Notes on the $K$-theory of Complex Grassmannians

Lenny Ng

18.979

May 1997

We wish to study the $K$-theory of the Grassmannian $G(k, n)$ of complex $k$-planes in $\mathbb{C}^n$. The Chern character gives a ring isomorphism

$$\text{ch} : K(G(k, n)) \otimes \mathbb{Q} \longrightarrow H^*(G(k, n), \mathbb{Q})$$

between the rationalized $K$-theory of $G(k, n)$ and the rational cohomology ring of $G(k, n)$. The CW-complex structure of $G(k, n)$ (see, e.g., [3]) immediately implies that, as an additive group, $K(G(k, n)) \otimes \mathbb{Q} = \mathbb{Q}^{(n \choose k)}$ is a free group on $\binom{n}{k}$ generators. Our goal is to make this description more concrete.

Note that $G(k, n) = U(n)/\left(U(k) \times U(n-k)\right) = St(k, n)/U(k)$ is the quotient by $U(k)$ of the complex Stiefel manifold $St(k, n)$ of partial orthonormal $k$-frames in $\mathbb{C}^n$. Thus any finite-dimensional representation $\rho : U(k) \longrightarrow GL(W)$ of $U(k)$ gives rise to the natural vector bundle $St(k, n) \times_\rho W \longrightarrow G(k, n)$, where $St(k, n) \longrightarrow G(k, n)$ is viewed as a principal $U(k)$-bundle. This vector bundle, which we will denote by $\Phi(\rho)$, is most concretely visualized as a quotient by $U(k)$ of the trivial $U(k)$-equivariant vector bundle $St(k, n) \times W(\rho) \longrightarrow St(k, n)$. (Here $\rho$ denotes the action of $U(k)$ on $W$ by $\rho$.) This trivial vector bundle is simply the pullback under the natural map $St(k, n) \longrightarrow G(k, n)$ of $\Phi(\rho)$.

It is an easy exercise to verify that given two representations $\rho_1, \rho_2$ of $U(k)$, $\Phi(\rho_1 \otimes \rho_2) = \Phi(\rho_1) \otimes \Phi(\rho_2)$. Hence the map $\rho \mapsto \Phi(\rho)$ gives a semiring homomorphism from representations of $U(k)$ to vector bundles over $G(k, n)$, which extends by universality to a ring homomorphism

$$\Phi : R(U(k)) \otimes \mathbb{Q} \longrightarrow K(G(k, n)) \otimes \mathbb{Q}$$

from the rationalized representation (or character) ring of $U(k)$ to the rationalized $K$-theory of $G(k, n)$. Using techniques not understood by me, $\Phi$ can be shown to be surjective. We thus obtain a very concrete method by which elements of $K(G(k, n))$ can be constructed. It is then of interest to compute $\ker(\Phi)$.

We first need to understand the representation ring of $U(n)$. For a full account, see [2]; we will give an abridged answer here. A maximal torus in $U(k)$ is given by $T(k) = \{\text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_k})\}$. The representation ring of $T(k)$ is easily seen to be $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}]$, where the monomial $x_1^{e_1} \cdots x_k^{e_k}$ corresponds to the one-dimensional representation for which
the \( j \)-th factor of \( T(k) = (S^1)^k \) acts on \( \mathbb{C} \) via the standard weight \( e_j \) representation; in other words,
\[
\text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_k}) \cdot z = e^{i(e_1\theta_1 + \cdots + e_k\theta_k)}z.
\]
The Weyl group of \( U(k) \) is the full symmetric group \( S_k \) on \( k \) elements, and thus the rationalized representation ring of \( U(k) \) is
\[
R(U(k)) \otimes \mathbb{Q} = \mathbb{Q}[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}]^{S_k} = \mathbb{Q}[\sigma_1, \ldots, \sigma_k, \sigma_k^{-1}],
\]
the subgroup of finite Laurent series in \( x_1, \ldots, x_k \) which is invariant under permutations of the indeterminates. Here \( \sigma_i \) is the \( i \)-th elementary symmetric polynomial in \( x_1, \ldots, x_k \).

Thus, given an element \( f = f(x_1, \ldots, x_k) \) of \( R(U(k)) \otimes \mathbb{Q} \), that is, a symmetric finite Laurent series in \( x_1, \ldots, x_k \) with rational coefficients, we obtain an element \( \Phi(f) \) of \( K(G(k,n)) \otimes \mathbb{Q} \). To better understand the structure of \( \Phi(f) \), we look at its pullback bundle over \( St(k,n)/T(k) \). Note that the projection \( St(k,n) \to G(k,n) \) factors into
\[
St(k,n) \to St(k,n)/T(k) \xrightarrow{\pi} G(k,n),
\]
where \( \pi \) is defined to be the second projection. There is a map
\[
\tilde{\Phi} : R(T(k)) \otimes \mathbb{Q} \to K(St(k,n)/T(k)) \otimes \mathbb{Q}
\]
constructed along the same lines as \( \Phi \), and a map \( \pi^* : R(U(k)) \otimes \mathbb{Q} \to R(T(k)) \otimes \mathbb{Q} \) defined canonically by restriction. Then an element \( f \) of the representation ring of \( U(k) \) naturally gives rise to two \( K \)-classes over \( St(k,n)/T(k) \), namely the bundle \( \tilde{\Phi}(\pi^*(f)) \) and the pullback bundle \( \pi^*(\Phi(f)) \).

**Lemma 1** \( \tilde{\Phi}(\pi^*(f)) = \pi^*(\Phi(f)) \).

**Proof.** Straightforward; left as an exercise. \( \square \)

Thus we obtain the following diagram:
\[
\begin{array}{ccc}
St(k,n) \times W(f) & \to & \tilde{\Phi}(\pi^*(f)) \\
\downarrow & & \downarrow \\
St(k,n) & \to & St(k,n)/T(k) \xrightarrow{\pi} G(k,n)
\end{array}
\]

Recall that any representation of \( T(k) \) splits into a direct sum of one-dimensional representations, of the form of the form \( x_1^{e_1} \cdots x_k^{e_k} \), where, as before, we will use \( x_j \) to denote the weight 1 representation associated to the \( j \)-th factor of \( T(k) \), and the exponents denote tensor product powers. Corresponding to \( x_1, \ldots, x_k \) are line bundles \( L_1, \ldots, L_k \) over \( St(k,n)/T(k) \). The following result is nearly immediate.

**Proposition 2** Let \( f(x_1, \ldots, x_k) = \sum a_{m_1\cdots m_k} x_1^{m_1} \cdots x_k^{m_k} \) be an element of \( R(U(k)) \otimes \mathbb{Q} \). Then
\[
\pi^*(\Phi(f)) = \sum a_{m_1\cdots m_k} L_1^{\otimes m_1} \cdots L_k^{\otimes m_k}.
\]
Proof. Use Lemma 1 and the fact that $\Phi$ is a ring homomorphism. □

A bit more notation is needed at this point. Let $S^*$ be the dual to the tautological $k$-plane bundle over $G(k,n)$, and let $\gamma_1, \ldots, \gamma_k$ be the Chern roots of $S^*$, so that the equation

$$(1 + \gamma_1) \cdots (1 + \gamma_k) = 1 + c_1(S^*) + \cdots + c_k(S^*)$$

is formally satisfied. Any symmetric function of $\gamma_1, \ldots, \gamma_k$ can be written in terms of their elementary symmetric functions $c_1(S^*), \ldots, c_k(S^*)$, and thus should be interpreted as an element of $H^*(G(k,n)) \otimes \mathbb{Q}$.

Lemma 3 $\Phi(x_1 + \cdots + x_k) = S^*$.

Proof. Note that $\sigma_1 = x_1 + \cdots + x_k$ corresponds to the standard representation $U(k) \hookrightarrow GL(k,\mathbb{C})$. This argument was done in class for $k = 1, n = 2$. Essentially, lift $\Phi(\sigma_1)$ to a $U(k)$-equivariant bundle $\Phi(\sigma_1)$ over $St(k,n)$, tensor it with the “tautological bundle” over $St(k,n)$ (i.e., the pullback of the tautological bundle over $G(k,n)$), and find a $U(k)$-equivariant nowhere vanishing section of this tensor product. □

We are now in a position to compute the kernel of $\Phi$ using the Chern character isomorphism. The following proposition is the crucial result in this direction.

Proposition 4 Suppose $f \in R(U(k)) \otimes \mathbb{Q}$. Then $\text{ch}(\Phi(f)) = f(e^{\gamma_1}, \ldots, e^{\gamma_k})$.

Proof. The following lemma will be useful.

Lemma 5 (Borel-Hirzebruch [1]) If $P \rightarrow B$ is a principal $G$-bundle, and $T$ is a maximal torus in $G$, then the bundle projection factors as $P \rightarrow P/T \rightarrow B$, and the induced map $H^*(B) \rightarrow H^*(P/T)$ is an injection; in fact, $H^*(B) = H^*(P/T)^W$, where $W$ is the Weyl group of $G$.

In our case, this lemma implies that $\pi^* : H^*(G(k,n)) \rightarrow H^*(St(k,n)/T(k))$ is an injection. Let $f(x_1, \ldots, x_k) = \sum a_{m_1 \cdots m_k}x_1^{m_1} \cdots x_k^{m_k}$, as before. By naturality of the Chern character and Proposition 2,

$$\pi^* \text{ch}(\Phi(f)) = \text{ch}(\pi^* \Phi(f)) = \sum a_{m_1 \cdots m_k} (\text{ch}L_1)^{m_1} \cdots (\text{ch}L_k)^{m_k}$$

$$= \sum a_{m_1 \cdots m_k} e^{m_1c_1(L_1)} \cdots e^{m_kc_k(L_k)}$$

$$= f(e^{c_1(L_1)}, \ldots, e^{c_k(L_k)}).$$

But Lemma 3 implies that $\pi^*(c(S^*)) = (1 + c_1(L_1)) \cdots (1 + c_1(L_k))$, where $c$ denotes the total Chern class, and so $c_1(L_1), \ldots, c_k(L_k)$ are formally the image under $\pi^*$ of the Chern roots $\gamma_1, \ldots, \gamma_k$. Since $\pi^*$ is injective as a map on cohomology, the proposition follows. □

We now apply this proposition to the simplest case, the case of $G(1,n) = \mathbb{C}P^{n-1}$. 3
Corollary 6 If $k = 1$, then

$$\ker(\Phi) = \{ f = \sum_j a_j x^j \in \mathbb{Q}[x, x^{-1}] : \sum_j j^i a_j = 0, i = 0, \ldots, n-1 \}$$

$$= \{ f : 0 = f(1) = f'(1) = \cdots = f^{(n-1)}(1) \}.$$

In terms of $\chi(f) : U(1) \to \mathbb{C}$, the character of $f$, this corollary says that $\chi(f)$ vanishes to order $n$ at $1 \in U(1)$.

These conditions on $f$ have an intuitive explanation. The condition that $\sum_j a_j = 0$ says simply that the “fibers” of $\Phi(f)$ have dimension 0. For the next condition, $\sum_j j a_j = 0$, recall that $K(S^2) = \mathbb{Z} \oplus \mathbb{Z}$, where one factor comes from the dimension of a vector bundle over $S^2$, and the other comes from the winding number of the clutching function given by this vector bundle. Then $\sum_j j a_j$ gives the winding number corresponding to the $K$-class $\Phi(f)$ restricted to $S^2 = \mathbb{C}P^1 \subset \mathbb{C}P^{n-1}$. The higher conditions ($i \geq 2$) correspond to higher winding numbers.

**Proof of Corollary 6.** The kernel of $\Phi$ is precisely the set of $f$ so that $f(e^{\gamma_1}) = \text{ch}(\Phi(f)) = 0$. But $\gamma_1 = c_1(S^*)$ generates $H^*(\mathbb{C}P^{n-1})$ with $(\gamma_1)^n = 0$; therefore, if $f = \sum_j a_j x^j$, then

$$f(e^{\gamma_1}) = \sum_j a_j e^{j\gamma_1} = \sum_{i=0}^{n-1} \left( \sum_j j^i a_j \right) \frac{(\gamma_1)^i}{i!}.$$

The corollary follows. $\square$

**References**

