{A}^-)^{\text{comm}}, \{\ell_k^-\})$.*

The remainder of this subsection is devoted to proving Proposition 5.12. In light of Proposition 5.10, it suffices to verify the L_2 morphism property. To this end, define the map $F : (\mathcal{A}^+)^{\text{comm}} \otimes (\mathcal{A}^+)^{\text{comm}} \rightarrow (\mathcal{A}^-)^{\text{comm}}$ by:

$$F(x_1, x_2) = f_1\ell_2^+(x_1, x_2) - \ell_2^-(f_1x_1, f_1x_2) - f_2(\ell_1^+x_1, x_2) - (-1)^{|x_1|} f_2(x_1, \ell_1^+x_2) - \ell_1^- f_2(x_1, x_2).$$

We want to show that $F \equiv 0$.

Since f_1 is an algebra map and ℓ_2^\pm and f_2 satisfy the Leibniz rule, it suffices to show that $F(x_1, x_2) = 0$ for generators $x_1, x_2 \in \{q_1, \dots, q_n, t_1, \dots, t_s\}$. We break the verification of $F(x_1, x_2) = 0$ for generators x_1, x_2 into several cases.

CASE 1: $x_1 = q_i$ AND $x_2 = q_j$ FOR SOME $i, j > 2$.

From the definition of F, f_1, f_2 , to show that $F(q_i, q_j) = 0$ we need to prove:

$$\ell_2^-(q_i, q_j) = f_1\ell_2^+(q_i, q_j) - f_2(\ell_1^+q_i, q_j) - (-1)^{|q_i|} f_2(q_i, \ell_1^+q_j).$$

Note that we can write $\ell_2^\pm(q_i, q_j) = (\ell_2^\pm)^{\text{SFT}}(q_i, q_j) + (\ell_2^\pm)^{\text{str}}(q_i, q_j)$, where $(\ell_2^\pm)^{\text{str}}(q_i, q_j)$ is the string contribution to ℓ_2^\pm and

$$(\ell_2^\pm)^{\text{SFT}}(q_i, q_j) = (-1)^{|q_i|} \{\{h_2^\pm, q_i\}, q_j\}$$

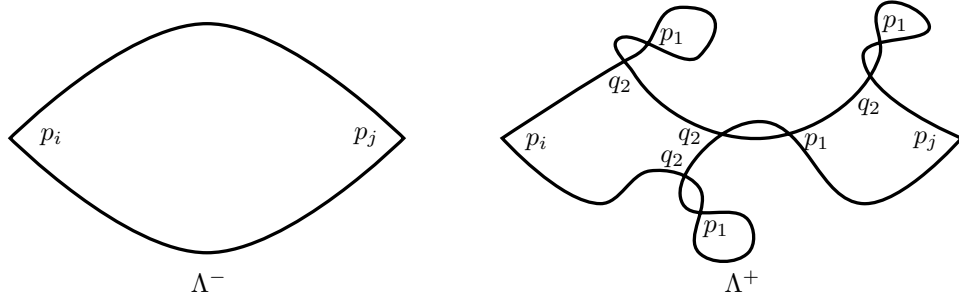


FIGURE 16. Following a disk in Δ_{p_i, p_j}^- from Λ^- to Λ^+ . On the right, a possibility for the corresponding picture in Λ^+ . The core (“spine”) is a chain of two disks, one containing p_i and one containing p_j , joined by a $p_1 q_2$ bigon. There may also be some number of branches consisting of disks in $\Delta_{p_1}^{+'}$ joined to the core by a $p_1 q_2$ bigon.

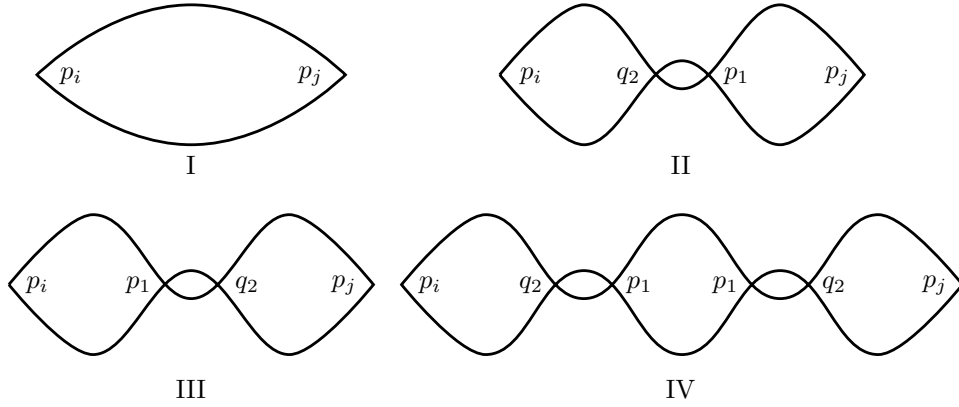


FIGURE 17. The four possibilities for the picture in Λ^+ corresponding to a disk in Λ^- in Δ_{p_i, p_j}^- . Each of these pictures may have branches (not depicted) given by disks in $\Delta_{a_1}^{+'}$ joined to the picture by a bigon.

is the SFT contribution (counting disks with positive punctures at p_i and p_j) to ℓ_2^\pm . The string contributions $(\ell_2^-)^{\text{str}}(q_i, q_j)$ and $(\ell_2^+)^{\text{str}}(q_i, q_j)$ are equal since the configuration of base points relative to the endpoints of the Reeb chords a_i, a_j is the same for Λ^+ and Λ^- . Thus we wish to show:

$$(9) \quad (\ell_2^-)^{\text{SFT}}(q_i, q_j) = f_1(\ell_2^+)^{\text{SFT}}(q_i, q_j) - f_2(\ell_1^+ q_i, q_j) - (-1)^{|q_i|} f_2(q_i, \ell_1^+ q_j).$$

To this end, consider a disk Δ contributing to $(\ell_2^-)^{\text{SFT}}(q_i, q_j)$: this is a disk in $\Delta(\Lambda^-)$ with positive punctures at p_i and p_j . As in the proof of Proposition 5.10, follow Δ from Λ^- through the Reidemeister II move to Λ^+ . The move pinches Δ into a union of disks in Λ^+ connected by $p_1 q_2$ bigons; see Figure 16 for an illustration. We call this chain of disks C . Up

to sign, the contribution of Δ to $(\ell_2^-)^{\text{SFT}}(q_i, q_j)$ is the product of the negative corners of Δ . This is equal to the product of the negative corners of C , not counting the q_2 corners where disks are linked by bigons. We will show that this last product is a term on the right hand side of (9), and conversely that every right hand term in (9) comes from some such configuration C . It will follow that there is a correspondence between the terms on the two sides of (9), not including signs. We will then check that the signs of corresponding terms match.

Consider a chain C of disks in Λ^+ corresponding to Δ as above. Some of these disks may be disks in $\Delta_{p_1}^+$, with a single positive puncture at p_1 ; we call these disks “branches”. If we remove these branches, then what remains is either a disk with positive punctures at p_i and p_j , or a chain of disks connecting a disk with positive puncture at p_i and a disk with positive puncture at p_j , with the chain linked by p_1q_2 bigons. Since Δ has only positive punctures at p_i and p_j , each pinch through the Reidemeister move introduces a p_1, q_2 pair of punctures, and each disk in C must have at least one positive puncture, C must fall into one of the four categories shown in Figure 17.

Up to sign, the contribution of Δ to $(\ell_2^-)^{\text{SFT}}(q_i, q_j)$ is the product of the negative corners of Δ . We claim that, first up to sign, there is a corresponding term on the right hand side of (9). Which term depends on whether the configuration in Λ^+ corresponding to Δ , which we denote by C , is of type I, II, III, or IV, in the terminology of Figure 17. In each case, the negative corners of Δ are the same as the negative corners of C , not including any q_2 corners.

If C is of type I, then the product of the negative corners of C , without the branches but including any q_2 adjacent to a branch, is a term in $\ell_2^+(q_i, q_j)$; thus, with the branches, the product of the negative corners besides q_2 's is a term in $f_1\ell_2^+(q_i, q_j)$. If C is of type II, write Δ_1 and Δ_2 for the p_iq_2 disk and the p_1p_j disk, respectively, where each disk may contain q_2 corners attached to branches. The product of the negative corners of Δ_2 is a term in $\ell_2^+(q_1, q_j)$; attaching branches and taking the product of all negative corners besides q_2 's then gives a term in $f_1\ell_2^+(q_1, q_j)$ and thus a term in $f_2(q_2, q_j)$ by the definition of f_2 . On the other hand, if the product of the negative corners of Δ_1 is vq_2 for some v , then attaching branches to Δ_1 and taking the product of all negative corners besides q_2 's gives a term in $f_1(v)$. The total product of negative corners between Δ_1 , Δ_2 , and their branches is thus a term in $f_1(v)f_2(q_2, q_j)$ and consequently a term in $f_2(\ell_1^+(q_i), q_j)$. Similarly, if C is of type III, then the product of the negative corners of C besides q_2 's is a term in $f_2(q_i, \ell_1^+(q_j))$.

If C is of type IV, there are two contributions to the right hand side of (9) coming from C . One comes from $f_2(\ell_1^+q_i, q_j)$: $f_2(q_2, q_j)$ contains a term of the form $f_1(\ell_2^+(q_1, q_1)\{p_2, \ell_1^+(q_j)\})$ which corresponds to the product of the negative corners of the p_1p_1 and q_2p_j disks in C along with their attached

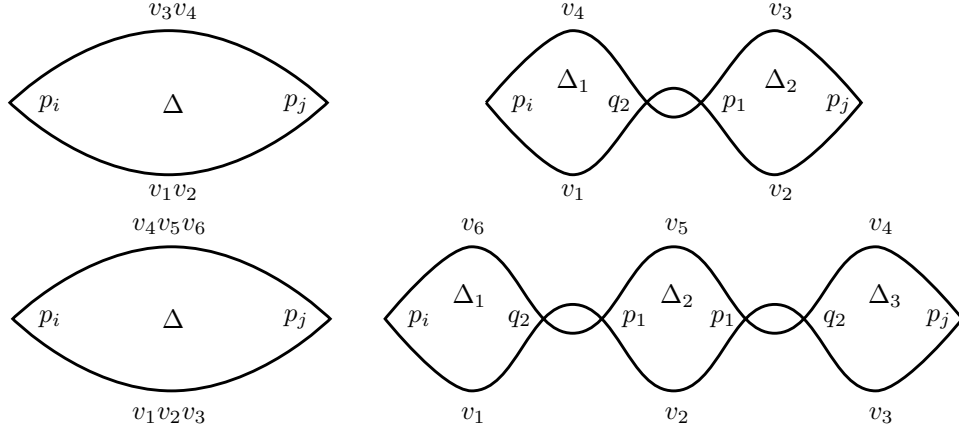


FIGURE 18. Top row: a disk Δ for Λ^- with corresponding type II configuration C for Λ^+ ; bottom row: a disk Δ for Λ^- with corresponding type IV configuration C for Λ^- . For the diagrams in Λ^+ , the v_i 's denote the product of the disk corners along the indicated arcs; for the corresponding diagrams in Λ^- , the products of negative corners are products of v_i 's as shown.

branches, and passing from $f_2(q_2, q_j)$ to $f_2(\ell_1^+ q_i, q_j)$ is multiplication by the negative corners of the $p_i q_2$ disk along with its branches. There is symmetrically another contribution from C coming from $f_2(q_i, \ell_1^+ q_j)$. Each of these contributions is multiplied by $\frac{1}{2}$, and (again up to sign) it follows that the contribution of C to the right hand side of (9) is equal to the contribution of Δ to the left hand side.

Conversely, every term on the right hand side of (9) comes from some chain C of disks in Λ^+ , and passing C through the Reidemeister II move to Λ^- gives an honest disk in Λ^- and thus a term on the left hand side of (9). Thus we have verified (9) up to signs.

Finally, we check that the signs agree. For type I, this is clear from the definition of ℓ_2^\pm . We will check type II (type III is very similar) and type IV.

For type II, we label the negative corners of Δ and C as shown in Figure 18, where for simplicity we remove all branches from C and correspondingly ignore the action of the f_1 map on the right hand side of (9). (As shown earlier, adding the branches is precisely the same as applying f_1 .) If we write $\text{sgn}(\Delta)$ for the product of the SFT orientation signs of the corners of Δ according to Section 2.6 (cf. Figure 6), then the disk Δ contributes $\epsilon_1(\text{sgn} \Delta) v_1 v_2 p_j v_3 v_4 p_i$ to h_2^- and thus

$$\epsilon_1(\text{sgn} \Delta) (-1)^{|q_i|} \{ \{ v_1 v_2 p_j v_3 v_4 p_i, q_i \}, q_j \} = \epsilon_1(\text{sgn} \Delta) (-1)^{|p_j|(|v_3|+|v_4|)+|q_i|} v_1 v_2 v_3 v_4$$

to $\ell_2^-(q_i, q_j)$. Similarly, the disk Δ_2 contributes $\epsilon_2(\text{sgn } \Delta_2)(-1)^{|p_j||v_3|+|q_1|}v_2v_3$ to $\ell_2^+(q_1, q_j)$ and the disk Δ_1 contributes $\epsilon_1(\text{sgn } \Delta_1)v_1q_2v_4$ to $\ell_1^+(q_i)$. From the definition of f_2 and the Leibniz rule, it follows that the configuration C contributes

$$-\epsilon_1\epsilon_2(\text{sgn } \Delta_1)(\text{sgn } \Delta_2)(-1)^{|p_j||v_3|+|q_1|+|v_4||q_j|+|p_1|}v_1v_2v_3v_4$$

to $-f_2(\ell_1^+q_i, q_j)$. But we see from Figure 14 that $\text{sgn } \Delta = -\epsilon_2(\text{sgn } \Delta_1)(\text{sgn } \Delta_2)$, and thus the above expression is equal to $\epsilon_1(\text{sgn } \Delta)(-1)^{|p_j||v_3|+|q_1|+|v_4||q_j|+|p_1|}v_1v_2v_3v_4$. Since $|p_k|+|q_k| = -1$ for all k and $|p_i|+|v_1|+|q_2|+|v_4| = -2$ (since $|h_2^+| = -2$ and Δ_1 gives a term in h_2^+), it follows readily that this expression is equal to the contribution of Δ to $\ell_2^-(q_i, q_j)$.

For type IV, label as shown in Figure 18. It is readily checked that Δ_1 contributes $\epsilon_1(\text{sgn } \Delta_1)v_1q_2v_6$ to $\ell_1^+q_i$; Δ_2 contributes $\epsilon_2(\text{sgn } \Delta_2)(-1)^{|q_1|+|v_5||p_1|}v_2v_5$ to $\ell_2^+(q_1, q_1)$; and Δ_3 contributes $\epsilon_3(\text{sgn } \Delta_3)(-1)^{|q_2|+|v_4||p_j|}q_2v_3v_4$ to $\ell_1^+q_j$. From this it is straightforward to calculate that the chain of disks contributes

$$-\frac{1}{2}\epsilon_2\epsilon_3(\text{sgn } \Delta_2)(\text{sgn } \Delta_3)(-1)^{|q_1|+|v_5||p_j|+|v_4||p_j|+|q_2|}v_2v_3v_4v_5$$

to the $-\frac{1}{2}\ell_2^+(q_1, q_1)\{p_2, \ell_1^+(q_j)\}$ term in $f_2(q_2, q_j)$, and consequently it contributes

$$\begin{aligned} & \frac{1}{2}\epsilon_1\epsilon_2\epsilon_3(\text{sgn } \Delta_1)(\text{sgn } \Delta_2)(\text{sgn } \Delta_3)(-1)^{|q_1|+(|v_4|+|v_5|+|v_6|)|p_j|+|q_2|+|v_1|}v_1v_2v_3v_4v_5v_6 \\ &= \frac{1}{2}\epsilon_1(\text{sgn } \Delta)(-1)^{|q_i|+(|v_4|+|v_5|+|v_6|)|p_j|}v_1v_2v_3v_4v_5v_6 \end{aligned}$$

to $-f_2(\ell_1^+q_i, q_j)$, where the equality follows from the fact that $(-1)^{|q_1|+1}(\text{sgn } \Delta) = \epsilon_2\epsilon_3(\text{sgn } \Delta_1)(\text{sgn } \Delta_2)(\text{sgn } \Delta_3)$, as can be deduced from Figure 14. Similarly it can be computed that the chain of disks also contributes the same quantity, with the same sign, to $-(-1)^{|q_i|}f_2(q_i, \ell_1^+q_j)$. These sum together to give

$$\epsilon_1(\text{sgn } \Delta)(-1)^{|q_i|+(|v_4|+|v_5|+|v_6|)|p_j|}v_1v_2v_3v_4v_5v_6,$$

which is exactly the contribution of the original disk Δ to $(\ell_2^-)^{\text{SFT}}(q_i, q_j)$. This completes the verification of (9) and finishes Case 1.

The remaining cases are easier to handle than Case 1. We first note the following result, which is readily checked from the definition of F , the L_∞ relations for $\{\ell_k^\pm\}$, and the chain-map property of f_1 from Proposition 5.10:

Lemma 5.13. *For any x_1, x_2 , we have $F(x_2, x_1) = (-1)^{|x_1||x_2|+1}F(x_1, x_2)$ and*

$$\ell_1^-F(x_1, x_2) = F(\ell_1^+x_1, x_2) + (-1)^{|x_1|}F(x_1, \ell_1^+x_2).$$

In particular, by the symmetry of F , it suffices to check that $F(x_1, x_2) = 0$ in the following cases: $x_1 = t_i$ and x_2 is arbitrary; $x_1 = q_1$ and x_2 is some q_j ; and $x_1 = q_2$ and x_2 is some q_j .

CASE 2: $x_1 = t_i$ FOR SOME i . By symmetry, we can assume $x_1 = t_i$, in which case

$$F(t_i, x_2) = f_1 \ell_2^+(t_i, x_2) - \ell_2^-(t_i, f_1 x_2).$$

We first consider the case where $x_2 \in \{q_3, \dots, q_n, t_1, \dots, t_s\}$, i.e., $x_2 \neq q_1, q_2$. By definition, when t_i is one of the inputs to ℓ_2^\pm , the result is precisely the string portion of ℓ_2^\pm : $\ell_2^+(t_i, x_2) = (\ell_2^+)^{\text{str}}(t_i, x_2) = \alpha^+ t_i x_2$ and $\ell_2^-(t_i, f_1 x_2) = \ell_2^-(t_i, x_2) = (\ell_2^-)^{\text{str}}(t_i, x_2) = \alpha^- t_i x_2$ for some $\alpha^\pm \in \frac{1}{2}\mathbb{Z}$. Furthermore, since the placement of base points relative to x_2 is the same in Λ^+ and Λ^- , we see from Proposition 3.8 that $\alpha^+ = \alpha^-$. We conclude that $F(t_i, x_2) = f_1(\alpha^+ t_i x_2) - \alpha^- t_i x_2 = 0$ in this case.

If $x_2 = q_1$, then since $\ell_2^+(t_i, q_1)$ is a multiple of $t_i q_1$ and $f_1(q_1) = 0$, we have $F(t_i, q_1) = 0$. Finally, if $x_2 = q_2$, it follows from Lemma 5.13 that $\ell_1^- F(t_i, q_1) = F(t_i, \ell_1^+ q_1) = \epsilon(F(t_i, q_2) - F(t_i, v))$. Since v does not involve q_2 , we now know that $F(t_i, v) = 0$, whence $F(t_i, q_2) = 0$.

CASE 3: $x_1 = q_1, x_2 \in \{q_1, \dots, q_n\}$. We further break this down into three subcases.

If $x_2 = q_j$ for $j > 2$, then we have

$$F(q_1, q_j) = f_1 \ell_2^+(q_1, q_j) - f_2(\ell_1^+ q_1, q_j) + (-1)^{|q_1||q_j|} f_2(\ell_1^+ q_j, q_1).$$

The second term on the right hand side is $-\epsilon f_2(q_2 - v, q_j) = -\epsilon f_2(q_2, q_j)$ while the third term is $(-1)^{|q_1|+1} f_2(q_1, \ell_1^+ q_j) = -\frac{\epsilon}{2} f_1(\ell_2^+(q_1, q_1)\{p_2, \ell_1^+ q_j\})$ by Lemma 5.11. It follows from the definition of $f_2(q_2, q_j)$ that $F(q_1, q_j) = 0$.

If $x_2 = q_1$, note that $F(q_1, q_1)$ is trivially 0 by symmetry if $|q_1|$ is even. If $|q_1|$ is odd, then

$$\begin{aligned} F(q_1, q_1) &= f_1 \ell_2^+(q_1, q_1) + 2f_2(q_1, \ell_1^+ q_1) \\ &= f_1 \ell_2^+(q_1, q_1) - \epsilon f_1(\ell_2^+(q_1, q_1)\{p_2, \ell_1^+ q_1\}) \\ &= 0 \end{aligned}$$

where the second equality comes from Lemma 5.11 and the third equality uses $\ell_1^+ q_1 = \epsilon(q_2 - v)$.

Finally, if $x_2 = q_2$, we want to show that $F(q_1, q_2) = 0$. If $|q_1|$ is odd, then from Lemma 5.13 and the fact that $F(q_1, q_1) = 0$, we have $0 = 2F(q_1, \ell_1^+ q_1) = 2\epsilon(F(q_1, q_2) - F(q_1, v))$. Since v does not involve q_2 , we know from the above cases that $F(q_1, v) = 0$, whence $F(q_1, q_2) = 0$. If $|q_1|$ is even, then $\ell_1^+(q_1, q_1) = 0$ and thus $f_2(q_2, q_1) = 0$, and we have

$$\begin{aligned} F(q_1, q_2) &= f_1 \ell_2^+(q_1, q_2) - f_2(\ell_1^+ q_1, q_2) - \ell_1^- f_2(q_1, q_2) \\ &= f_1 \ell_2^+(q_1, q_2) + \epsilon f_2(q_2, v) - \epsilon f_2(q_2, q_2) \\ &= f_1 \ell_2^+(q_1, q_2) + f_1 \ell_2^+(q_1, v) - \epsilon f_2(q_2, q_2) \\ &= 0, \end{aligned}$$

where the third equality comes from Lemma 5.11 and the final equality is the definition of $f_2(q_2, q_2)$.

CASE 4: $x_1 = q_2$, $x_2 \in \{q_1, \dots, q_n\}$. As in Case 2, we note from Lemma 5.13 that

$$\ell_1^- F(q_1, x_2) = F(\ell_1^+ q_1, x_2) + (-1)^{|q_1|} F(q_1, \ell_1^+ x_2) = \epsilon F(q_2, x_2) - \epsilon F(v, x_2) + (-1)^{|q_1|} F(q_1, \ell_1^+ x_2).$$

Since v does not involve q_2 , if $x_2 \neq q_2$ then all of $F(q_1, x_2)$, $F(v, x_2)$, and $F(q_1, \ell_1^+ x_2)$ are 0 by previous cases; thus $F(q_2, x_2) = 0$ if $x_2 \neq q_2$. Finally, if $x_2 = q_2$, then $F(v, q_2) = 0$ since we have established that $F(x_2, q_2) = 0$ if $x_2 \neq q_2$. Thus again all of $F(q_1, q_2)$, $F(v, q_2)$, and $F(q_1, \ell_1^+ x_2)$ are 0, and it follows that $F(q_2, q_2) = 0$.

This completes the proof of Proposition 5.12 and thus the proof of invariance.

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