

REPRESENTATIONS, SHEAVES, AND LEGENDRIAN $(2, m)$ TORUS LINKS

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ABSTRACT. We study an A_∞ category associated to Legendrian links in \mathbb{R}^3 whose objects are n -dimensional representations of the Chekanov–Eliashberg differential graded algebra of the link. This representation category generalizes the positive augmentation category and we conjecture that it is equivalent to a category of sheaves of microlocal rank n constructed by Shende, Treumann, and Zaslow. We establish the cohomological version of this conjecture for a family of Legendrian $(2, m)$ torus links.

1. INTRODUCTION

1.1. Main results and conjectures. Legendrian contact homology [Eli98, EGH00] is a powerful invariant of Legendrian links which associates a differential graded algebra $(\mathcal{A}_\Lambda, \partial_\Lambda)$, often called the *Chekanov–Eliashberg DGA*, to a link Λ in a contact manifold Y . This noncommutative algebra is generated freely by Reeb chords of Λ , and its differential counts certain holomorphic disks in $\mathbb{R} \times Y$ with boundary on $\mathbb{R} \times \Lambda$.

Chekanov [Che02] defined this invariant combinatorially for Legendrian links in the standard contact \mathbb{R}^3 , and introduced the use of *augmentations* to extract more computable information from the DGA. These are DGA morphisms $\epsilon : (\mathcal{A}_\Lambda, \partial_\Lambda) \rightarrow (\mathbb{k}, 0)$, where \mathbb{k} is the base ring, and they produce chain complexes $LCC_*^\epsilon(\Lambda) = (\ker \epsilon / (\ker \epsilon)^2, \partial^\epsilon)$ for what is called *linearized contact homology*. The set of homologies $\{LCH_*^\epsilon(\Lambda)\}$ over all ϵ is an effective invariant of Λ , which has been heavily studied because it is much easier to use in practice than the full DGA.

In recent years, it has become clear that these invariants have a much richer structure than simply a set: they can be organized into an A_∞ category [BC14], and even a unital A_∞ category $Aug_+(\Lambda, \mathbb{k})$ [NRS⁺15], which is an invariant of Λ up to A_∞ equivalence. The objects of this *augmentation category* are the augmentations of $(\mathcal{A}_\Lambda, \partial_\Lambda)$, and the morphism spaces $\text{Hom}(\epsilon_1, \epsilon_2)$ are an appropriate generalization of the linearized cochain complexes.

Meanwhile, Shende, Treumann, and Zaslow [STZ17] (building on [GKS12]) defined for each Legendrian link Λ in \mathbb{R}^3 a differential graded category $\mathcal{S}h(\Lambda, \mathbb{k})$, whose objects are constructible sheaves of \mathbb{k} -modules on the plane with singular support controlled by the front projection of Λ . Evidence suggested that in some cases, the

sheaves with “microlocal rank 1” corresponded bijectively to augmentations of Λ , and this was no coincidence: the main theorem of [NRS⁺15] asserts that the full subcategory $\mathcal{S}h_1(\Lambda, \mathbb{k})$ of such sheaves is equivalent to $\mathcal{A}ug_+(\Lambda, \mathbb{k})$ over any field \mathbb{k} . This provides a link between the disparate worlds of holomorphic curve invariants in contact geometry and homological algebra invariants.

The goal of this paper, then, is to begin investigating natural generalizations of this theorem. Beyond augmentations, a natural way to get a handle on the Chekanov–Eliashberg DGA is by studying its *representations*, and in fact we expect that the finite-dimensional representation theory of this DGA over a field \mathbb{k} is contained in $\mathcal{S}h(\Lambda, \mathbb{k})$. Specifically, it is reasonable to guess that objects of the analogous category $\mathcal{S}h_n(\Lambda, \mathbb{k})$ of sheaves with microlocal rank n should correspond not to augmentations, but rather to n -dimensional representations of $(\mathcal{A}_\Lambda, \partial_\Lambda)$, cf. [NR13], with an augmentation being a 1-dimensional representation. Two possible ways to generalize the A_∞ structure of $\mathcal{A}ug_+(\Lambda, \mathbb{k})$ to an A_∞ category of n -dimensional representations were proposed in [CDGG16].

In the present paper we pick out one of these constructions in particular and compare the resulting category to the sheaf category $\mathcal{S}h_n(\Lambda, \mathbb{k})$ in a family of examples.

More precisely, given any integer $n \geq 1$, a field \mathbb{k} , and a Legendrian link $\Lambda \subset \mathbb{R}^3$, we construct an A_∞ category called the *representation category* $\mathcal{R}ep_n(\Lambda, \mathbb{k})$. The objects of this category are n -dimensional representations of the Chekanov–Eliashberg differential graded algebra $(\mathcal{A}_\Lambda, \partial_\Lambda)$ associated to Λ , that is, algebra maps

$$\rho : \mathcal{A}_\Lambda \rightarrow \text{Mat}_n(\mathbb{k}),$$

where $\text{Mat}_n(\mathbb{k})$ is the \mathbb{k} -algebra of $n \times n$ matrices over \mathbb{k} , such that ρ satisfies $\rho \circ \partial_\Lambda = 0$. When $n = 1$, the category $\mathcal{R}ep_1(\Lambda, \mathbb{k})$ agrees with the positive augmentation category $\mathcal{A}ug_+(\Lambda, \mathbb{k})$ from [NRS⁺15]. The definition of $\mathcal{R}ep_n(\Lambda, \mathbb{k})$ is given in Section 2, though it is essentially contained in the more general discussion of “noncommutative augmentation categories” in [CDGG16].

If the main result of [NRS⁺15] is that the augmentation category $\mathcal{A}ug_+(\Lambda, \mathbb{k})$ is equivalent to the sheaf category $\mathcal{S}h_1(\Lambda, \mathbb{k})$, then we expect that even beyond just considering objects, the representation category $\mathcal{R}ep_n(\Lambda, \mathbb{k})$ should itself correspond on the sheaf side to the category $\mathcal{S}h_n(\Lambda, \mathbb{k})$ of constructible sheaves with microsupport on Λ and microlocal rank n . We formulate the following conjecture.

Conjecture 1.1 (“Representations are sheaves”). *Let Λ be a Legendrian link in \mathbb{R}^3 , and let \mathbb{k} be a field. Then for any $n \geq 1$, there is an A_∞ equivalence of A_∞ categories*

$$\mathcal{R}ep_n(\Lambda, \mathbb{k}) \xrightarrow{\sim} \mathcal{S}h_n(\Lambda, \mathbb{k}).$$

In particular, on the level of cohomology, Conjecture 1.1 would immediately imply the following result.

Corollary 1.2. *If Λ is a Legendrian link, then the cohomology categories $H^*\mathcal{R}ep_n(\Lambda, \mathbb{k})$ and $H^*\mathcal{S}h_n(\Lambda, \mathbb{k})$ are equivalent.*

In this paper, we will not attempt to prove Conjecture 1.1 or even Corollary 1.2 in general. However, we will prove the following special case of Corollary 1.2.

Theorem 1.3. *For $m \geq 1$, let Λ_m be the Legendrian $(2, m)$ torus link given by the rainbow closure of the braid $\sigma_1^m \in B_2$, as shown in Figure 1, equipped with its standard binary Maslov potential. Then the cohomology categories $H^*\mathcal{R}ep_n(\Lambda, \mathbb{k})$ and $H^*\mathcal{S}h_n(\Lambda, \mathbb{k})$ are equivalent.*

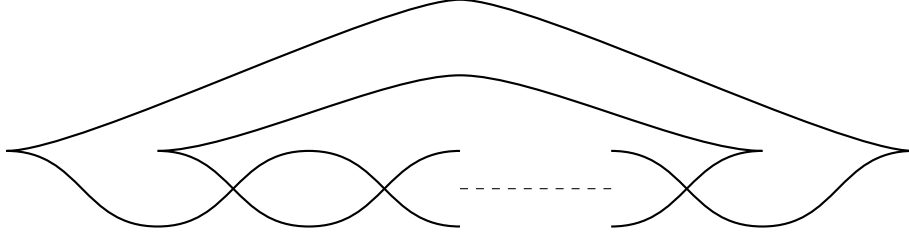


FIGURE 1. The front projection of the Legendrian link Λ_m , given as the rainbow closure of the 2-braid σ_1^m .

To prove Theorem 1.3, we will explicitly compute the representation and (cohomological) sheaf categories associated to Λ_m in a fair amount of detail, and show that they are identical. The computation on the sheaf side is likely to seem excessively detailed to a reader who is comfortable with the underlying sheaf theory. However, one of the goals of this paper is to break down the abstract sheaf calculation for Λ_m into computations that essentially reduce to elementary linear algebra, with the hope of providing a concrete entry point into the sheaf theory for contact geometers.

Remark 1.4. Ideally, we would prove that there is an A_∞ equivalence between $\mathcal{R}ep_n(\Lambda_m)$ and $\mathcal{S}h_n(\Lambda_m)$ themselves, instead of only working at the level of cohomology. However, our computation does not descend to this level because we compute Ext groups in $H^*(\mathcal{S}h_n(\Lambda_m))$ in terms of extensions rather than working directly with complexes in $\mathcal{S}h_n(\Lambda_m)$. One could hope to prove the desired equivalence by computing the appropriate Hochschild cohomology groups of these categories: if they are formal then the dg and A_∞ structures are irrelevant for these examples, and if not then the deformation theory of these categories may still be simple enough to deduce their equivalence, by the sort of strategy outlined in [Sei02].

Remark 1.5. It should be possible to define the representation category $\mathcal{R}ep_n$ for Legendrian submanifolds in arbitrary 1-jet spaces. This would follow the definition of the positive augmentation category $\mathcal{A}ug_+$ for these Legendrians, which as of now has not been completely written down, though see [EL17] for work in this direction. We would then expect Conjecture 1.1 to hold for Legendrians in all 1-jet spaces.

Note also that there is no particular reason to separate sheaves by their microlocal rank, just as there is no reason to only consider n -dimensional representations of $(\mathcal{A}_\Lambda, \partial_\Lambda)$ for a single value of n . One could consider the category $\mathcal{S}h_{<\infty}(\Lambda, \mathbb{k})$ of all sheaves with finite microlocal rank, and similarly build a category $\mathcal{R}ep_{<\infty}(\Lambda, \mathbb{k})$ of

finite-dimensional representations. The computations in the present paper should extend to this more general setting with no extra difficulties other than careful bookkeeping. However, we will stick to comparing $\mathcal{R}ep_n$ and $\mathcal{S}h_n$ for fixed n , as this is already enough to motivate Conjecture 1.1 and suggest that our construction of $\mathcal{R}ep_n$ is the correct one.

1.2. Geometric motivation. Here we discuss some geometric intuition behind Conjecture 1.1. Our discussion is intended to provide some heuristic motivation and is consequently a sketch omitting full technical details.

First note that an exact Lagrangian filling L of a Legendrian submanifold Λ induces an augmentation of the DGA for Λ . By arguments of Seidel and Ekholm [Ekh12], there is an exact triangle relating: the wrapped Floer homology of L , $HW(L)$, as defined in [AS10]; the linearized contact homology of Λ ; and the homology of L , computed as the infinitesimally wrapped Floer homology of L . Hence when $HW(L)$ vanishes, we have the so-called Seidel–Ekholm–Dimitroglou–Rizell isomorphism between linearized contact homology and $H_*(L)$. This occurs for instance when L is in the symplectization of the contact manifold, or more generally L lies in a Weinstein filling and avoids its Lagrangian skeleton, see [CDGG17, Proposition 7.6]. In [EL17, Theorem 4] it is shown that when $HW(L) = 0$ this isomorphism lifts to the level of categories, leading an equivalence between some version of the augmentation category of Λ and the infinitesimally wrapped Fukaya category.

On the sheaf side, consider the following example: let N be a submanifold of a manifold Q with smooth boundary $\partial\bar{N}$ such that $N = \{f > 0\}$ and $\partial\bar{N} = \{f = 0\}$ for a smooth function f . The boundary at infinity of the microsupport of the constant sheaf on N (i.e. $i_*\mathbb{C}_N$, where $i : N \rightarrow Q$ is inclusion) is the cosphere bundle of $\partial\bar{N}$, and its derived self hom space is the de Rham complex $\Omega^*(N)$ (see [NZ09, Lemma 4.4.1]). The graph of $d(\ln f|_N)$ is a Lagrangian filling of this cosphere bundle, and it has the topology of N , so its Floer homology is quasi-isomorphic to the hom space in the sheaf category. This is a particularly relevant example: it follows from work of Nadler–Zaslow that such sheaves generate the full category $\mathcal{S}h(Q)$, which allows them to prove that the category of sheaves corresponds to the derived infinitesimally wrapped category (see [NZ09] and [Nad09]).

Combining the two sides, we expect that the augmentation and sheaf categories for Λ should be equivalent when $HW = 0$. This last condition is why Conjecture 1.1 is stated for Λ in \mathbb{R}^3 (which might more generally be replaced by a 1-jet space), as opposed to a unit cotangent bundle ST^*Q as is more natural for the sheaf picture.

Now instead of exact Lagrangian fillings of Λ , suppose that we consider exact Lagrangian fillings L equipped with rank n local systems. On the sheaf side, this leads to the category $\mathcal{S}h_n(\Lambda, \mathbb{k})$. On the DGA side, the filling L produces a “universal” augmentation from the DGA $(\mathcal{A}_\Lambda, \partial_\Lambda)$ to the group ring $\mathbb{Z}[\pi_1(L)]$; this is the standard augmentation corresponding to L , which counts holomorphic disks with boundary on L and positive end at a Reeb chord of Λ , but now each such disk contributes the homotopy class of its boundary. Composing this with a rank n local

system on L leads to an n -dimensional representation of $(\mathcal{A}_\Lambda, \partial_\Lambda)$. Collecting these representations into a category yields the representation category $\mathcal{R}ep_n(\Lambda, \mathbb{k})$. As before, one could then guess that the representation and sheaf categories should be equivalent, and this is the content of Conjecture 1.1.

Remark 1.6. One caveat: the discussion above would produce a correspondence between derived categories, whereas Conjecture 1.1 is for the original A_∞ categories before triangulation. However, the main result of [NRS⁺15] establishes Conjecture 1.1 for $n = 1$ without passing to the derived categories, and we believe that this should continue to hold for arbitrary n . We note that the methods of [NRS⁺15] do not immediately generalize to the setting $n \geq 2$; one key difference is that the local computations in that paper rely on a classification of simple representations of a particular quiver algebra that does not extend to higher rank representations.

One should be able to define an explicit A_∞ functor from $\mathcal{R}ep$ to $\mathcal{S}h$ yielding the equivalence in Conjecture 1.1. This builds on a construction of Viterbo for Lagrangians in cotangent bundles, and a map along these lines has been suggested independently by F. Bourgeois and V. Shende. Roughly speaking, the idea is as follows: given an n -dimensional representation ρ of the DGA of a Legendrian $\Lambda \subset J^1(Q) \subset ST^*(Q \times \mathbb{R})$, for any $q_0 \in Q \times \mathbb{R}$ one can consider the linearized contact homology complex generated by Reeb chords from Λ to the Legendrian cotangent fiber over q_0 , linearized with respect to ρ . Then we map ρ to a sheaf whose stalk over q_0 is this complex, which has microsupport on Λ and microlocal rank n .

Note that this construction works for Legendrians in $ST^*(Q \times \mathbb{R})$ and not just $J^1(Q)$. However, to extend the map on objects to a map on morphisms, one again needs some vanishing condition on HW , as is guaranteed e.g. by restricting to $J^1(Q)$. Without this vanishing, as an example, the conormal in $ST^*\mathbb{R}^2$ of a point in \mathbb{R}^2 is the microsupport of the skyscraper sheaf on the point whose hom space is 1-dimensional (corresponding to the homology of the disk filling), whereas the corresponding hom space in the augmentation category is 2-dimensional (the homology of S^1); the difference comes from the fact that the cotangent fiber has nonzero wrapped homology in the sector T^*D^2 , see [GPS17].

Much of the preceding discussion is speculative both on the analytic side (for general Weinstein manifolds, thought of as the cotangent of arboreal singularities) and on the algebraic side. Note that recently the sheaf formulation of wrapped Floer homology (as suggested in [Nad16] for an approach) has been used in [GPS18, Section 6.4] to prove an equivalence between some sheaf category and modules over an algebra which is conjectured in [EL17] to be related to some version of the Chekanov–Eliashberg algebra. It is not currently fully clear to us what the precise relation is between Conjecture 1.1 and the results from [GPS18].

1.3. Organization. The remainder of the paper is organized as follows. In Section 2 we recall the relevant construction from [CDGG16], and give details of the construction of $\mathcal{R}ep_n$ and its invariance under Legendrian isotopy (following [NRS⁺15]).

Then in Section 3 we proceed to compute this category for a family of Legendrian $(2, m)$ torus links. In Section 4 we recall the construction of the categories $\mathcal{S}h(\Lambda)$ and the definition of microlocal rank. In Section 5 we compute most of the cohomology of the sheaf category for the same family of Legendrian knots, leaving the vanishing of Ext^2 and higher to Section 6. Finally, in Section 7 we gather all of these computations together and prove Theorem 1.3.

Acknowledgments. During our exploration of the sheaf category we benefited greatly from conversations with Steven Boyer, Stéphane Guillermou, Dan Rutherford, Vivek Shende, David Treumann, Nicolas Vichery, and Eric Zaslow; we also thank the referee for helpful feedback. The first author is partially supported by the ANR projects COSPIN (ANR-13-JS01-0008-01) and QUANTACT (ANR-16-CE40-0017). The second author is partially supported by NSF grants DMS-1406371 and DMS-1707652. Some of the early work for this paper stemmed from discussions at the 2016 workshop “Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4” at the Banff International Research Station.

2. THE REPRESENTATION CATEGORY OF A LEGENDRIAN LINK

In this section we describe the representation category $\mathcal{R}ep_n(\Lambda, \mathbb{k})$ associated to a Legendrian link Λ and prove some elementary properties of this A_∞ category. The construction of the representation category is due to [CDGG16], who mention it briefly in the context of a more general discussion of “noncommutative augmentation categories”. Since this particular category is of central importance to this paper, we will review its definition in a way that we hope can be read in a relatively self-contained way. However, we will assume that the reader is familiar with Chekanov–Eliashberg differential graded algebras associated to Legendrian links, about which there is now an extensive literature; a particular treatment that is especially convenient for our discussion can be found in [NRS⁺15].

Remark 2.1. The portion of [CDGG16] that is most relevant to this paper is [CDGG16, Appendix A], which discusses noncommutative augmentation categories associated to Legendrian links. There are two categories proposed there that both generalize the positive augmentation category $\mathcal{A}ug_+$ to n -dimensional representations, corresponding to two general constructions of A_∞ categories (“Case I”, “Case II”) presented in [CDGG16, §4]. Case II produces a category denoted in [CDGG16] by $\mathcal{R}ep_+(\Lambda, n)$, and this uses the fact that $\text{Mat}_n(\mathbb{k})$ is a Hermitian algebra; however, this category is not unital (see [CDGG16, Remark A.3]) and we do not consider it here. Instead we use the Case I category denoted in [CDGG16] by $\mathcal{R}ep_+(\Lambda, S)$, in the special case where $S = \text{Mat}_n(\mathbb{k})$. The treatment of $\mathcal{R}ep_+(\Lambda, S)$ in [CDGG16] is rather cursory and we will expand on its definition below.

2.1. Twisting by noncommutative augmentations. Before defining the representation category, we review a general technique given in [CDGG16] for producing an A_∞ algebra by dualizing a Chekanov–Eliashberg differential graded algebra

(or, more generally, a semi-free DGA in the sense of [NRS⁺15]) equipped with an augmentation. When the augmentation takes values in a field \mathbb{k} , this story is well-studied; in the context of Legendrian knots, see e.g. [CKE⁺11, BC14, NRS⁺15]. In [CDGG16, section 4.1], this dualizing procedure is generalized to the case when the augmentation takes values in a possibly noncommutative algebra over \mathbb{k} , and this is the version that we review here.

Let \mathbb{k} be a field, and $\{c_1, \dots, c_p\}$ a finite set. For an algebra A over \mathbb{k} (which for us will typically be $\text{Mat}_n(\mathbb{k})$), we will denote by M_A the free A - A -bimodule with generating set $\{c_1, \dots, c_p\}$. We assume that M_A is equipped with a grading coming from assigning degrees to each of the c_i 's and that elements of the coefficient ring have degree 0. We will consider the following tensor algebra:

$$\mathcal{A}_A = \mathcal{T}_A(M_A) = \bigoplus_{n=0}^{\infty} M_A^{\boxtimes n},$$

where $\boxtimes = \otimes_A$ (with $M_A^{\boxtimes 0} = A$) and multiplication in \mathcal{A}_A is given by concatenation.

The grading on M_A induces a grading on \mathcal{A}_A . A differential ∂ on \mathcal{A}_A is a degree -1 map satisfying the (signed) Leibniz rule $\partial(xy) = (\partial x)y + (-1)^{|x|}x(\partial y)$ along with $\partial(\alpha) = 0$ for all $\alpha \in A$. This differential is called *augmented* if all constant terms in ∂ vanish: more precisely, ∂ maps \mathcal{A}_A to $\bigoplus_{n=1}^{\infty} M_A^{\boxtimes n} \subset \mathcal{A}_A$.

If \mathcal{A}_A is equipped with an augmented differential ∂ , we can dualize this differential graded algebra to an A_{∞} algebra. Let M^{\vee} denote the dual of M_A , $M^{\vee} = \text{Hom}_{A-A}(M, A)$. An element of M^{\vee} is uniquely determined by where it sends c_1, \dots, c_p in A , and so we may identify M^{\vee} with the free A -module generated by $c_1^{\vee}, \dots, c_p^{\vee}$, where if $\phi \in M^{\vee}$ satisfies $\phi(c_i) = a_i$ then ϕ is identified with $\sum a_i c_i^{\vee} \in M^{\vee}$. We assign each generator the grading $|c_i^{\vee}| = |c_i| + 1$, where $|c_i|$ is the grading of c_i as an element of \mathcal{A} .

Given an augmented differential $\partial : \mathcal{A}_A \rightarrow \mathcal{A}_A$, we can dualize the k -th order part of ∂ to obtain a \mathbb{k} -multilinear map

$$\mu_k : (M^{\vee})^{\otimes k} \rightarrow M^{\vee},$$

where the tensor product is over \mathbb{k} and not A .

A k -th order term in $\partial(c_i)$,

$$\partial(c_i) = \alpha_k c_{i_k} \alpha_{k-1} c_{i_{k-1}} \cdots \alpha_1 c_{i_1} \alpha_0 + \cdots$$

with $\alpha_j \in A$, contributes a term to μ_k :

$$\mu_k(a_{i_1} c_{i_1}^{\vee}, \dots, a_{i_k} c_{i_k}^{\vee}) = (-1)^{\sigma} (\alpha_k a_{i_k} \alpha_{k-1} a_{i_{k-1}} \cdots \alpha_1 a_{i_1} \alpha_0) c_i^{\vee} + \cdots,$$

where the sign is determined by

$$(1) \quad \sigma = \frac{k(k-1)}{2} + \left(\sum_{p < q} |c_{i_p}^{\vee}| |c_{i_q}^{\vee}| \right) + |c_{i_2}^{\vee}| + |c_{i_4}^{\vee}| + \cdots.$$

Since $\partial^2 = 0$, the μ_k operations satisfy the A_∞ relations; here we use the A_∞ sign conventions of [NRS⁺15, §2.3] (in particular, the Koszul sign rule) rather than those of [CDGG16], and hence the signs are taken from [NRS⁺15, §3.1].

Proposition 2.2. $(M^\vee, \{\mu_k\})$ is an A_∞ algebra over \mathbb{k} .

Proof. This follows exactly as in [CDGG16, Theorem 4.11], except that the signs have been determined using [NRS⁺15, §3.1]. \square

The remainder of this subsection is devoted to a description of how to construct an augmented differential ∂ on \mathcal{A}_A (and thus an A_∞ algebra) from a general differential graded algebra equipped with an augmentation. (When $A = \mathbb{k}$, this story is well-known, cf. [Che02]). To this end, let R be a second \mathbb{k} -algebra, producing a tensor algebra \mathcal{A}_R in the same way as the definition of \mathcal{A}_A , and suppose that we have a differential ∂ on \mathcal{A}_R satisfying the Leibniz rule and sending $R \subset \mathcal{A}_R$ to 0. We will consider augmentations to the first \mathbb{k} -algebra A :

Definition 2.3. An *augmentation* of $(\mathcal{A}_R, \partial)$ to A is a DGA map

$$\epsilon : (\mathcal{A}_R, \partial) \rightarrow (A, 0),$$

i.e., a graded \mathbb{k} -algebra map $\epsilon : \mathcal{A}_R \rightarrow A$ such that $\epsilon \circ \partial = 0$.

Remark 2.4. To be specific, the case that will be of interest to us is when $(\mathcal{A}_R, \partial)$ is the Chekanov–Eliashberg DGA of a Legendrian link Λ in \mathbb{R}^3 (more precisely, the fully noncommutative DGA with multiple base points, tensored with \mathbb{k}). In this case the generating set $\{c_1, \dots, c_p\}$ is the set of Reeb chords of the link, and the ring R is the group ring $\mathbb{k}[F_q]$ where F_q is the free group with q generators. In other words, \mathcal{A}_R is generated as a \mathbb{k} -algebra by the Reeb chords c_1, \dots, c_p of Λ along with elements $t_1^{\pm 1}, \dots, t_q^{\pm 1}$, where t_i and t_j do not commute for $i \neq j$. As usual for Legendrian contact homology, the grading on \mathcal{A}_R comes from giving Reeb chords degrees corresponding to their Conley–Zehnder indices, and the differential $\partial : \mathcal{A}_R \rightarrow \mathcal{A}_R$ counts holomorphic disks with boundary on $\mathbb{R} \times \Lambda$.

Furthermore, we will be primarily interested in the case $A = \text{Mat}_n(\mathbb{k})$; then we will call an augmentation of $(\mathcal{A}_R, \partial)$ to A a *representation* of \mathcal{A}_R , and we will usually use ρ instead of ϵ to denote the representation. See Definition 2.6 below.

Now suppose that we have a differential graded algebra $(\mathcal{A}_R, \partial)$ equipped with an augmentation ϵ to A . Then ∂ induces a differential ∂_A on \mathcal{A}_A by replacing all occurrences of $\gamma \in R$ in ∂ by $\epsilon(\gamma) \in A$, and ϵ then induces an augmentation of $(\mathcal{A}_A, \partial_A)$ to A . One can then twist ∂_A by this augmentation to produce an augmented differential on \mathcal{A}_A .

We now work out this scheme in more detail. Define a \mathbb{k} -algebra map $\phi_\epsilon : \mathcal{A}_R \rightarrow \mathcal{A}_A$ by its action on generators:

$$\begin{aligned} \phi_\epsilon(c_i) &= c_i + \epsilon(c_i) & i &= 1, \dots, p \\ \phi_\epsilon(\gamma) &= \epsilon(\gamma) & \gamma &\in R \subset \mathcal{A}_R. \end{aligned}$$

We can then define a map $\partial_\epsilon : \mathcal{A}_A \rightarrow \mathcal{A}_A$ by

$$\begin{aligned} \partial_\epsilon(c_i) &= \phi_\epsilon(\partial c_i) & i = 1, \dots, p \\ \partial_\epsilon(\alpha) &= 0 & \alpha \in A, \end{aligned}$$

where ∂_ϵ is extended to all of \mathcal{A}_A by the Leibniz rule. We now have the following proposition, which by the preceding discussion allows us to produce an A_∞ algebra from $(\mathcal{A}_A, \partial_\epsilon)$.

Proposition 2.5. *The map ∂_ϵ is an augmented differential on \mathcal{A}_A .*

Proof. First note that $\partial_\epsilon \circ \phi_\epsilon = \phi_\epsilon \circ \partial$: the two sides agree on generators of \mathcal{A}_A by construction, and both sides behave the same way under products. It follows that $\partial_\epsilon^2(c_i) = \partial_\epsilon^2(\phi_\epsilon(c_i)) = \phi_\epsilon \partial^2(c_i) = 0$, and so $\partial_\epsilon^2 = 0$ on all of \mathcal{A}_A since ∂_ϵ satisfies the Leibniz rule. Furthermore, the constant term of $\partial_\epsilon c_i$ is by construction $\epsilon(\partial c_i)$, which is 0 since ϵ is an augmentation; thus ∂_ϵ is augmented. \square

In what follows, we will need to compare the A_∞ structures arising from equivalent (“stable tame”) DGAs. To do this, we observe that the above construction of an augmented differential ∂_ϵ from an augmentation ϵ is natural, in the following sense. Suppose that we have a DGA map $f : (\mathcal{A}_R, \partial) \rightarrow (\mathcal{A}'_R, \partial)$, where $\mathcal{A}_R, \mathcal{A}'_R$ are both tensor algebras but with possibly different generating sets $\{c_1, \dots, c_p\}, \{c'_1, \dots, c'_p\}$; in particular, f restricts to the identity map on R . Further suppose that we have an augmentation ϵ' of $(\mathcal{A}'_R, \partial)$ to A ; this induces an augmentation $\epsilon = \epsilon' \circ f$ of $(\mathcal{A}_R, \partial)$ to A . We construct maps $\phi_\epsilon : \mathcal{A}_R \rightarrow \mathcal{A}_A$, $\phi_{\epsilon'} : \mathcal{A}'_R \rightarrow \mathcal{A}'_A$, $\partial_\epsilon : \mathcal{A}_A \rightarrow \mathcal{A}_A$, $\partial_{\epsilon'} : \mathcal{A}'_A \rightarrow \mathcal{A}'_A$ as above, where $\mathcal{A}_A, \mathcal{A}'_A$ have the same generating sets as $\mathcal{A}_R, \mathcal{A}'_R$ respectively.

Define a \mathbb{k} -algebra morphism $f_A : \mathcal{A}_A \rightarrow \mathcal{A}'_A$ by

$$\begin{aligned} f_A(c_i) &= \phi_{\epsilon'} f(c_i) - \epsilon(c_i) & i = 1, \dots, p \\ f_A(\alpha) &= \alpha & \alpha \in A. \end{aligned}$$

We claim that f_A intertwines the differentials ∂_ϵ and $\partial_{\epsilon'}$. Indeed, note that we have $f_A \circ \phi_\epsilon = \phi_{\epsilon'} \circ f$ since these two compositions agree on generators of \mathcal{A}_R :

$$f_A \phi_\epsilon(c_i) = f_A(c_i + \epsilon(c_i)) = \phi_{\epsilon'} f(c_i) - \epsilon(c_i) + \epsilon(c_i) = \phi_{\epsilon'} f(c_i)$$

and $f_A \phi_\epsilon(\gamma) = \epsilon(\gamma) = \phi_{\epsilon'}(\gamma) = \phi_{\epsilon'} f(\gamma)$ for $\gamma \in R$. As in the proof of Proposition 2.5, we have $\partial_{\epsilon'} \circ \phi_{\epsilon'} = \phi_{\epsilon'} \circ \partial$, whence

$$f_A \partial_\epsilon(c_i) = f_A \phi_\epsilon \partial(c_i) = \phi_{\epsilon'} f \partial(c_i) = \phi_{\epsilon'} \partial f(c_i) = \partial_{\epsilon'} \phi_{\epsilon'} f(c_i) = \partial_{\epsilon'} f_A(c_i),$$

and we conclude that $f_A \circ \partial_\epsilon = \partial_{\epsilon'} \circ f_A$.

Now the constant term in $f_A(c_i)$ is $\epsilon' f(c_i) - \epsilon(c_i) = 0$, and so the map f_A has no constant terms: that is, the component of $f_A|_{M_A}$ mapping to $(M'_A)^{\boxtimes 0}$ is 0. Thus we can write $f_A|_M = f_1 \oplus f_2 \oplus \dots \oplus f_k \oplus \dots$ where each f_k has image in $(M'_A)^{\boxtimes k}$. Dualizing all the f_i as in the construction of the μ_k , we get a family of maps $\{F_k\}_{k \in \mathbb{N}}$ which forms an A_∞ algebra morphism $((M')^\vee, \{\mu'_k\}) \rightarrow (M^\vee, \{\mu_k\})$.

The whole construction is functorial, so that if f is an isomorphism then it induces an isomorphism of A_∞ algebras.

2.2. The representation category. Following [BC14, CDGG16], one can expand the construction from Section 2.1 to construct an A_∞ category whose objects are augmentations $\epsilon : \mathcal{A} \rightarrow A$, such that the self-homs of ϵ form the A_∞ algebra given above. In the case where $A = \text{Mat}_n(\mathbb{k})$, this produces an A_∞ category that we might term the “negative representation category”, following the terminology of [NRS⁺15]. We will instead be interested in a “positive” version of the representation category, which requires a bit more work to set up and which we will now describe. Throughout this subsection, we assume that $A = \text{Mat}_n(\mathbb{k})$, though the discussion works for more general algebras as well. In the case $n = 1$, the representation category $\mathcal{R}ep_1(\Lambda, \mathbb{k})$ will precisely be the positive augmentation category $\mathcal{A}ug_+(\Lambda, \mathbb{k})$ from [NRS⁺15].

Definition 2.6. Let (\mathcal{A}, ∂) be the Chekanov–Eliashberg DGA of a Legendrian link Λ equipped with base points. An n -dimensional *representation* of (\mathcal{A}, ∂) is an augmentation $\rho : \mathcal{A} \rightarrow \text{Mat}_n(\mathbb{k})$, meaning a homomorphism of graded \mathbb{k} -algebras (here $\text{Mat}_n(\mathbb{k})$ lives entirely in grading 0) with $\rho \circ \partial = 0$.

Remark 2.7. One can more generally consider DGA maps from (\mathcal{A}, ∂) to $(\text{End } V, \delta)$, where V is a graded \mathbb{k} -vector space and δ is induced from a differential on V . The representations that we consider in this paper correspond to $\delta = 0$ and $V = \mathbb{k}^n$ supported in degree 0. The general DGA maps are of interest when studying augmentations of satellites [LR18]; one can similarly construct a category out of these, but to simplify the exposition we omit the details here.

Given a Legendrian link Λ with base points $*_1, \dots, *_q$, we can now define the representation category $\mathcal{R}ep_n(\Lambda, \mathbb{k})$, fleshing out the definition in [CDGG16]. Let c_1, \dots, c_p be the Reeb chords of Λ , and let (\mathcal{A}, ∂) be the Chekanov–Eliashberg DGA of Λ . Note again that \mathcal{A} has invertible generators $t_1^{\pm 1}, \dots, t_q^{\pm 1}$ for each base point which do not commute with the c_i or with each other, except that $t_i t_i^{-1} = t_i^{-1} t_i = 1$.

Definition 2.8. The representation category $\mathcal{R}ep_n(\Lambda, \mathbb{k})$ is the A_∞ category over \mathbb{k} defined as follows.

- (1) The objects of $\mathcal{R}ep_n(\Lambda, \mathbb{k})$ are n -dimensional representations of (\mathcal{A}, ∂) .
- (2) Given two objects ρ_1, ρ_2 , the morphism space $\text{Hom}(\rho_1, \rho_2)$ is the free graded $\text{Mat}_n(\mathbb{k})$ -module generated by $c_1^\vee, \dots, c_p^\vee, x_1^\vee, y_1^\vee, \dots, x_q^\vee, y_q^\vee$, the formal duals to the Reeb chords of Λ along with $2q$ additional generators. These have gradings $|c_i^\vee| = |c_i| + 1$, $|x_j^\vee| = 1$, and $|y_j^\vee| = 0$.
- (3) We have \mathbb{k} -multilinear maps

$$\mu_k : \text{Hom}(\rho_{k-1}, \rho_k) \otimes \cdots \otimes \text{Hom}(\rho_1, \rho_2) \otimes \text{Hom}(\rho_0, \rho_1) \rightarrow \text{Hom}(\rho_0, \rho_k)$$

defined by dualizing the augmented differential of the $(k + 1)$ -copy of Λ with respect to the pure augmentation $\epsilon = (\rho_0, \dots, \rho_k)$, as in the previous subsection.

To be more precise, we construct μ_k by using the *Lagrangian projection* $(k+1)$ -copy Λ_f^{k+1} of Λ as described in [NRS⁺15, §4.2.2]. Here we choose q base points $*_1, \dots, *_q$ on Λ with at least one base point on each component of Λ , and then choose a Morse function f on Λ with $2q$ critical points consisting of 1 minimum and 1 maximum in a small neighborhood of each $*_j$. We then use f to produce $k+1$ parallel copies of Λ by perturbing Λ in the normal direction according to f .

Each x_j^\vee and y_j^\vee correspond to crossings between different components of the $(k+1)$ -copy near $*_j$ at the maximum and minimum, respectively. Labeling the copies $\Lambda_1, \dots, \Lambda_{k+1}$ from top to bottom and letting z be any of c_i, x_j, y_j , we let $z^{i,j}$ denote the corresponding Reeb chord of Λ_f^{k+1} from Λ_j to Λ_i . Each Λ_i has base points $*_1^i, \dots, *_q^i$.

The DGA of Λ_f^{k+1} is described explicitly by [NRS⁺15, Proposition 4.14], which we restate below as Proposition 3.3, and the pure augmentation ϵ satisfies $\epsilon(t_j^i) = \rho_i(t_j)$, $\epsilon(c_i^{i,j}) = \rho_i(c_i)$ if $i = j$, and $\epsilon(z) = 0$ for all other generators. Twisting the differential of this DGA by the pure augmentation ϵ , each k -th order term of the form

$$\partial^\epsilon(w^{1,k+1}) = \alpha_k z_k^{1,2} \alpha_{k-1} z_{k-1}^{2,3} \cdots \alpha_1 z_1^{k,k+1} \alpha_0 + \dots$$

then contributes

$$\mu_k(a_1 z_1^\vee, \dots, a_k z_k^\vee) = (-1)^\sigma (\alpha_k a_k \alpha_{k-1} a_{k-1} \cdots \alpha_1 a_1 \alpha_0) w^\vee + \dots$$

with the sign $(-1)^\sigma$ as in equation (1).

Proposition 2.9. *($\mathcal{R}ep_n(\Lambda, \mathbb{k}), \{\mu_k\}$) is a unital A_∞ category over \mathbb{k} .*

Proof. This category is defined as $\mathcal{R}ep_+(\Lambda, \text{Mat}_n(\mathbb{k}))$ in [CDGG16, §A.2.2] (but with sign conventions as in [NRS⁺15]). There it is explained how the construction of a consistent sequence of DGAs from [NRS⁺15] adapts to the case where the objects are representations instead of just augmentations; then we can pass from this sequence to $\mathcal{R}ep_n(\Lambda, \mathbb{k})$ by [NRS⁺15, Definition 3.16], and it is an A_∞ category by [NRS⁺15, Proposition 3.17]. As observed in [CDGG16, §A.2.2], the existence of a unit $e_\rho = -\sum_{j=1}^q y_j^\vee$ in each $\text{Hom}(\rho, \rho)$ essentially follows from [NRS⁺15, Proposition 3.28]; this unit is closed as a consequence of [EES09, Theorem 5.5]. \square

Proposition 2.10. *Fix n and \mathbb{k} . Up to A_∞ equivalence, the A_∞ category $\mathcal{R}ep_n(\Lambda, \mathbb{k})$ does not depend on the choice of Morse function f . Moreover, if Λ and Λ' are Legendrian isotopic, then $\mathcal{R}ep_n(\Lambda, \mathbb{k})$ and $\mathcal{R}ep_n(\Lambda', \mathbb{k})$ are A_∞ equivalent.*

Proof. This follows exactly as in [NRS⁺15, Theorem 4.20]; we outline the steps here.

First, following [NRS⁺15, Proposition 4.21], we suppose that Λ and Λ' are related by a Legendrian isotopy carrying the Morse function f on Λ to f' on Λ' , and that f (and hence f') has exactly one local maximum on each component. Then the Legendrian isotopy relating these links induces a stable tame isomorphism of DGAs, which induces a consistent family of DGA morphisms (in the sense of [NRS⁺15,

Definition 3.19)) as shown in [NRS⁺15, Proposition 3.29]. Using the construction of A_∞ morphisms at the end of Section 2.1, we produce an A_∞ equivalence from this consistent family by following the proofs of [NRS⁺15, Propositions 3.29 and 4.21] verbatim: tame isomorphisms of DGAs are easily seen to give A_∞ equivalences, and then stabilization is treated by considering the functor associated to the inclusion $i : \mathcal{A} \rightarrow S(\mathcal{A})$ and doing the same verification by hand as in [BC14, Theorem 2.14] and [NRS⁺15, Proposition 4.21].

Next, we repeat [NRS⁺15, Proposition 4.22] to argue that for fixed Λ , the category $\mathcal{R}ep_n(\Lambda, \mathbb{k})$ is independent of the choice of f . More precisely, each of two operations induces a consistent sequence of DGA morphisms, and hence an A_∞ functor which is readily shown to be an equivalence. These operations are replacing a single base point with multiple base points along a small arc of Λ , and pushing a base point through a crossing of Λ . A small amount of extra bookkeeping is needed in this setting because the coefficients do not commute in general, unlike in the case of $\mathcal{A}ug_+(\Lambda, \mathbb{k})$, but the adaptation is straightforward. \square

We conclude this section with a discussion of isomorphism in the representation category. Here isomorphism of objects is defined as usual: ρ_1, ρ_2 are isomorphic if they are isomorphic in the cohomology category $H^*\mathcal{R}ep_n(\Lambda, \mathbb{k})$, in the sense that there are closed morphisms between ρ_1 and ρ_2 that compose to give the identity in cohomology.

Proposition 2.11. *Let $\rho_1, \rho_2 : \mathcal{A} \rightarrow \text{Mat}_n(\mathbb{k})$ be two representations of the Chekanov–Eliashberg DGA (\mathcal{A}, ∂) . If there is an invertible $M \in \text{Mat}_n(\mathbb{k})$ such that $\rho_2 = M^{-1}\rho_1M$, then ρ_1 is isomorphic to ρ_2 .*

Conversely, suppose that Λ has a single base point and no Reeb chords in degree -1 or -2 . If ρ_1 is isomorphic to ρ_2 , then there is an invertible M such that $\rho_2 = M^{-1}\rho_1M$.

Remark 2.12. In the case $n = 1$ of Proposition 2.11, we see that if Λ has one base point and no Reeb chords in degree -1 or -2 , then no two distinct augmentations are isomorphic in $\mathcal{A}ug_+(\Lambda, \mathbb{k})$, cf. [NRS⁺15, Proposition 5.17].

In the proof of Proposition 2.11 and in subsequent sections we will make use of a certain *link grading* $r \times c$ from [NRS⁺15, Definition 4.12], which we recall here.

Definition 2.13. Let Λ be an oriented, pointed Legendrian link with base points $*_1, \dots, *_q$, at least one of which lies on each component of Λ . We write U_i for the component of $\Lambda \setminus \{*_1, \dots, *_q\}$ with initial endpoint $*_i$. Then for each Reeb chord generator c of the DGA (\mathcal{A}, ∂) , we define $r(c)$ and $c(c)$ to be the unique integers for which the upper and lower endpoints of c lie on $U_{r(c)}$ and $U_{c(c)}$ respectively. We also define $r(t_i)$ and $c(t_i)$ so that the base point $*_i$ is preceded by $U_{r(t_i)}$ and followed by $U_{c(t_i)}$. (By definition $c(t_i) = i$.)

Proof of Proposition 2.11. In order to compute the differential μ_1 on each $\text{Hom}(\rho_i, \rho_j)$, we read off the DGA of Λ_f^2 from [NRS⁺15, Proposition 4.14] (restated for convenience as Proposition 3.3 below). Letting c_1, \dots, c_p denote the Reeb chords of Λ_i and (r, c) the link grading, in $\text{Hom}(\rho_1, \rho_2)$ we have

$$\begin{aligned} \mu_1(u_i y_i^\vee) &= \sum_{r(c_k)=i} u_i \rho_2(c_k) c_k^\vee - \sum_{c(c_k)=i} \rho_1(c_k) u_i c_k^\vee \\ &\quad + \sum_{r(t_k)=i} \rho_1(t_k)^{-1} u_i \rho_2(t_k) x_k^\vee - \sum_{c(t_k)=i} u_i x_k^\vee. \end{aligned}$$

(Here we have eliminated a sign $(-1)^{|a_k|}$ in front of $\rho_1(c_k) u_i c_k^\vee$ by observing that $\rho_1(c_k) = 0$ if $|c_k| \neq 0$.) Summing over all i , we have

$$\begin{aligned} \mu_1 \left(\sum_{i=1}^q u_i y_i^\vee \right) &= \sum_{k=1}^p (u_{r(c_k)} \rho_2(c_k) - \rho_1(c_k) u_{c(c_k)}) c_k^\vee \\ &\quad + \sum_{k=1}^q (\rho_1(t_k)^{-1} u_{r(t_k)} \rho_2(t_k) - u_{c(t_k)}) x_k^\vee, \end{aligned}$$

so $f = \sum_i u_i y_i^\vee \in \text{Hom}(\rho_1, \rho_2)$ is a degree-0 cycle if and only if $u_{r(z)} \rho_2(z) = \rho_1(z) u_{c(z)}$ for all $z = c_1, \dots, c_p, t_1, \dots, t_q$. Similarly, $g = \sum_i u'_i y_i^\vee \in \text{Hom}(\rho_2, \rho_1)$ is a degree-0 cycle if and only if $u'_{r(z)} \rho_1(z) = \rho_2(z) u'_{c(z)}$ for all z .

For the composition μ_2 , we examine the DGA of Λ_f^3 : the only terms with degree 2 in the y_k variables have the form $\partial y_k^{13} = y_k^{12} y_k^{23}$, from which we have $\mu_2(u y_k^\vee, u' y_k^\vee) = 0$ if $i \neq j$ and $\mu_2(u y_k^\vee, u' y_k^\vee) = -u' u y_k^\vee$. Thus any two elements $f \in \text{Hom}^0(\rho_1, \rho_2)$ and $g \in \text{Hom}^0(\rho_2, \rho_1)$ of the form $f = \sum u_i y_i^\vee$ and $g = \sum u'_i y_i^\vee$ have compositions

$$\begin{aligned} \mu_2(g, f) &= - \sum_{i=1}^q u_i u'_i y_i^\vee \in \text{Hom}^0(\rho_1, \rho_1), \\ \mu_2(f, g) &= - \sum_{i=1}^q u'_i u_i y_i^\vee \in \text{Hom}^0(\rho_2, \rho_2), \end{aligned}$$

which are equal to the units $e_{\rho_j} = - \sum_{i=1}^q y_i^\vee$ if and only if $u'_i = u_i^{-1}$ for all $i = 1, \dots, q$.

Putting these facts together, we see that the pair (f, g) as above gives an isomorphism $\rho_1 \cong \rho_2$ if and only if there are invertible matrices u_1, \dots, u_q such that

$$(2) \quad \rho_2(z) = u_{r(z)}^{-1} \rho_1(z) u_{c(z)}$$

for all generators z of (\mathcal{A}, ∂) . If ρ_1 and ρ_2 are conjugate, with $\rho_2 = M^{-1} \rho_1 M$, then we take $u_i = M$ for all i to see that indeed $\rho_1 \cong \rho_2$.

For the converse, we suppose that Λ has no Reeb chords in degree -1 or -2 and that the cycles $f \in \text{Hom}^0(\rho_1, \rho_2)$ and $g \in \text{Hom}^0(\rho_2, \rho_1)$ satisfy $[\mu_2(g, f)] = [e_{\rho_1}]$ and $[\mu_2(f, g)] = [e_{\rho_2}]$ in cohomology. By assumption, each $\text{Hom}^0(\rho_i, \rho_j)$ is spanned by

the y_i^\vee , so we must have $f = \sum u_i y_i^\vee$ and $g = \sum u'_i y_i^\vee$ for some matrices u_i and u'_i . Moreover, we have $\text{Hom}^{-1}(\rho_j, \rho_j) = 0$ for all j , so any cycle in $\text{Hom}^0(\rho_j, \rho_j)$ which is homologous to e_{ρ_j} must in fact be equal to it. In particular, we have $\mu_2(g, f) = e_{\rho_1}$ and $\mu_2(f, g) = e_{\rho_2}$, and it follows exactly as above that equation (2) holds. The assumption that Λ has exactly one base point implies that the link grading collapses to $r = c = 1$; setting $M = u_1$, we conclude that $\rho_2 = M^{-1}\rho_1M$ as claimed. \square

When Λ has multiple base points $*_1, \dots, *_q$, we can think of each representation $\rho : \mathcal{A} \rightarrow \text{Mat}_n(\mathbb{k})$ as an action of (\mathcal{A}, ∂) on $V^{\oplus q} = V_1 \oplus \dots \oplus V_q$, where $V = \mathbb{k}^n$ and each $\rho(z)$ is an element of $\text{Hom}(V_{c(z)}, V_{r(z)})$. In other words, for each generator z of \mathcal{A} the endomorphism $\rho(z)$ is a $qn \times qn$ matrix, viewed as a $q \times q$ matrix of $n \times n$ blocks whose only nonzero block is in position $(r(z), c(z))$. For more on this perspective, see [NRS⁺15, §3.2].

In this case, if Λ has no Reeb chords in degree -1 or -2 , equation (2) says that two representations ρ_1, ρ_2 are isomorphic in $H^*\mathcal{R}ep_n(\Lambda, \mathbb{k})$ if and only if they are identified by some simultaneous change of bases for V_1, \dots, V_q , or equivalently iff they are conjugate by some invertible $qn \times qn$ matrix such that all off-diagonal $n \times n$ blocks are zero. When Λ is a knot with several base points, this seems like a much weaker notion of equivalence than in Proposition 2.11, but this is explained by the fact that the additional base points introduce extra $t_i^{\pm 1}$ generators into the DGA and hence there are generally many more representations.

3. THE REPRESENTATION CATEGORY OF LEGENDRIAN $(2, m)$ TORUS LINKS

In this section, we will use Λ_m to denote the Legendrian $(2, m)$ torus link in \mathbb{R}^3 whose Lagrangian (xy) projection is shown in Figure 2. This is Legendrian isotopic to the Λ_m shown in Figure 1 in the Introduction, with the advantage that the Chekanov–Eliashberg DGA for this version of Λ_m is rather simple. For the purposes of computing this DGA, place base points $*_1, *_2$ along the loops to the right of b_1, b_2 (note that if m is odd, this is one more base point than necessary).

Let a_1, a_2, \dots denote a collection of noncommuting variables. Define two sequences of polynomials $\{P_m\}_{m=0}^\infty$ and $\{Q_m\}_{m=0}^\infty$ as follows:

$$\begin{aligned} P_0 &= 1 \\ P_1(a_1) &= a_1 \\ P_m(a_1, \dots, a_m) &= P_{m-1}(a_1, \dots, a_{m-1})a_m + P_{m-2}(a_1, \dots, a_{m-2}) \\ Q_0 &= 1 \\ Q_1(a_1) &= -a_1 \\ Q_m(a_1, \dots, a_m) &= -Q_{m-1}(a_2, \dots, a_m)a_1 + Q_{m-2}(a_3, \dots, a_m). \end{aligned}$$

Thus $P_2(a_1, a_2) = 1 + a_1a_2$, $Q_2(a_1, a_2) = 1 + a_2a_1$, $P_3(a_1, a_2, a_3) = a_1 + a_3 + a_1a_2a_3$, $Q_3(a_1, a_2, a_3) = -a_1 - a_3 - a_3a_2a_1$, and so forth.

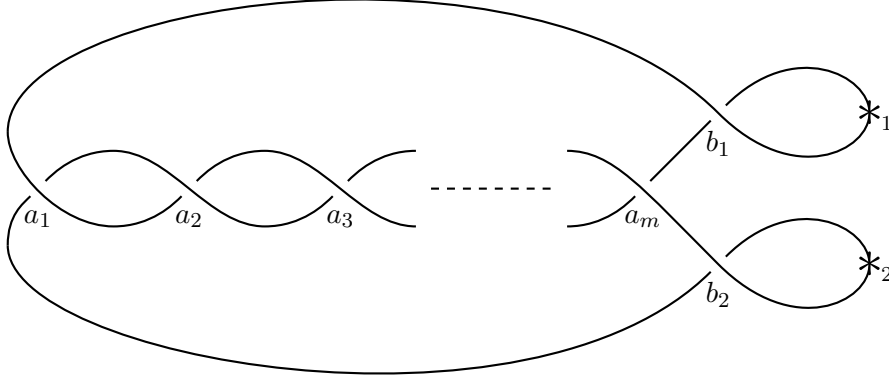


FIGURE 2. A Lagrangian projection of the Legendrian $(2, m)$ torus link Λ_m .

The Chekanov–Eliashberg DGA $(\mathcal{A}(\Lambda_m), \partial)$ for Λ_m is then generated by b_1, b_2 in degree 1 and $a_1, \dots, a_m, t_1^{\pm 1}, t_2^{\pm 1}$ in degree 0, with differential given by:

$$\begin{aligned}\partial(b_1) &= t_1^{-1} + P_m(a_1, \dots, a_m) \\ \partial(b_2) &= t_2 + Q_m(a_1, \dots, a_m)\end{aligned}$$

and the differential of all degree 0 generators is 0.

Remark 3.1. If m is even and Λ_m is a 2-component link, the grading on $\mathcal{A}(\Lambda_m)$ is not well-defined, but rather depends on a choice of Maslov potential on Λ_m : the general grading is $|b_1| = |b_2| = 1$ and $|a_i| = (-1)^i \ell$ for some fixed $\ell \in \mathbb{Z}$. In this case, we choose a Maslov potential such that $\ell = 0$. This corresponds to a choice of Maslov potential on the front projection from Figure 1 that is binary, cf. the sheaf computation in Section 5.1.

An object of the representation category $\mathcal{R}ep_n(\Lambda_m, \mathbb{k})$ is determined by where it sends the degree 0 generators of $(\mathcal{A}(\Lambda_m), \partial)$: that is, it is given by a choice of matrices $A_1, \dots, A_m, T_1, T_2 \in \text{Mat}_n(\mathbb{k})$ such that

$$\begin{aligned}P_m(A_1, \dots, A_m) &= -T_1^{-1} \\ Q_m(A_1, \dots, A_m) &= -T_2.\end{aligned}$$

We now have the following result, generalizing Sylvester’s determinant identity (which is the $m = 2$ case).

Proposition 3.2. *Let $A_1, \dots, A_m \in \text{Mat}_n(\mathbb{k})$. Then*

$$\det(P_m(A_1, \dots, A_m)) = (-1)^{mn} \det(Q_m(A_1, \dots, A_m)).$$

Proof. By the recurrence relation for P_m , we have the following matrix identities, where the matrices are presented in block form with each entry being an $n \times n$ matrix:

(3)

$$(P_{m-1}(A_1, \dots, A_{m-1}) \quad P_m(A_1, \dots, A_m)) = \begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & A_m \end{pmatrix}$$

(4)
$$(Q_m(A_1, \dots, A_m) \quad Q_{m-1}(A_2, \dots, A_m)) = \begin{pmatrix} -A_m & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -A_1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Multiplying (4) on the right by $\begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & A_m \end{pmatrix}$ and combining with (3) yields:

$$\begin{aligned} & \begin{pmatrix} 1 & 0 \\ P_{m-1}(A_1, \dots, A_{m-1}) & P_m(A_1, \dots, A_m) \end{pmatrix} \\ &= \begin{pmatrix} Q_m(A_1, \dots, A_m) & Q_{m-1}(A_2, \dots, A_m) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & A_m \end{pmatrix}. \end{aligned}$$

Now take the determinant of both sides. \square

By Proposition 3.2, the objects of $\mathcal{R}ep_n(\Lambda_m, \mathbb{k})$ are in correspondence with

(5)
$$\{(A_1, \dots, A_m) \in \text{Mat}_n(\mathbb{k}) \mid P_m(A_1, \dots, A_m) \text{ is invertible}\};$$

then T_1 and T_2 are determined by $T_1 = -P_m(A_1, \dots, A_m)^{-1}$ and $T_2 = -Q_m(A_1, \dots, A_m)$.

We now turn to the morphisms and A_∞ operations in $\mathcal{R}ep_n(\Lambda_m, \mathbb{k})$. Let $\rho = (A_1, \dots, A_m)$ and $\rho' = (A'_1, \dots, A'_m)$ be two objects in $\mathcal{R}ep_n(\Lambda_m, \mathbb{k})$. The graded hom spaces between ρ and ρ' are the free $\text{Mat}_n(\mathbb{k})$ -modules with generators given as follows:

$$\begin{aligned} \text{Hom}^2(\rho, \rho') &= \langle b_1^\vee, b_2^\vee \rangle \\ \text{Hom}^1(\rho, \rho') &= \langle a_1^\vee, \dots, a_m^\vee, x_1^\vee, x_2^\vee \rangle \\ \text{Hom}^0(\rho, \rho') &= \langle y_1^\vee, y_2^\vee \rangle. \end{aligned}$$

To calculate the μ_1 maps, we appeal to [NRS⁺15, Proposition 4.14], which gives a combinatorial formula for the DGA of the k -copy of Λ in terms of the DGA of Λ , and which for convenience we recall here.

Proposition 3.3 ([NRS⁺15, Proposition 4.14]). *Let the DGA (\mathcal{A}, ∂) of Λ be generated by Reeb chords c_1, \dots, c_p along with $t_1^{\pm 1}, \dots, t_q^{\pm 1}$ corresponding to base points $*_1, \dots, *_q$. The DGA $(\mathcal{A}^k, \partial^k)$ for the k -copy of Λ is given as follows. As an algebra, \mathcal{A}^k is generated by:*

- $\{c_\ell^{ij}\}_{1 \leq i, j \leq k, 1 \leq \ell \leq p}$, with $|c_\ell^{ij}| = |c_\ell|$;
- $\{(t_\ell^i)^{\pm 1}\}_{1 \leq i \leq k, 1 \leq \ell \leq q}$, with $|(t_\ell^i)^{\pm 1}| = 0$;
- $\{x_\ell^{ij}, y_\ell^{ij}\}_{1 \leq i < j \leq k, 1 \leq \ell \leq q}$, with $|x_\ell^{ij}| = 0$ and $|y_\ell^{ij}| = -1$.

To state the differential, define $k \times k$ matrices: C_1, \dots, C_p with $(C_\ell)_{ij} = c_\ell^{ij}$, and $X_1, \dots, X_q, Y_1, \dots, Y_q, \Delta_1, \dots, \Delta_q$ with

$$(X_\ell)_{ij} = \begin{cases} 1 & i = j \\ x_\ell^{ij} & i < j \\ 0 & i > j \end{cases} \quad (Y_\ell)_{ij} = \begin{cases} y_\ell^{ij} & i < j \\ 0 & i \geq j \end{cases} \quad (\Delta_\ell)_{ij} = \begin{cases} t_\ell^i & i = j \\ 0 & i \neq j. \end{cases}$$

Then ∂^k is given entry-by-entry by:

$$\begin{aligned} \partial^k(C_\ell) &= \Phi(\partial(c_\ell)) + Y_{r(c_\ell)}C_\ell - (-1)^{|c_\ell|}C_\ell Y_{c(c_\ell)} \\ \partial^k(X_\ell) &= \Delta_\ell^{-1}Y_{r(t_\ell)}\Delta_\ell X_\ell - X_\ell Y_{c(t_\ell)} \\ \partial^k(Y_\ell) &= Y_\ell^2, \end{aligned}$$

where $\Phi : \mathcal{A} \rightarrow \text{Mat}_k(\mathcal{A}^k)$ is the ring homomorphism given by $\Phi(c_\ell) = C_\ell$, $\Phi(t_\ell) = \Delta_\ell X_\ell$, $\Phi(t_\ell^{-1}) = X_\ell^{-1}\Delta_\ell^{-1}$, and the link grading $r(\cdot), c(\cdot)$ was defined in Definition 2.13.

In our case, we have link grading $c(t_1) = 1$, $c(t_2) = 2$, $(r(t_1), r(t_2)) = \begin{cases} (2, 1) & m \text{ odd} \\ (1, 2) & m \text{ even} \end{cases}$,

$(r(a_j), c(a_j)) = \begin{cases} (1, 2) & j \text{ odd} \\ (2, 1) & j \text{ even} \end{cases}$, and $(r(b_j), c(b_j)) = \begin{cases} (1, 2) & m \text{ odd} \\ (j, j) & m \text{ even} \end{cases}$. This leads

to the following formulas for the differential ∂^2 in the 2-copy of Λ :

$$\partial^2(b_1^{12}) = -x_1^{12}(t_1^2)^{-1} + y_{r(b_1)}^{12}b_1^{22} + b_1^{11}y_{c(b_1)}^{12} + \sum_{j=1}^m P_{j-1}(a_1^{11}, \dots, a_{j-1}^{11})a_j^{12}P_{m-j}(a_{j+1}^{22}, \dots, a_m^{22})$$

$$\partial^2(b_2^{12}) = t_2^1 x_2^{12} + y_{r(b_2)}^{12}b_2^{22} + b_2^{11}y_{c(b_2)}^{12} - \sum_{j=1}^m Q_{m-j}(a_{j+1}^{11}, \dots, a_m^{11})a_j^{12}Q_{j-1}(a_1^{22}, \dots, a_{j-1}^{22})$$

$$\partial^2(a_j^{12}) = y_{r(a_j)}^{12}a_j^{22} - a_j^{11}y_{c(a_j)}^{12}$$

$$\partial^2(x_1^{12}) = (t_1^1)^{-1}y_{r(t_1)}^{12}t_1^2 - y_1^{12}$$

$$\partial^2(x_2^{12}) = (t_2^1)^{-1}y_{r(t_2)}^{12}t_2^2 - y_2^{12}$$

$$\partial^2(y_1^{12}) = \partial^2(y_2^{12}) = 0.$$

Augmenting by substituting $a_j^{11} = A_j$, $a_j^{22} = A'_j$, $b_j^{11} = b_j^{22} = 0$, $t_j^1 = T_j$, $t_j^2 = T'_j$ and dualizing gives the following formulas for μ_1 :

$$\begin{aligned} \mu_1(z_1 b_1^\vee) &= \mu_1(z_2 b_2^\vee) = 0 \\ \mu_1(w_j a_j^\vee) &= P_{j-1}(A_1, \dots, A_{j-1})w_j P_{m-j}(A'_{j+1}, \dots, A'_n)b_1^\vee \\ &\quad - Q_{n-j}(A_{j+1}, \dots, A_n)w_j Q_{j-1}(A'_1, \dots, A'_{j-1})b_2^\vee \\ \mu_1(v_1 x_1^\vee) &= -v_1(T_1')^{-1}b_1^\vee \\ \mu_1(v_2 x_2^\vee) &= T_2 v_2 b_2^\vee \end{aligned}$$

$$\begin{aligned}\mu_1(u_1 y_1^\vee) &= \sum_{j \text{ odd}} u_1 A'_j a_j^\vee - \sum_{j \text{ even}} A_j u_1 a_j^\vee - u_1 x_1^\vee + \begin{cases} T_2^{-1} u_1 T'_2 x_2^\vee & m \text{ odd} \\ T_1^{-1} u_1 T'_1 x_1^\vee & m \text{ even} \end{cases} \\ \mu_1(u_2 y_2^\vee) &= - \sum_{j \text{ odd}} A_j u_2 a_j^\vee + \sum_{j \text{ even}} u_2 A'_j a_j^\vee - u_2 x_2^\vee + \begin{cases} T_1^{-1} u_2 T'_1 x_1^\vee & m \text{ odd} \\ T_2^{-1} u_2 T'_2 x_2^\vee & m \text{ even.} \end{cases}\end{aligned}$$

From this we can calculate the cohomology of $\text{Hom}(\rho, \rho')$ with respect to μ_1 .

Proposition 3.4. *Let $\rho = (A_1, \dots, A_m)$ and $\rho' = (A'_1, \dots, A'_m)$ be objects in $\text{Rep}_n(\Lambda_m, \mathbb{k})$. Then we have:*

$$H^0 \text{Hom}(\rho, \rho') \cong \{(u_1, u_2) \in (\text{Mat}_n(\mathbb{k}))^2 \mid u_1 A'_j = A_j u_2 \text{ for } j \text{ odd and } A_j u_1 = u_2 A'_j \text{ for } j \text{ even}\};$$

$$H^1 \text{Hom}(\rho, \rho') \cong (\text{Mat}_n(\mathbb{k}))^m /$$

$$\{(u_1 A'_1 - A_1 u_2, u_2 A'_2 - A_2 u_1, u_1 A'_3 - A_3 u_2, \dots, u_1 A'_m - A_m u_2) \mid u_1, u_2 \in \text{Mat}_n(\mathbb{k})\}$$

if m is odd, and

$$H^1 \text{Hom}(\rho, \rho') \cong (\text{Mat}_n(\mathbb{k}))^m /$$

$$\{(u_1 A'_1 - A_1 u_2, u_2 A'_2 - A_2 u_1, u_1 A'_3 - A_3 u_2, \dots, u_2 A'_m - A_m u_1) \mid u_1, u_2 \in \text{Mat}_n(\mathbb{k})\}$$

if m is even; and $H^2 \text{Hom}(\rho, \rho') = 0$.

Proof. As $\mu_1(-v T'_1 x_1^\vee) = v b_1^\vee$ and $\mu_1(T_2^{-1} v x_2^\vee) = v b_2^\vee$, the statement for H^2 is clear. For $H^1 \text{Hom}(\rho, \rho')$, note that any degree 1 cycle can be written as $\sum_j w_j a_j^\vee + v_1 x_1^\vee + v_2 x_2^\vee$, where w_1, \dots, w_m are unconstrained and v_1, v_2 are determined by w_1, \dots, w_m ; to get the desired isomorphism, project away the x_1^\vee and x_2^\vee terms. Finally, for $H^0 \text{Hom}(\rho, \rho')$, note that by the formula for μ_1 , the statement that $\mu_1(u_1 y_1^\vee + u_2 y_2^\vee) = 0$ is equivalent to the conditions $u_1 A'_j = A_j u_2$ etc. given in the statement of the proposition, along with the conditions

$$\begin{aligned}u_1 &= T_1^{-1} u_2 T'_1 & u_2 &= T_2^{-1} u_1 T'_2 & m \text{ odd} \\ u_1 &= T_1^{-1} u_1 T'_1 & u_2 &= T_2^{-1} u_2 T'_2 & m \text{ even,}\end{aligned}$$

but these extra conditions are superfluous due to the equalities $T_1^{-1} = -P_m(A_1, \dots, A_m)$, $(T'_1)^{-1} = -P_m(A'_1, \dots, A'_m)$, $T_2 = -Q_m(A_1, \dots, A_m)$, $T'_2 = -Q_m(A'_1, \dots, A'_m)$ and Lemma 3.5 below. \square

Lemma 3.5. *Suppose for some $m \geq 1$ that $A_1, \dots, A_m, A'_1, \dots, A'_m, u_1, u_2$ are elements of $\text{Mat}_n(\mathbb{k})$ satisfying*

$$u_1 A'_j = A_j u_2 \text{ for all odd } j \quad \text{and} \quad A_j u_1 = u_2 A'_j \text{ for all even } j.$$

Then we have

$$\begin{aligned} u_1 P_m(A'_1, \dots, A'_m) &= \begin{cases} P_m(A_1, \dots, A_m)u_2, & m \text{ odd} \\ P_m(A_1, \dots, A_m)u_1, & m \text{ even}, \end{cases} \\ Q_m(A_1, \dots, A_m)u_2 &= \begin{cases} u_1 Q_m(A'_1, \dots, A'_m), & m \text{ odd} \\ u_2 Q_m(A'_1, \dots, A'_m), & m \text{ even}. \end{cases} \end{aligned}$$

Proof. We prove this by induction on m , the cases $m = 1, 2$ being straightforward once we recall that $P_1(a_1) = a_1$, $P_2(a_1, a_2) = 1 + a_1 a_2$, $Q_1(a_1) = -a_1$, and $Q_2(a_1, a_2) = 1 + a_2 a_1$. We let (i, j) be $(1, 2)$ if m is odd and $(2, 1)$ if m is even; in either case we have $A_m u_j = u_i A'_m$.

For the first equation, multiplying the defining recurrence for P_m on the left by u_1 gives

$$\begin{aligned} u_1 P_m(A'_1, \dots, A'_m) &= (u_1 P_{m-1}(A'_1, \dots, A'_{m-1}))A'_m + u_1 P_{m-2}(A'_1, \dots, A'_{m-2}) \\ &= (P_{m-1}(A_1, \dots, A_{m-1})u_i)A'_m + P_{m-2}(A_1, \dots, A_m)u_j \\ &= P_{m-1}(A_1, \dots, A_{m-1})(A_m u_j) + P_{m-2}(A_1, \dots, A_m)u_j \\ &= P_m(A_1, \dots, A_m)u_j \end{aligned}$$

as desired. For the second equation, we use the recurrence

$$Q_m(A_1, \dots, A_m) = -A_m Q_{m-1}(A_1, \dots, A_{m-1}) + Q_{m-2}(A_1, \dots, A_{m-2}),$$

which is easily seen to be equivalent to the one which defines Q_m ; multiplying on the right by u_2 then gives $Q_m(A_1, \dots, A_m)u_2 = u_i Q_m(A'_1, \dots, A'_m)$ by an identical argument. \square

Remark 3.6. A summary of Proposition 3.4 and its proof is as follows. Since $\mu_1(d_1 x_1^\vee) = -d_1(T'_1)^{-1}b_1^\vee$ and $\mu_1(d_2 x_2^\vee) = T_2 d_2 b_2^\vee$ and T'_1, T_2 are invertible, to compute cohomology with respect to μ_1 we can cancel all multiples of x_1^\vee with all multiples of b_1^\vee , and similarly x_2^\vee with b_2^\vee . The resulting quotient complex is supported only in degrees 0 and 1 and is given by the map

$$\begin{aligned} (\text{Mat}_n(\mathbb{k}))^2 &\rightarrow (\text{Mat}_n(\mathbb{k}))^m \\ (u_1, u_2) &\mapsto (u_1 A'_1 - A_1 u_2, u_2 A'_2 - A_2 u_1, u_1 A'_3 - A_3 u_2, \dots). \end{aligned}$$

Then $H^0 \text{Hom}(\rho, \rho')$ and $H^1 \text{Hom}(\rho, \rho')$ are the kernel and cokernel of this map, respectively.

To conclude the computation of the cohomology category $H^* \mathcal{R}ep_n(\Lambda, \mathbb{k})$, we need to compute the composition maps μ_2 . For degree reasons, the only possible nonzero μ_2 maps are

$$\begin{aligned} \mu_2 : H^0 \text{Hom}(\rho', \rho'') \otimes H^0 \text{Hom}(\rho, \rho') &\rightarrow H^0 \text{Hom}(\rho, \rho'') \\ \mu_2 : H^0 \text{Hom}(\rho', \rho'') \otimes H^1 \text{Hom}(\rho, \rho') &\rightarrow H^1 \text{Hom}(\rho, \rho'') \\ \mu_2 : H^1 \text{Hom}(\rho', \rho'') \otimes H^0 \text{Hom}(\rho, \rho') &\rightarrow H^1 \text{Hom}(\rho, \rho''). \end{aligned}$$

Let $\rho = (A_1, \dots, A_m)$, $\rho' = (A'_1, \dots, A'_m)$, $\rho'' = (A''_1, \dots, A''_m)$ be objects in $\mathcal{R}ep_n(\Lambda, \mathbb{k})$. From [NRS⁺15, Proposition 4.14], we read off the parts of the 3-copy differential ∂^3 that are relevant for calculating μ_2 :

$$\begin{aligned}\partial^3(a_j^{13}) &= y_{r(a_j)}^{12} a_j^{23} - a_j^{12} y_{c(a_j)}^{23} + \dots \\ \partial^3(x_j^{13}) &= (t_j^1)^{-1} y_{r(t_j)}^{12} t_j^2 x_j^{23} - x_j^{12} y_{c(t_j)}^{23} + \dots \\ \partial^3(y_j^{13}) &= y_j^{12} y_j^{23}.\end{aligned}$$

Dualizing these differentials gives the following. For $H^0 \otimes H^0$, we have:

$$\mu_2(u'_1 y_1^\vee + u'_2 y_2^\vee, u_1 y_1^\vee + u_2 y_2^\vee) = -u_1 u'_1 y_1^\vee - u_2 u'_2 y_2^\vee.$$

In terms of the isomorphism for H^0 from Proposition 3.4, we have:

$$(6) \quad \mu_2((u'_1, u'_2), (u_1, u_2)) = -(u_1 u'_1, u_2 u'_2).$$

For $H^0 \otimes H^1$, we have:

$$\mu_2(u'_1 y_1^\vee + u'_2 y_2^\vee, \sum_j w_j a_j^\vee + v_1 x_1^\vee + v_2 x_2^\vee) = - \sum_{j \text{ odd}} w_j u'_2 a_j^\vee - \sum_{j \text{ even}} w_j u'_1 a_j^\vee - v_1 u'_1 x_1^\vee - v_2 u'_2 x_2^\vee.$$

In terms of the isomorphisms for H^0 and H^1 from Proposition 3.4, we have:

$$(7) \quad \mu_2((u'_1, u'_2), (w_1, \dots, w_m)) = -(w_1 u'_2, w_2 u'_1, w_3 u'_2, \dots).$$

Similarly, for $H^1 \otimes H^0$, we find

$$(8) \quad \mu_2((w'_1, \dots, w'_m), (u_1, u_2)) = -(u_1 w'_1, u_2 w'_2, u_1 w'_3, \dots).$$

4. THE CATEGORY $\mathcal{S}h(\Lambda, \mathbb{k})$

In this Section we recall some of the theory of microsupport of constructible sheaves from [KS94], its application to symplectic and contact geometry as in [GKS12], and the construction of the category $\mathcal{S}h(\Lambda, \mathbb{k})$ from [STZ17]. We do not intend to be exhaustive, recalling only those facts which are necessary in order to have a working definition of microsupport and compute the category for our particular Legendrian $(2, m)$ torus knots.

4.1. Dg version of the derived category. We describe here the construction of the dg-quotient of categories of complexes of sheaves as described in [Dri04]. Letting X be a topological space, we denote by $\widetilde{\mathcal{S}h}(X, \mathbb{k})$ the dg-category of complexes of sheaves of \mathbb{k} -modules on X with constructible homology. Let \widetilde{A} be any full subcategory of $\widetilde{\mathcal{S}h}(X, \mathbb{k})$ stable under degree shift and whose objects contain all acyclic complexes. We denote by A the quotient category by acyclic complexes as described in [Dri04, Section 3]. Briefly, the objects of A are those of \widetilde{A} , and if \mathcal{F} is acyclic then $\text{hom}_A(\mathcal{F}, \mathcal{F})$ is equipped with one additional morphism $\varepsilon_{\mathcal{F}}$ of degree -1 with differential $d\varepsilon_{\mathcal{F}} = \text{id}_{\mathcal{F}}$.

Remark 4.1. This extra morphism has the following effect. Let $f \in \text{hom}(\mathcal{F}, \mathcal{F}')$ be a closed morphism which induces an isomorphism in the cohomological category (i.e. a quasi-isomorphism). Then the complex of sheaves $\mathcal{F}_f := \text{cone}(f)$ is acyclic. We denote by $i_{\mathcal{F}}, i_{\mathcal{F}'}, p_{\mathcal{F}}, p_{\mathcal{F}'}$ the canonical inclusions into and projections out of \mathcal{F}_f . Note that $i_{\mathcal{F}'}$ and $p_{\mathcal{F}}$ are closed morphisms, whereas $di_{\mathcal{F}} = i_{\mathcal{F}'} \circ f$ and $dp_{\mathcal{F}'} = f \circ p_{\mathcal{F}}$. Thus in A the map $g := p_{\mathcal{F}} \circ \varepsilon \circ i_{\mathcal{F}'}$ is a closed map such that $fg + I_{\mathcal{F}'} = d(p_{\mathcal{F}'} \circ \varepsilon \circ i_{\mathcal{F}'})$ and $gf + I_{\mathcal{F}} = d(p_{\mathcal{F}} \circ \varepsilon \circ i_{\mathcal{F}})$, which means that f is invertible up to homotopy. In particular, $[f]$ is invertible in the homology category $H_*(A)$.

Indeed the category $H^0(\mathcal{S}h(X, \mathbb{k}))$ coincides with the derived category of constructible sheaves of \mathbb{k} -modules on X , and we have $H^i(\text{hom}(\mathcal{F}, \mathcal{F}')) = \text{Ext}^i(\mathcal{F}, \mathcal{F}')$.

4.2. Combinatorial description of constructible sheaves. Let \mathcal{S} be a stratification of the plane \mathbb{R}^2 , viewed as a category where there is a unique map from a to b iff $a \subset \bar{b}$. For a given stratum $a \in \mathcal{S}$, we let the star of a , denoted $s(a)$, be the union of all strata whose closure contains a . Here we will only consider stratifications which are *regular cell complexes*, i.e. all strata and their stars are contractible in $S^2 = \mathbb{R}^2 \cup \infty$.

We denote by $\widetilde{\text{Fun}}(\mathcal{S}, \mathbb{k})$ the category of functors from \mathcal{S} into chain complexes of \mathbb{k} -modules (with acyclic stalk on the non compact strata). As in Section 4.1, given any full subcategory \widetilde{A} whose objects contain all functors valued in acyclic complexes, we denote by A its quotient by those functors.

An object $\mathcal{F} \in \mathcal{S}h(\mathbb{R}^2, \mathbb{k})$ whose homology sheaves are constructible with respect to the stratification \mathcal{S} leads to an object $\Gamma_{\mathcal{S}}(\mathcal{F})$ defined by $\Gamma_{\mathcal{S}}(\mathcal{F})(a) = \mathcal{F}(s(a))$. This extends to a functor from $\mathcal{S}h_{\mathcal{S}}(X, \mathbb{k})$ to $\text{Fun}(\mathcal{S}, \mathbb{k})$. It follows from [Nad09, Lemma 2.3.3] that for regular cell complexes, this map is a quasi-equivalence.

4.3. Microsupport. In order to avoid notational confusion involving the sheaf cohomology of complexes of sheaves, given a sheaf \mathcal{F} of (possibly graded) \mathbb{k} -vector spaces, we denote its i -th sheaf cohomology group by $R^i\Gamma(\mathcal{F})$, the total group being $R\Gamma(\mathcal{F}) = \bigoplus_i R^i\Gamma(\mathcal{F})$.

Let $\mathcal{F} \in \text{Ob}(\mathcal{S}h(X, \mathbb{k}))$. For $(x, p) \in T^*X$, we denote by $C_{x,p}$ the set of functions $f : X \rightarrow \mathbb{R}$ such that $f(x) = 0$ and $df_x = p$. A point (x, p) is *characteristic* for \mathcal{F} if there is some $f \in C_{x,p}$ and a sequence of positive numbers $\varepsilon \rightarrow 0$ for which the restriction-induced map

$$R\Gamma_{B(x,\varepsilon)}(\mathcal{F}) \rightarrow R\Gamma_{\{f < 0\} \cap B(x,\varepsilon)}(\mathcal{F})$$

is not an isomorphism. We denote by $SS(\mathcal{F})$ the closure of the set of characteristic points for \mathcal{F} . We denote by $SS_{\bullet}(\mathcal{F})$ the set of points $(x, p) \in SS(\mathcal{F})$ such that $p \neq 0$. It follows from results in [KS94] that if $SS_{\bullet}(\mathcal{F})$ is a submanifold of T^*X , then it is a conical Lagrangian for the standard symplectic structure. Its projectivization is a Legendrian submanifold of $S(T^*X)$.

Given a Legendrian submanifold Λ of $S(T^*X)$, we denote by $\mathcal{S}h(\Lambda, \mathbb{k})$ the full subcategory of $\mathcal{S}h(X, \mathbb{k})$ generated by objects \mathcal{F} whose homology sheaves are constructible with respect to its front projection $\Pi(\Lambda) \subset X$ and such that $SS_\bullet(\mathcal{F})$ has projectivization inside Λ ; if X is a product $Y \times \mathbb{R}$, with z the coordinate on \mathbb{R} , then we will also insist that \mathcal{F} has acyclic stalk for $z \ll 0$. It follows from [GKS12, Theorem 3.7] and [STZ17, Theorem 4.10] that $\mathcal{S}h(\Lambda, \mathbb{k})$ is invariant under Legendrian isotopies of Λ .

We wish to study invariants of Legendrians in \mathbb{R}^3 , which does not have the form $S(T^*X)$. However, a 1-jet space $J^1(Y)$ naturally embeds in the subspace of $S(T^*(Y \times \mathbb{R}))$ consisting of “graphical” (i.e., non-vertical) contact elements, so that we can talk about sheaves on $Y \times \mathbb{R}$ with microsupport in a Legendrian in $J^1(Y)$. In the case $Y = \mathbb{R}$ we thus get an invariant $\mathcal{S}h(\Lambda, \mathbb{k})$ of $\Lambda \subset J^1(\mathbb{R}) = \mathbb{R}^3$, as desired. This is the perspective used throughout [STZ17], and the embedding $i : \mathbb{R}^3 \hookrightarrow S(T^*\mathbb{R}^2)$ is explicitly described in [STZ17, §2.1.1]. Its image is equal to the set of “downward” covectors, and it identifies the front (xz) projection of $\Lambda \subset \mathbb{R}^3$ with the projection of $i(\Lambda) \subset S(T^*\mathbb{R}^2)$ to the base \mathbb{R}^2 . Thus $\mathcal{S}h(\Lambda, \mathbb{k})$ consists of complexes of sheaves on \mathbb{R}^2 whose homology is constructible with respect to the front projection $\Pi(\Lambda) \subset \mathbb{R}^2$, and [STZ17] gives a very concrete interpretation of the singular support condition, which we now describe.

Assume that $\Lambda \subset \mathbb{R}^3$ is generic so that its front $\Pi(\Lambda)$ has only cusps and double points as singularities. Then $\Pi(\Lambda)$ induces a stratification of \mathbb{R}^2 where the top-dimensional strata are the regions of the complement of the front, the 0-dimensional strata are cusps and double points of the front projection, and the remainder of the front forms the 1-dimensional strata. We denote by \mathcal{S}_Λ this stratification. When this stratification is a regular cell complex, the condition for \mathcal{F} to have microsupport in Λ translates to the following conditions for $\Gamma_{\mathcal{S}_\Lambda}(\mathcal{F})$:

- (1) **Arcs:** Near an arc a we have a map f going to the upper 2-dimensional stratum U and a map g going to the lower 2-dimensional stratum D ; we require g to be a quasi-isomorphism.

$$\begin{array}{c}
 \mathcal{F}(U) \\
 \uparrow f \\
 \mathcal{F}(s(a)) \\
 \cong \downarrow \\
 \mathcal{F}(D)
 \end{array}$$

FIGURE 3. The microsupport condition near an arc a .

- (2) **Cusps:** Near a cusp c we have an upper arc u and a lower arc d , and regions I and O which are “inside” the cusp and “outside” the cusp respectively.

We require the maps from c to u , from b to O , and from a to I to be quasi-isomorphisms as shown in Figure 4.

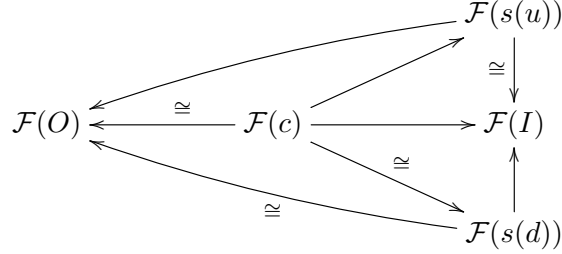


FIGURE 4. The microsupport condition near a cusp.

- (3) **Crossings:** Near a crossing x we have four 2-dimensional strata, labeled N , S , E , and W ; and four arcs nw , ne , sw , and se . We require all downward maps to be quasi-isomorphisms, as in Figure 5. We also require the total

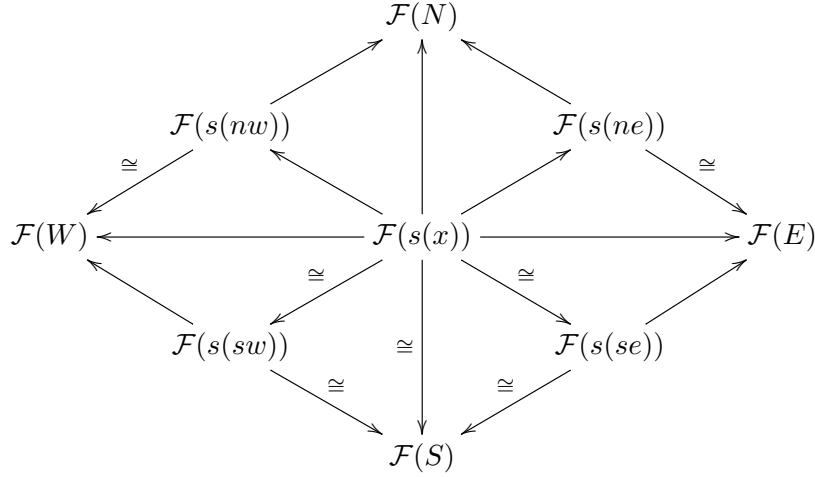


FIGURE 5. The microsupport condition near a crossing c .

complex

$$\text{Tot}(\mathcal{F}(s(x)) \rightarrow \mathcal{F}(s(nw)) \oplus \mathcal{F}(s(ne)) \rightarrow \mathcal{F}(N))$$

to be acyclic.

4.4. Microlocal rank. Fix a Maslov potential μ for Λ , viewed as an integer $\mu(a)$ assigned to each arc a of the stratification. Following [STZ17, Section 5.1], an object

\mathcal{F} of $\mathcal{S}h(\Lambda, \mathbb{k})$ is said to have *microlocal rank* n if for any arc a of the stratification, if f is the chain map of Figure 3 then

$$H^i(\text{Cone}(f)) \cong \begin{cases} \mathbb{k}^n & i = -\mu(a) \\ 0 & i \neq -\mu(a). \end{cases}$$

We note for future reference that this condition is satisfied for $\text{Cone}(0 \rightarrow \mathbb{k}^n)$ for $\mu(a) = 0$, and for $\text{Cone}(\mathbb{k}^n \rightarrow 0)$ for $\mu(a) = 1$, where \mathbb{k}^n is in degree 0 in each case.

We denote by $\mathcal{S}h_n(\Lambda, \mathbb{k})$ the full subcategory of $\mathcal{S}h(\Lambda, \mathbb{k})$ which consists of objects of microlocal rank n . (In [STZ17] this category is labeled $\mathcal{C}_n(\Lambda)$.) Then the equivalences of categories provided by [GKS12, STZ17] in proving the invariance of $\mathcal{S}h(\Lambda, \mathbb{k})$ carry sheaves of microlocal rank n to sheaves of microlocal rank n , so $\mathcal{S}h_n(\Lambda, \mathbb{k})$ is a Legendrian isotopy invariant of Λ .

5. THE COHOMOLOGY SHEAF CATEGORY FOR Λ_m

In this section, we compute the objects and morphisms for the cohomology category $H^*\mathcal{S}h_n(\Lambda_m, \mathbb{k})$ of the sheaf category $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$ for the Legendrian knot Λ_m . The version of Λ_m that we will use here is the rainbow closure of the 2-braid $\sigma_1^m \in B_2$, as shown in Figure 1, equipped with the binary Maslov potential given by 0 on the two lower strands and 1 on the two upper strands. This is not identical to the version of Λ_m from Section 3, which we used to compute the representation category $\mathcal{R}ep_n(\Lambda_m, \mathbb{k})$; however, they are Legendrian isotopic, and both $\mathcal{R}ep_n$ and $\mathcal{S}h_n$ are invariant (up to A_∞ equivalence) under Legendrian isotopy. The advantage of considering the rainbow closure front for the sheaf computation is that, as explained in [STZ17], this front has a binary Maslov potential and all objects are legible.

The objects of $H^*\mathcal{S}h_n(\Lambda_m, \mathbb{k})$, which are the same as the objects of $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$, are presented in Section 5.1; as in [STZ17], these can be depicted as a collection of \mathbb{k} -vector spaces with certain linear maps between them. The spaces of morphisms in $H^*\mathcal{S}h_n(\Lambda_m, \mathbb{k})$ are given by Ext groups between objects. For Legendrian links such as Λ_m with a binary Maslov potential, the only possible nonzero Ext groups are Ext^0 , Ext^1 , and Ext^2 : as explained in [STZ17, Section 7.2.1], in this setting all objects correspond to $\mathbb{k}[x, y]$ -modules, and $\mathbb{k}[x, y]$ admits a bimodule resolution of length 3. We calculate Ext^0 and Ext^1 in Sections 5.3 and 5.4, respectively, and then we describe their composition in Section 5.5. The final Ext group, Ext^2 , vanishes, but the argument for this is more delicate and we leave it to Section 6.

5.1. Objects in $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$. Here we describe the objects in the category $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$ where Λ_m is equipped with the obvious binary Maslov potential where the upper and lower strands at any cusp have potentials 1 and 0 respectively.

First, we note that any sheaves \mathcal{F} in $\text{Ob}(\mathcal{S}h_n(\Lambda_m, \mathbb{k}))$ must have homology concentrated in degree 0, as shown in [STZ17, Proposition 5.17]. This implies that the map

$$(\cdots \rightarrow C_{-1} \rightarrow \ker d_0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots) \rightarrow (\cdots \rightarrow C_{-1} \rightarrow C_0 \rightarrow C_1 \rightarrow \cdots)$$

is a quasi-isomorphism, and so is

$$(\cdots \rightarrow C_{-1} \rightarrow \ker d_0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots) \rightarrow (\cdots \rightarrow 0 \rightarrow H_0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots).$$

It follows from Remark 4.1 that any object \mathcal{F} is homotopy equivalent to its 0-th homology sheaf and thus we can restrict to objects which are actual sheaves.

The sheaf is zero over the unbounded 2-dimensional stratum, and thus from the cusp condition (Figure 4) we get that the sheaf is also zero over the star of the two “outside” cusps and the star of the strand below them. Since the sheaf has microlocal rank n , the region I just inside these cusps has $\mathcal{F}(I) = V$ for some vector space V of dimension n . Note that the topmost strand u of the front is the upward strand for both cusps, and by the microsupport condition, we have an isomorphism $\mathcal{F}(s(U)) \xrightarrow{\cong} \mathcal{F}(I)$. We can use this isomorphism to change $\mathcal{F}(s(U))$ to an isomorphic vector space, with the result that the map $\mathcal{F}(s(U)) \rightarrow \mathcal{F}(I)$ is the identity.

We next consider the inside cusps of Λ_m , which we label c_l (left) and c_r (right), and which have a common upper strand u and a common inner 2-dimensional stratum I (different from I in the previous paragraph). From the microsupport condition at the cusp (see Figure 4), for each of $c = c_l$ and $c = c_r$ we have an isomorphism $\mathcal{F}(s(c)) \xrightarrow{\cong} \mathcal{F}(s(d))$, and by replacing $\mathcal{F}(s(d))$ by an isomorphic vector space we can arrange that this isomorphism is the identity map. The microlocal rank condition implies that we can identify the vector space $\mathcal{F}(I)$ with $V \oplus V$; we replace $\mathcal{F}(s(u))$ by an isomorphic vector space so that the isomorphism $\mathcal{F}(s(u)) \rightarrow \mathcal{F}(I) = V \oplus V$ is the identity. Finally, for both c_l and c_r we have an isomorphism $\mathcal{F}(s(d)) \rightarrow \mathcal{F}(O)$, where O is the same region in both cases; we change $\mathcal{F}(O)$ so that this isomorphism is the identity for c_l , but we do not assume that the same is true for c_r (we will use the extra freedom to define the map ϕ_{m+1} below). The sheaf around the cusps c_l and c_r is now given as in Figure 6.

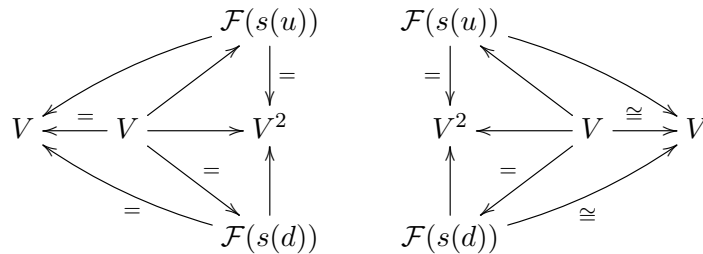


FIGURE 6. The microsupport condition near the inside cusps c_l (left) and c_r (right).

We now turn our attention to crossings. Note that at all crossings, the stalk of \mathcal{F} along the south 2-dimensional stratum is zero. In addition, there are $m - 1$ arcs in the front of Λ_m that form the ne arc for one crossing and the nw arc for the crossing to its right; by changing the vector spaces associated to each of these arcs by an

isomorphism, we can arrange that for all crossings, the maps $\mathcal{F}(s(nw)) \rightarrow \mathcal{F}(W)$ and $\mathcal{F}(s(ne)) \rightarrow \mathcal{F}(E)$ are the identity. The sheaf around each crossing is now given as in Figure 7, where the bottom half of the sheaf as in Figure 5 is zero and has been omitted. The acyclicity condition at the crossing is then equivalent to the condition that $f_1 \oplus f_2$ is an isomorphism.

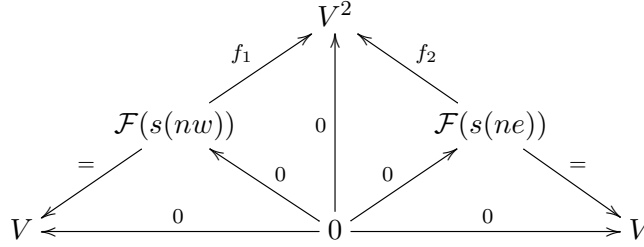


FIGURE 7. The sheaf near a crossing of Λ_m .

Altogether, the data needed to describe a sheaf with microsupport on Λ_m are:

- For each inner Maslov-0 one dimensional stratum a with U the two dimensional stratum above it, a map $\mathcal{F}(s(a)) \rightarrow \mathcal{F}(U)$.
- Associated to the “inside” Maslov-1 one dimensional stratum, a map from V^2 to V .

These data must satisfy the following:

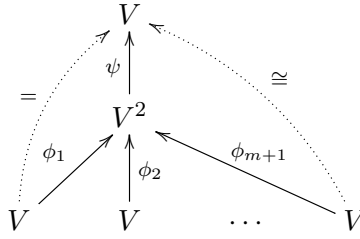
- If two one-dimensional strata meet at a crossing, the direct sum of the two associated maps is an isomorphism.
- At the inner left cusp, the composition of the map associated to the lower strand with the one associated to the upper strand is the identity.
- At the inner right cusp, the composition of the map associated to the lower strand with the one associated to the upper strand is an isomorphism.

All this discussion implies that an object of the sheaf category $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$ corresponds to a n -dimensional vector space V and a choice of maps $\phi_1, \dots, \phi_{m+1} : V \rightarrow V^2$ and $\psi : V^2 \rightarrow V$ such that the following conditions hold:

- $\psi \circ \phi_1 = \text{Id}$ and $\psi \circ \phi_{m+1}$ is an isomorphism;
- for $1 \leq i \leq m$, the images of ϕ_i and ϕ_{i+1} are complementary in V^2 : in particular, ϕ_i is injective for each i .

See Figure 8. In Section 5.2 we will demonstrate this construction in detail for the Hopf link Λ_2 .

We now reformulate these conditions. Since $\psi \circ \phi_1 = \text{Id}$, we can choose a basis of V^2 so that $\phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\psi = \begin{pmatrix} 0 & 1 \end{pmatrix}$.


 FIGURE 8. An object of $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$.

Now the images of ϕ_1 and ϕ_2 are complementary, which means that we can choose a basis for the domain of ϕ_2 for which $\phi_2 = \begin{pmatrix} 1 \\ A_1 \end{pmatrix}$ for some linear $A_1 : V \rightarrow V$. Similarly, for $3 \leq j \leq m + 1$, the image of ϕ_{j-1} is complementary to the image of ϕ_{j-2} as well as to the image of ϕ_j , and so we can inductively choose bases for ϕ_j and linear maps $A_{j-1} : V \rightarrow V$ such that $\phi_j = \phi_{j-2} + \phi_{j-1} \circ A_{j-1}$.

Thus we obtain a sequence of endomorphisms of V , A_1, \dots, A_m , such that if we construct the following chain of maps:

$$(9) \quad V \oplus V \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A_m \end{pmatrix}} \dots \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A_2 \end{pmatrix}} V \oplus V \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix}} V^2$$

then for each $k = 1, \dots, m$, the sum $\phi_k \oplus \phi_{k+1}$ is the composition of the final k arrows in this chain.

It follows from (3) that the matrix for ϕ_k is $\begin{pmatrix} P_{k-1}(A_2, \dots, A_k) \\ P_k(A_1, \dots, A_k) \end{pmatrix}$. The inner right cusp now dictates that $\psi \circ \phi_{m+1}$ is an isomorphism on V ; since $\psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, this simply states that $P_m(A_1, \dots, A_m)$ is invertible. We conclude the following:

Proposition 5.1. *The objects in $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$ are in correspondence with choices of $A_1, \dots, A_m \in \text{End } V$ for which $P_m(A_1, \dots, A_m) \in \text{End } V$ is invertible. Any such (A_1, \dots, A_m) determines an object as in Figure 8 by the conditions that $\psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the final k arrows in (9) compose to give $\phi_k \oplus \phi_{k+1}$.*

5.2. Example: the Hopf link. Here we illustrate the identification of objects in $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$ with the configuration shown in Figure 8, in the case $m = 2$ (the Hopf link). This link is shown in Figure 9, where we have labeled the strata as determined by the front projection: the U_i are 2-dimensional, the a_i and b_i are 1-dimensional, and the x_i and y_i are 0 dimensional.

The category \mathcal{S} corresponding to this stratification as in Section 4.2 is the transitive closure of the left hand diagram of Figure 10, in which we omit the identity self-morphisms for clarity.

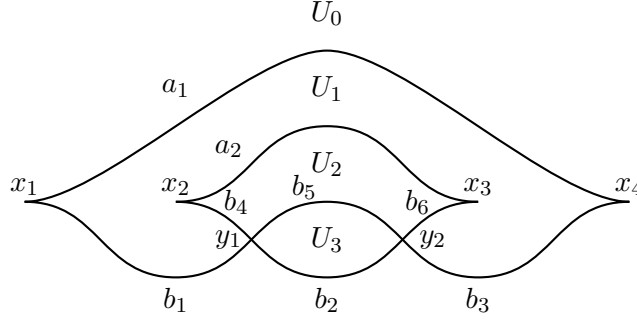


FIGURE 9. The front projection of the Hopf link.

Applying the functor $\Gamma_{\mathcal{S}}$ to an object \mathcal{F} in the sheaf category leads to the right hand diagram in Figure 10. Here by applying automorphisms of the blue vector spaces, we can turn isomorphisms into identity maps as shown. (There is one more isomorphism between V and V that we could also turn into the identity, but as discussed in Section 5.1, it is more convenient to leave this as an arbitrary isomorphism for the purposes of identifying objects in the sheaf category with matrices A_1, A_2 .) The information in the right hand diagram is then encoded by the red maps $\phi_1, \phi_2, \phi_3, \psi$; that is, precisely the picture in Figure 8 for $m = 2$.

5.3. Ext^0 in the sheaf category. We next turn to Ext^0 between two objects in $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$. Given two objects, a morphism between them in Ext^0 is determined by a morphism between the strata in the stratification determined by the front of Λ_m . In terms of the diagram from Figure 8, a morphism is a choice of $u_0, u_1, \dots, u_{m+1} : V \rightarrow V$ and $v : V^2 \rightarrow V^2$ such that the following diagram commutes:

$$(10) \quad \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} V \xrightarrow{u_0} V \\ \psi \uparrow \qquad \qquad \qquad \uparrow \psi' \\ V^2 \xrightarrow{v} V^2 \\ \phi_1 \nearrow \qquad \qquad \qquad \nwarrow \phi_{m+1} \\ V \qquad \qquad \qquad \dots \qquad \qquad \qquad V \end{array} \\ \phi_2 \uparrow \qquad \qquad \qquad \uparrow \phi'_2 \\ V \qquad \qquad \qquad \dots \qquad \qquad \qquad V \end{array} \\ \dots \\ \phi_{m+1} \nearrow \qquad \qquad \qquad \nwarrow \phi'_{m+1} \\ V \qquad \qquad \qquad \dots \qquad \qquad \qquad V \end{array} \\ \begin{array}{c} \xrightarrow{u_1} \qquad \xrightarrow{u_2} \qquad \xrightarrow{u_{m+1}} \end{array} \end{array}$$

Now describe two objects in $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$ by (A_1, \dots, A_m) and (A'_1, \dots, A'_m) as in Proposition 5.1. Then a morphism from (A_1, \dots, A_m) to (A'_1, \dots, A'_m) is a choice of

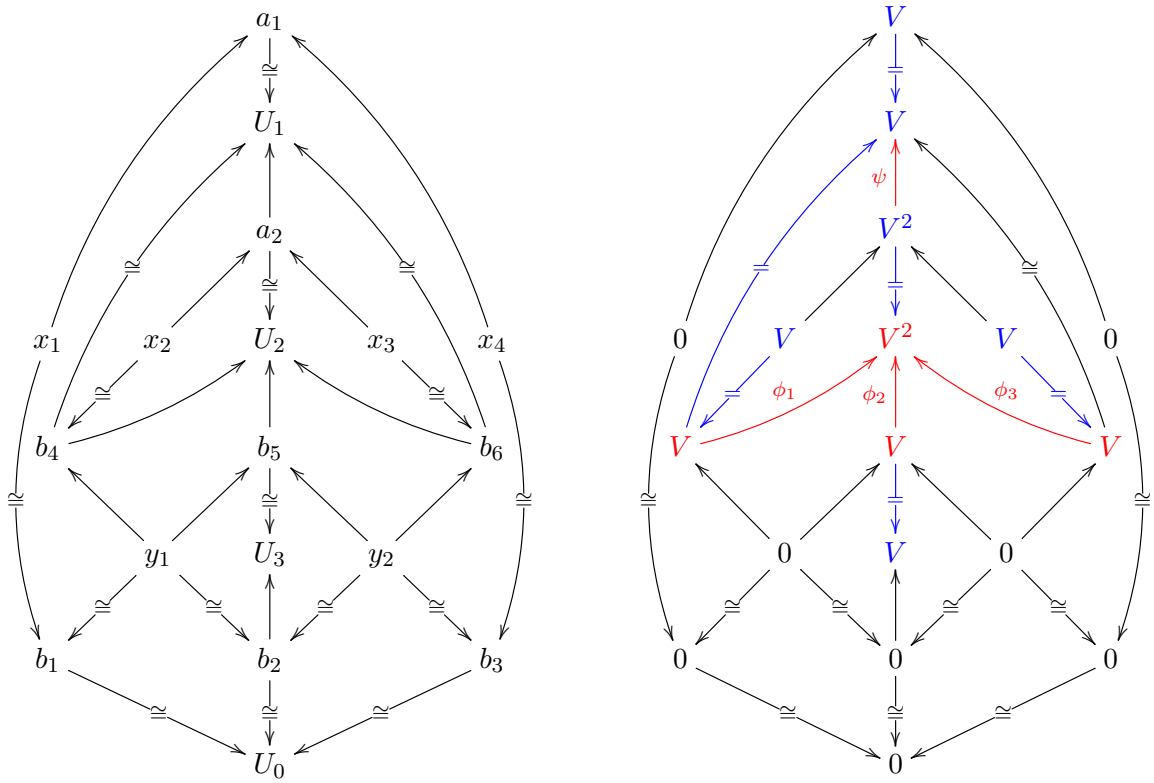


FIGURE 10. Left: the stratification \mathcal{S} of the front of the Hopf link as a category; right: the result of applying the functor $\Gamma_{\mathcal{S}}$ to an object in the sheaf category. There is also a (zero) map from a_1 to U_0 , omitted for readability.

$u_0, u_1, \dots, u_{m+1}, v$ such that $u_0 \circ \psi = \psi' \circ v$ and the following diagram commutes:

$$\begin{array}{ccccccc}
 V \oplus V & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A_m \end{pmatrix}} & \cdots & \longrightarrow & V \oplus V & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A_2 \end{pmatrix}} & V \oplus V & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix}} & V^2 \\
 \downarrow \begin{pmatrix} u_m & 0 \\ 0 & u_{m+1} \end{pmatrix} & & & & \downarrow \begin{pmatrix} u_2 & 0 \\ 0 & u_3 \end{pmatrix} & & \downarrow \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} & & \downarrow v \\
 V \oplus V & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A'_m \end{pmatrix}} & \cdots & \longrightarrow & V \oplus V & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A'_2 \end{pmatrix}} & V \oplus V & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A'_1 \end{pmatrix}} & V^2.
 \end{array}$$

The commutativity of the k -th square from the right (for $2 \leq k \leq m$) gives the pair of equations $u_{k+1} = u_{k-1}$ and $A'_k u_{k+1} = u_k A_k$. The commutativity of the rightmost square gives $v = \begin{pmatrix} u_2 & 0 \\ A'_1 u_2 - u_1 A_1 & u_1 \end{pmatrix}$, whence the relation $u_0 \circ \psi = \psi' \circ v$ combined with

$\psi = \psi' = \begin{pmatrix} 0 & 1 \end{pmatrix}$ implies $u_0 = u_1$ and $A'_1 u_2 - u_1 A_1 = 0$. Since Ext^0 is given by morphisms between objects in $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$, we conclude the following result.

Proposition 5.2. *Let $\mathcal{F} = (A_1, \dots, A_m)$ and $\mathcal{F}' = (A'_1, \dots, A'_m)$ denote objects in $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$. Then we have:*

$$\text{Ext}^0(\mathcal{F}, \mathcal{F}') \cong \{(u_1, u_2) \in (\text{End } V)^2 : A'_k u_1 = u_2 A_k \text{ for all even } k \text{ and} \\ A'_k u_2 = u_1 A_k \text{ for all odd } k\}.$$

Explicitly, given u_1, u_2 , we can construct an element of Ext^0 as depicted in (10), as follows: define $u_0 = u_1$, $u_k = u_1$ for $k \geq 1$ odd, $u_k = u_2$ for $k \geq 2$ even, and $v = \begin{pmatrix} u_2 & 0 \\ 0 & u_1 \end{pmatrix}$.

5.4. Ext^1 in the sheaf category. We now turn our attention to Ext^1 . We use the description of $\text{Ext}^1(\mathcal{F}, \mathcal{F}')$ as extensions \mathcal{G} of \mathcal{F} by \mathcal{F}' (see [HS71, Theorem III.2.4] or [Har77, Exercise 6.1]), i.e. as short exact sequences

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{G} \rightarrow \mathcal{F} \rightarrow 0.$$

Let $\mathcal{F} = (A_1, \dots, A_m)$ and $\mathcal{F}' = (A'_1, \dots, A'_m)$ be objects in $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$. Let \mathcal{G} denote an extension of \mathcal{F} by \mathcal{F}' . It follows from the triangle inequality [KS94, Proposition 5.1.3] that $SS_\bullet(\mathcal{G}) \subset SS_\bullet(\mathcal{F}) \cup SS_\bullet(\mathcal{F}')$. This implies that $SS_\bullet(\mathcal{G})$ has projectivization inside Λ . Furthermore, the microlocal rank of \mathcal{G} is $2n$, so we can represent \mathcal{G} by a diagram

$$(11) \quad \begin{array}{ccccc} & & V^2 & & \\ & & \uparrow \Psi & & \\ & & V^4 & & \\ \Phi_1 \nearrow & & \uparrow \Phi_2 & \nwarrow \Phi_{m+1} & \\ V^2 & & V^2 & \dots & V^2. \end{array}$$

The extension

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{G} \rightarrow \mathcal{F} \rightarrow 0$$

is given componentwise by a collection of short exact sequences $0 \rightarrow V \rightarrow V^2 \rightarrow V \rightarrow 0$ and one short exact sequence $0 \rightarrow V^2 \rightarrow V^4 \rightarrow V^2 \rightarrow 0$. We can choose bases for the components V^2, V^4 of \mathcal{G} such that all maps from \mathcal{F}' to \mathcal{G} are given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (for $V \rightarrow V^2$) and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ (for $V^2 \rightarrow V^4$), while all maps from \mathcal{G} to \mathcal{F} are given by $\begin{pmatrix} 0 & 1 \end{pmatrix}$ (for $V^2 \rightarrow V$) and $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (for $V^4 \rightarrow V^2$).

As before, we represent \mathcal{F} by a sequence of maps

$$V \oplus V \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A_m \end{pmatrix}} \dots \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A_2 \end{pmatrix}} V \oplus V \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix}} V^2$$

and similarly for \mathcal{F}' . (In each case the top map from V^2 to V is given by $\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and the A_i determine $\phi_1, \dots, \phi_{m+1}$.) We can analogously represent \mathcal{G} by $\Psi : V^4 \rightarrow V^2$ along with a sequence of maps

$$(12) \quad V^2 \oplus V^2 \xrightarrow{\Omega_m} \dots \xrightarrow{\Omega_2} V^2 \oplus V^2 \xrightarrow{\Omega_1} V^4$$

so that for $k = 1, \dots, m$, $\Phi_k \oplus \Phi_{k+1}$ is the composition of the final k arrows. Note in particular that for each $k \geq 2$, $\Omega_k : V^2 \oplus V^2 \rightarrow V^2 \oplus V^2$ sends the second V^2 factor to the first V^2 factor by the identity map; that is, the matrix for Ω_k is of the form $\begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$.

Now let \mathcal{G} be an extension of \mathcal{F} by \mathcal{F}' . The top maps in $\mathcal{F}', \mathcal{G}, \mathcal{F}$ give the commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & V^2 & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & V \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \uparrow & & \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \Psi \uparrow & & \uparrow \begin{pmatrix} 0 & 1 \end{pmatrix} \\ V^2 & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} & V^4 & \xrightarrow{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} & V^2, \end{array}$$

whence $\Psi = \begin{pmatrix} 0 & u_0 & 1 & v_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for some $u_0, v_0 \in \text{End } V$. The commutativity involving the rest of the maps $\Phi_1, \dots, \Phi_{m+1}$ can be expressed as the following commutative diagram:

$$\begin{array}{ccccccc} V \oplus V & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A'_m \end{pmatrix}} & \dots & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A'_2 \end{pmatrix}} & V \oplus V & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A'_1 \end{pmatrix}} & V^2 \\ \downarrow & & & & \downarrow & & \downarrow \\ V^2 \oplus V^2 & \xrightarrow{\Omega_m} & \dots & \xrightarrow{\Omega_2} & V^2 \oplus V^2 & \xrightarrow{\Omega_1} & V^4 \\ \downarrow & & & & \downarrow & & \downarrow \\ V \oplus V & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A_m \end{pmatrix}} & \dots & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A_2 \end{pmatrix}} & V \oplus V & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix}} & V^2, \end{array}$$

where all the vertical maps between the first and second rows are $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and all the vertical maps between the second and third rows are $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Commutativity of the k -th set of boxes from the right in the above diagram is now easily seen to imply that:

$$\Omega_k = \begin{pmatrix} 0 & * & 1 & * \\ 0 & 0 & 0 & 1 \\ 1 & * & A'_k & * \\ 0 & 1 & 0 & A_k \end{pmatrix}.$$

Along with the restriction on the form of Ω_k for $k \geq 2$, we conclude that extensions of \mathcal{F} by \mathcal{F}' are in correspondence with maps $\Psi, \Omega_1, \dots, \Omega_m$ where

$$\Psi = \begin{pmatrix} 0 & v_0 & 1 & w_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Omega_1 = \begin{pmatrix} 0 & x & 1 & v_1 \\ 0 & 0 & 0 & 1 \\ 1 & y & A'_1 & w_1 \\ 0 & 1 & 0 & A_1 \end{pmatrix} \quad \Omega_k = \begin{pmatrix} 0 & 0 & 1 & v_k \\ 0 & 0 & 0 & 1 \\ 1 & 0 & A'_k & w_k \\ 0 & 1 & 0 & A_k \end{pmatrix}, \quad 2 \leq k \leq m$$

for some $x, y, v_0, \dots, v_m, w_0, \dots, w_m \in \text{End } V$.

To compute $\text{Ext}^1(\mathcal{F}, \mathcal{F}')$, we need to quotient the space of extensions by equivalence. Suppose we have two extensions \mathcal{G} and \mathcal{G}' of \mathcal{F} by \mathcal{F}' . As above, represent these by matrices Ω_k and Ω'_k respectively, determined by $x, y, v_0, \dots, v_m, w_0, \dots, w_m$ and $x', y', v'_0, \dots, v'_m, w'_0, \dots, w'_m$ respectively. We wish to determine when \mathcal{G} and \mathcal{G}' are isomorphic; this happens precisely when there is some isomorphism $\mathcal{G} \rightarrow \mathcal{G}'$ such that the following diagram commutes:

$$(13) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{F} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{F} \longrightarrow 0. \end{array}$$

The map from \mathcal{G} to \mathcal{G}' can be expressed in components as follows:

$$(14) \quad \begin{array}{c} \begin{array}{c} \begin{array}{c} V^2 \xrightarrow{\quad \Upsilon_0 \quad} V^2 \\ \uparrow \Psi \quad \quad \quad \uparrow \Psi' \\ V^4 \xrightarrow{\quad \Theta \quad} V^4 \\ \uparrow \Phi_1 \quad \quad \quad \uparrow \Phi'_1 \\ V^2 \xrightarrow{\quad \Phi_{m+1} \quad} V^2 \quad \quad \quad V^2 \xrightarrow{\quad \Phi'_{m+1} \quad} V^2 \\ \uparrow \Phi_2 \quad \quad \quad \uparrow \Phi'_2 \\ V^2 \quad \quad \quad \dots \quad \quad \quad V^2 \quad \quad \quad \dots \quad \quad \quad V^2 \\ \uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow \\ V^2 \quad \quad \quad \dots \quad \quad \quad V^2 \quad \quad \quad \dots \quad \quad \quad V^2 \end{array} \\ \Upsilon_1 \quad \quad \quad \Upsilon_2 \quad \quad \quad \Upsilon_{m+1} \end{array} \end{array}$$

The commutativity of (13) implies that we have

$$\Theta = \begin{pmatrix} 1 & z_1 & 0 & z_2 \\ 0 & 1 & 0 & 0 \\ 0 & z_3 & 1 & z_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Upsilon_k = \begin{pmatrix} 1 & u_k \\ 0 & 1 \end{pmatrix}, \quad k \geq 0$$

for some $z_1, z_2, z_3, z_4, u_k \in \text{End } V$. The commutativity of (14) implies that $\Upsilon_0 \Psi = \Psi' \Theta$ and the following diagram commutes:

$$\begin{array}{ccccccc} V^2 \oplus V^2 & \xrightarrow{\Omega_m} & \dots & \xrightarrow{\Omega_2} & V^2 \oplus V^2 & \xrightarrow{\Omega_1} & V^4 \\ \Upsilon_m \oplus \Upsilon_{m+1} \downarrow & & & & \Upsilon_1 \oplus \Upsilon_2 \downarrow & & \Theta \downarrow \\ V^2 \oplus V^2 & \xrightarrow{\Omega'_m} & \dots & \xrightarrow{\Omega'_2} & V^2 \oplus V^2 & \xrightarrow{\Omega'_1} & V^4. \end{array}$$

Writing out the commutativity of this diagram and using the above formulas for Θ , Υ_k , Ω_k , and Ω'_k , we get the following equations:

$$(15) \quad \begin{aligned} x' &= x + z_2 & v'_1 &= v_1 - u_2 + z_1 + z_2 A_1 \\ y' &= y - u_1 + z_4 & w'_1 &= w_1 + z_3 - A_1 u_2 + z_4 A_1 \\ v'_0 &= v_0 - z_3 & v'_k &= v_k + u_{k-1} - u_{k+1} \\ w'_0 &= w_0 + u_0 - z_4 & w'_k &= w_k + u_k A_k - A'_k u_{k+1} \end{aligned}$$

(here $k \geq 2$).

We conclude that $\mathcal{G} = (x, y, v_k, w_k)$ and $\mathcal{G}' = (x', y', v'_k, w'_k)$ are isomorphic if and only if there exist z_1, z_2, z_3, z_4, u_k so that (15) holds. Now an inspection of (15) shows that every extension \mathcal{G} is isomorphic to one where $x = y = w_0 = 0$ and $v_k = 0$ for all $k \geq 0$; just choose z_j, u_k appropriately.

Thus for the purposes of calculating $\text{Ext}^1(\mathcal{F}, \mathcal{F}')$, we may restrict ourselves to extensions where $x = y = w_0 = v_k = 0$, which are determined by $(w_1, \dots, w_m) \in (\text{End } V)^m$. Now the extensions (w_1, \dots, w_m) and (w'_1, \dots, w'_m) are isomorphic if and only if there are z_j, u_k such that (15) hold (where $x = y = w_0 = v_k = x' = y' = w'_0 = v'_k = 0$). In this setting $v'_k = v_k + u_{k-1} - u_{k+1}$ gives $u_{k+1} = u_{k-1}$ for all $2 \leq k \leq m$, so that u_k is either u_1 or u_2 depending on the parity of k . The remaining equations in (15) become $z_4 = u_0 = u_1$, $z_1 = u_2$, $z_2 = z_3 = 0$, and $w'_k = w_k + u_k A_k - A'_k u_{k+1}$ for $1 \leq k \leq m$. We conclude the following result.

Proposition 5.3. *Let $\mathcal{F} = (A_1, \dots, A_m)$ and $\mathcal{F}' = (A'_1, \dots, A'_m)$ denote objects in $\text{Sh}_n(\Lambda_m, \mathbb{k})$. Then we have*

$$\text{Ext}^1(\mathcal{F}, \mathcal{F}') \cong (\text{End } V)^m / \{(u_1 A_1 - A'_1 u_2, u_2 A_2 - A'_2 u_1, u_1 A_3 - A'_3 u_2, \dots, u_1 A_m - A'_m u_2) \mid u_1, u_2 \in \text{End } V\}$$

if m is odd, and

$$\text{Ext}^1(\mathcal{F}, \mathcal{F}') \cong (\text{End } V)^m / \{(u_1 A_1 - A'_1 u_2, u_2 A_2 - A'_2 u_1, u_1 A_3 - A'_3 u_2, \dots, u_2 A_m - A'_m u_1) \mid u_1, u_2 \in \text{End } V\}$$

if m is even.

Explicitly, given $(w_1, \dots, w_m) \in (\text{End } V)^m$, we can construct an element of Ext^1 as depicted in (11), by the conditions that $\Psi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\Phi_1, \dots, \Phi_{m+1}$ are given as follows. Define $\Omega_1, \dots, \Omega_m$ by $\Omega_j = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & A'_j & w_j \\ 0 & 1 & 0 & A_j \end{pmatrix}$ for $1 \leq j \leq m$; these then

determine $\Phi_1, \dots, \Phi_{m+1}$ by the condition that the final k arrows in (12) compose to give $\Phi_k \oplus \Phi_{k+1}$.

5.5. Compositions in $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$. We describe now the compositions in the category $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$. In Section 6, we will show that all Ext^i groups vanish for $i > 1$, so we only have to give compositions of the form $u' \circ u$, $u' \circ e$, and $e' \circ u$ for $u \in \text{Ext}^0(\mathcal{F}, \mathcal{F}')$, $u' \in \text{Ext}^0(\mathcal{F}', \mathcal{F}'')$, $e \in \text{Ext}^1(\mathcal{F}, \mathcal{F}')$, and $e' \in \text{Ext}^1(\mathcal{F}', \mathcal{F}'')$.

In the first case this is obvious: the composition in Ext^0 of morphisms

$$\mathcal{F} \xrightarrow{u} \mathcal{F}' \xrightarrow{u'} \mathcal{F}''$$

given by $u = (u_1, u_2)$ and $u' = (u'_1, u'_2)$ is $u' \circ u = (u'_1 u_1, u'_2 u_2)$.

The only non-trivial compositions left involve degree 0 with degree 1. Let $\mathcal{F}, \mathcal{F}'$ and \mathcal{F}'' be three objects of $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$ and let $u : \mathcal{F} \rightarrow \mathcal{F}'$ be a morphism and $e' : \mathcal{F}'' \xrightarrow{i} \mathcal{G} \xrightarrow{p} \mathcal{F}'$ an extension. Then, following [HS71, §III.1], the extension $e' \circ u : \mathcal{F}'' \rightarrow u^* \mathcal{G} \rightarrow \mathcal{F}$ is given by the top row of the diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \curvearrowright & & \\ \mathcal{F}'' & \dashrightarrow & u^* \mathcal{G} & \longrightarrow & \mathcal{F} \\ \parallel & & \downarrow & & \downarrow u \\ \mathcal{F}'' & \xrightarrow{i} & \mathcal{G} & \xrightarrow{p} & \mathcal{F}', \end{array}$$

in which the right square is a pullback square whose universal property gives us the dotted arrow. (Concretely, the map $u^* \mathcal{G} \rightarrow \mathcal{G}$ restricts to an isomorphism

$$\ker(u^* \mathcal{G} \rightarrow \mathcal{F}) \xrightarrow{\sim} \ker(\mathcal{G} \xrightarrow{p} \mathcal{F}') = \text{Im}(\mathcal{F}'' \xrightarrow{i} \mathcal{G}),$$

which we use to lift the map $i : \mathcal{F}'' \rightarrow \mathcal{G}$ to $u^* \mathcal{G}$.)

Now $u \in \text{Ext}^0(\mathcal{F}, \mathcal{F}')$ is determined by $(u_1, u_2) \in (\text{End } V)^2$ as in Proposition 5.2, and $e' \in \text{Ext}^1(\mathcal{F}', \mathcal{F}'')$ is determined by $(w'_1, \dots, w'_m) \in (\text{End } V)^m$ as in Proposition 5.3. To calculate $e' \circ u$, it suffices to calculate the corresponding data (w_1, \dots, w_m) for $e' \circ u$ in terms of (u_1, u_2) and (w'_1, \dots, w'_m) .

The pullback $u^* \mathcal{G}$ is now given by the diagram in Figure 11, where all squares commute. From the definition of pullback, we get that the map U_i is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & u_i \end{pmatrix}$ and W by $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & u_1 \end{pmatrix}$. As in Section 5.4, from the relation $W \circ \Phi_k = \Phi'_k \circ U_k$ we obtain that $W \circ (\Phi_k \oplus \Phi_{k+1}) = (\Phi'_k \oplus \Phi'_{k+1}) \circ (U_k \oplus U_{k+1})$, which implies the commutativity of each square in the following diagram:

$$(16) \quad \begin{array}{ccccccc} V^2 \oplus V^2 & \xrightarrow{\Omega_m} & \dots & \xrightarrow{\Omega_2} & V^2 \oplus V^2 & \xrightarrow{\Omega_1} & V^4 \\ \downarrow U_m \oplus U_{m+1} & & & & \downarrow U_1 \oplus U_2 & & \downarrow W \\ V^2 \oplus V^2 & \xrightarrow{\Omega'_m} & \dots & \xrightarrow{\Omega'_2} & V^2 \oplus V^2 & \xrightarrow{\Omega'_1} & V^4. \end{array}$$

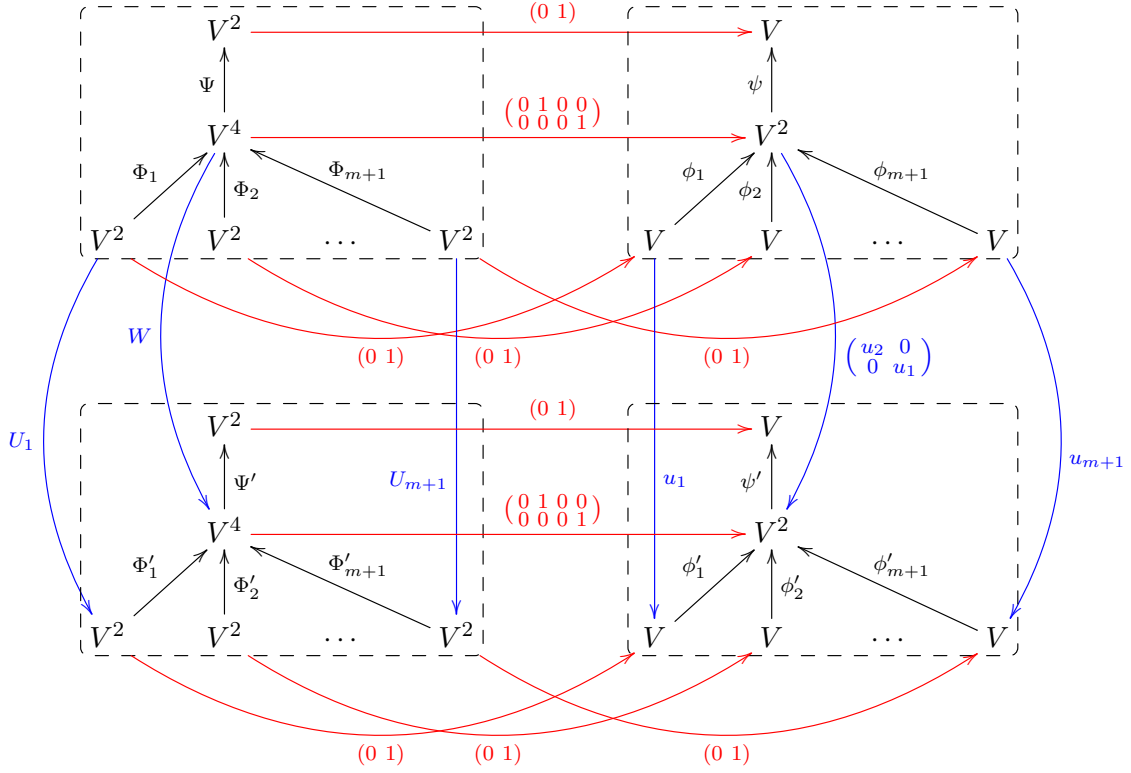


FIGURE 11. The pullback diagram for the morphisms $p : \mathcal{G} \rightarrow \mathcal{F}'$ and $u : \mathcal{F} \rightarrow \mathcal{F}'$.

Here, as in Proposition 5.3, we have $\Omega_k = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & A'_k & w_k \\ 0 & 1 & 0 & A_k \end{pmatrix}$ and $\Omega'_k = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & A'_k & w'_k \\ 0 & 1 & 0 & A'_k \end{pmatrix}$.

Plugging the formulas for $\Omega_k, \Omega'_k, U_k, W$ into the commutative squares in (16) yields $w'_k u_{k+1} = w_k$ for all k . It follows that the composition of the extension $e' = (w'_1, \dots, w'_m)$ with $u = (u_1, u_2)$ is $e' \circ u = (w'_1 u_2, w'_2 u_1, w'_3 u_2, \dots)$.

Similar computations for the pushforward of extensions imply that if $u' = (u'_1, u'_2) : \mathcal{F}' \rightarrow \mathcal{F}''$ is an element of $\text{Ext}^0(\mathcal{F}', \mathcal{F}'')$ and $e = (w_1, \dots, w_m) \in \text{Ext}^1(\mathcal{F}, \mathcal{F}')$, then $u' \circ e = (u'_1 w_1, u'_2 w_2, u'_1 w_3, \dots)$.

6. VANISHING OF Ext^2

Our goal in this section is to prove the following.

Proposition 6.1. *Let Λ be a rainbow braid closure and \mathbb{k} a field, and fix integers $r, s \geq 1$. Given $\mathcal{F} \in \text{Sh}_r(\Lambda, \mathbb{k})$ and $\mathcal{G} \in \text{Sh}_s(\Lambda, \mathbb{k})$, we have $\text{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for all $i \neq 0, 1$.*

Proof. Since Λ carries a binary Maslov potential, such \mathcal{F} and \mathcal{G} are equivalent to their zeroth cohomology sheaves (see the discussion in Section 5.1), so without loss of generality we can assume that \mathcal{F} and \mathcal{G} are honest sheaves of \mathbb{k} -modules.

We will first prove Proposition 6.5, asserting that $H^i(\mathbb{R}^2; \mathcal{H}om(\mathcal{F}, \mathcal{G})) = 0$ for all $i \geq 2$, in Section 6.1. In Section 6.2, we will then prove Proposition 6.6, asserting that $R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \simeq \mathcal{H}om(\mathcal{F}, \mathcal{G})$. Combining these facts, we see that $\text{Ext}^\bullet(\mathcal{F}, \mathcal{G})$ is quasi-isomorphic to

$$R\Gamma(\mathbb{R}^2, R\mathcal{H}om(\mathcal{F}, \mathcal{G})) \simeq R\Gamma(\mathbb{R}^2, \mathcal{H}om(\mathcal{F}, \mathcal{G})),$$

which is the sheaf cohomology of $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ and thus vanishes in degrees $i < 0$ and $i \geq 2$, as desired. \square

6.1. A computation of Čech cohomology. In this subsection we will prove that

$$H^i(\mathbb{R}^2; \mathcal{H}om(\mathcal{F}, \mathcal{G})) = 0 \text{ for all } i \geq 2,$$

where $\mathcal{F} \in \mathcal{S}h_r(\Lambda, \mathbb{k})$ and $\mathcal{G} \in \mathcal{S}h_s(\Lambda, \mathbb{k})$ are sheaves of \mathbb{k} -modules with microsupport on a rainbow braid closure Λ . Letting $\mathcal{H} = \mathcal{H}om(\mathcal{F}, \mathcal{G})$, we will compute the Čech cohomology of \mathcal{H} , using the fact that

$$\varinjlim_{\mathcal{U}} \check{H}^*(\mathcal{U}; \mathcal{H}) \cong H^*(\mathbb{R}^2; \mathcal{H})$$

as \mathcal{U} ranges over open coverings of \mathbb{R}^2 [God58, Théorème II.5.10.1]. It will therefore suffice to show that every \mathcal{U} admits a refinement \mathcal{V} such that $\check{H}^i(\mathcal{V}; \mathcal{H}) = 0$ for all $i \geq 2$.

We explain our construction of \mathcal{V} from a given open cover \mathcal{U} of \mathbb{R}^2 . We tile the plane by hexagons of side length $\epsilon > 0$, arranging by a small isotopy that the front projection of Λ only intersects these hexagons transversely along the midpoints of edges. We let $D \subset \mathbb{R}^2$ be a union of finitely many tiles D_1, \dots, D_n which is topologically a disk and which contains an open neighborhood of the front. By taking the side length ϵ sufficiently small and perturbing Λ slightly, we can also arrange that

- every tile $D_i \subset D$ is contained in some open set $U_i \in \mathcal{U}$;
- every tile contains at most one crossing or cusp;
- the two horizontal edges of a hexagon do not intersect the front;
- if a tile does not contain a crossing, then its boundary intersects the front at most twice, and it does so along consecutive edges if and only if it contains a cusp.

In other words, the tiles can only intersect Λ in the configurations shown in Figure 12, and they fit together to produce a front as in the left side of Figure 13. For some positive $\delta \ll \epsilon$, each tile D_i has a δ -neighborhood $V_i \subset U_i$, so we let

$$\mathcal{V} = \{V_1, \dots, V_n\} \cup \{U \cap (\mathbb{R}^2 \setminus D) \mid U \in \mathcal{U}\},$$

which is an open cover of the plane refining \mathcal{U} .

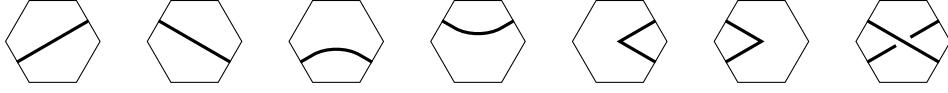


FIGURE 12. The possible ways in which the front for Λ can intersect a hexagonal tile.

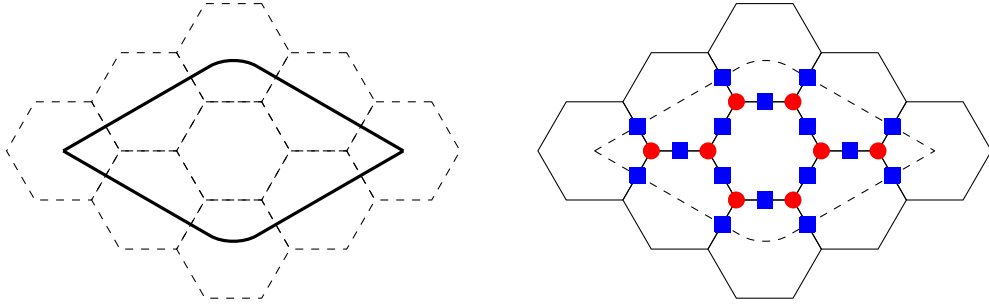


FIGURE 13. Left, the hexagonal tiling used to produce the refinement \mathcal{V} of the open cover \mathcal{U} . Right, the blue squares and red dots contribute generators to $\check{C}^1(\mathcal{V}; \mathcal{H})$ and $\check{C}^2(\mathcal{V}; \mathcal{H})$ respectively.

The Čech complex $\check{C}^*(\mathcal{V}; \mathcal{H})$ now admits a simple description. The sheaves \mathcal{F} and \mathcal{G} are zero outside D , and hence so is $\mathcal{H} = \text{Hom}(\mathcal{F}, \mathcal{G})$. Since arbitrary intersections of open sets which lie in $\mathbb{R}^2 \setminus D$ do not contribute to $\check{C}^*(\mathcal{V}; \mathcal{H})$, we can discard them. The remaining open sets are δ -neighborhoods of the hexagonal tiles D_1, \dots, D_n , which intersect in pairs along a neighborhood of each edge $e_{ij} = D_i \cap D_j$ of the tiling and in triples along a neighborhood of each vertex $v_{ijk} = D_i \cap D_j \cap D_k$, as shown in the right side of Figure 13. There are no nonempty $(k+1)$ -fold intersections in D for $k \geq 3$; hence $\check{C}^k(\mathcal{V}; \mathcal{H}) = 0$ for all such k and we have

$$\check{H}^2(\mathcal{V}; \mathcal{H}) = \text{coker}(d^1 : \check{C}^1(\mathcal{V}; \mathcal{H}) \rightarrow \check{C}^2(\mathcal{V}; \mathcal{H})).$$

Thus it suffices to show that d^1 is surjective.

The map d^1 can be described combinatorially as follows. Each triple intersection

$$V_{ijk} := V_i \cap V_j \cap V_k \ni v_{ijk}$$

lies in a two-dimensional stratum of the plane, so \mathcal{F} and \mathcal{G} and thus $\mathcal{H} = \text{Hom}(\mathcal{F}, \mathcal{G})$ are locally constant on V_{ijk} , from which

$$\mathcal{H}(V_{ijk}) = \text{Hom}(\mathcal{F}(V_{ijk}), \mathcal{G}(V_{ijk})).$$

A nonempty double intersection

$$V_{ij} := V_i \cap V_j$$

along an edge e_{ij} of the tiling contains two triple intersections V_{ijk_1} and V_{ijk_2} corresponding to either endpoint of e_{ij} . (This is a slight abuse of notation if one of the endpoints lies on ∂D , but in this case \mathcal{F} , \mathcal{G} , and \mathcal{H} are zero on any triple intersection

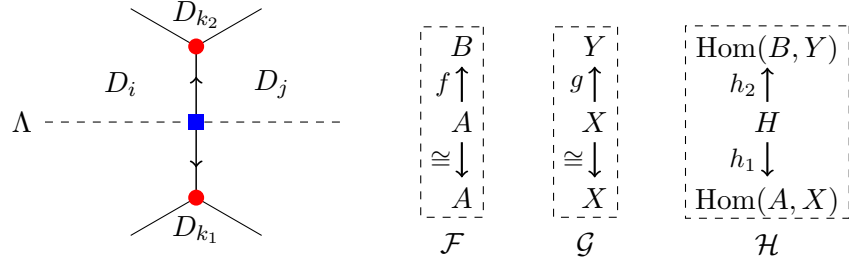


FIGURE 14. A portion of the Čech complex corresponding to the restriction maps $\mathcal{H}(V_{ij}) \rightarrow \mathcal{H}(V_{ijk_1})$, near an edge $e = D_i \cap D_j$ of the tiling which intersects Λ . The diagrams at right represent \mathcal{F} , \mathcal{G} , and \mathcal{H} near e .

near this endpoint and we can ignore it.) The restriction maps $\mathcal{H}(V_{ij}) \rightarrow \mathcal{H}(V_{ijk})$ fall into one of three cases:

- If Λ does not pass through e_{ij} , then \mathcal{F} , \mathcal{G} , \mathcal{H} are all locally constant near e_{ij} and the restriction maps

$$\mathcal{H}(V_{ijk_1}) \leftarrow \mathcal{H}(V_{ij}) \rightarrow \mathcal{H}(V_{ijk_2})$$

are both isomorphisms.

- If a strand of Λ with Maslov potential 0 passes through e_{ij} , then the restriction maps f, g depicted in Figure 14 are injective. In this case the local sections $H = \mathcal{H}(V_{ij})$ are in bijection with pairs $(\varphi : A \rightarrow X, \psi : B \rightarrow Y)$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\psi} & Y \\ f \uparrow & & \uparrow g \\ A & \xrightarrow{\varphi} & X \end{array}$$

commutes. The restriction maps h_1 and h_2 send (φ, ψ) to φ and ψ respectively. Then h_1 is surjective, since every φ can be lifted to some ψ ; and h_2 is injective with image

$$\text{Im}(h_2) = \{\psi : B \rightarrow Y \mid \text{Im}(\psi \circ f) \subset \text{Im}(g)\},$$

since every ψ coming from such a diagram determines the corresponding φ uniquely.

- If a strand of Λ with Maslov potential 1 passes through e_{ij} , then f, g are surjective in Figure 14, so a similar argument says that h_1 is injective with image

$$\text{Im}(h_1) = \{\varphi : A \rightarrow X \mid \ker(f) \subset \ker(g \circ \varphi)\}$$

and h_2 is surjective.

In Figure 15, we illustrate $d^1 : \check{C}^1 \rightarrow \check{C}^2$ for a particular Legendrian unknot Λ (drawn as a dotted curve) and tiling of the plane. Here each blue square along an

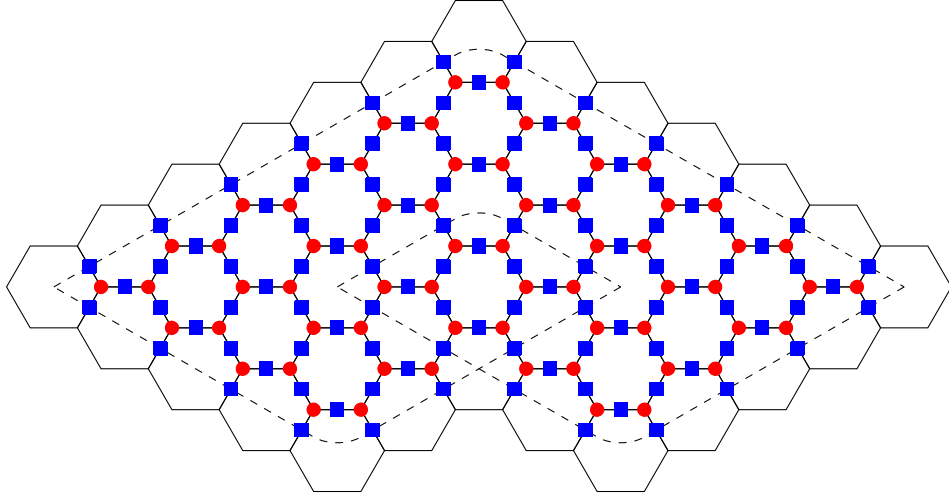


FIGURE 15. The Čech complex in degrees 1 and 2 for a Legendrian knot Λ .

edge e_{ij} contributes $\mathcal{H}(V_{ij})$ to $\check{C}^1(\mathcal{V}; \mathcal{H})$, each red dot at a vertex v_{ijk} contributes $\mathcal{H}(V_{ijk})$ to $\check{C}^2(\mathcal{V}; \mathcal{H})$, and each line segment connecting a blue square to a red dot gives a component $\mathcal{H}(V_{ij}) \rightarrow \mathcal{H}(V_{ijk})$ of the differential d^1 .

We now prove that d^1 is surjective, and hence that $\check{H}^2(\mathcal{V}; \mathcal{H}) = 0$, by playing the following game. We treat the blue squares and red dots in Figure 15 as vertices of a bipartite graph Γ , whose edges are given by the half-edges of hexagons which connect them. We call a blue square a *leaf* if it is adjacent to exactly one red dot. We produce a new graph Γ' by removing one of the following:

- a collection of leaves, or
- some collection of leaves on edges $\{e_{i_j i_l}\}$ and all of the red dots at vertices $\{v_{i_j i_l k_l}\}$ adjacent to them, but only if the corresponding restriction map

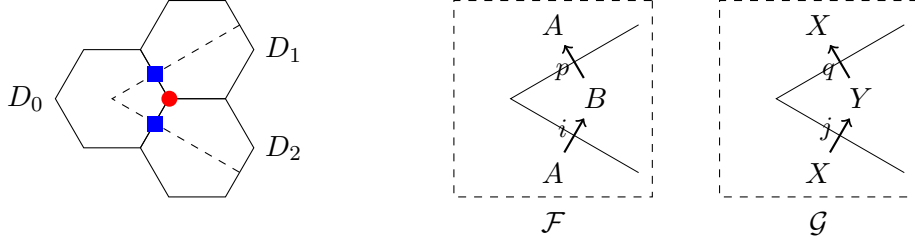
$$\bigoplus_l \mathcal{H}(V_{i_j i_l}) \rightarrow \bigoplus_l \mathcal{H}(V_{i_j i_l k_l})$$

is surjective. (The collection of vertices should be read as a set rather than a list, since some leaves may be adjacent to the same vertex.)

To any such graph Γ , we can associate a map

$$d_\Gamma : C^1(\Gamma) \rightarrow C^2(\Gamma)$$

by letting $C^1(\Gamma)$ be the direct sum of the $\mathcal{H}(V_{ij})$ corresponding to blue squares along edges e_{ij} , letting $C^2(\Gamma)$ be the direct sum of the $\mathcal{H}(V_{ijk})$ over vertices v_{ijk} with red dots, and taking d_Γ to be the sum of all of the corresponding restriction maps $\mathcal{H}(V_{ij}) \rightarrow \mathcal{H}(V_{ijk})$. If we have not yet removed any squares or dots, then d_Γ coincides with the original differential $d^1 : \check{C}^1(\mathcal{V}; \mathcal{H}) \rightarrow \check{C}^2(\mathcal{V}; \mathcal{H})$.

FIGURE 16. A portion of the Čech complex near a cusp of Λ .

Lemma 6.2. *If $d_{\Gamma'}$ is surjective, then so is d_{Γ} .*

Proof. Supposing that we have only removed a leaf corresponding to the edge e_{ij} and adjacent to the vertex e_{ijk} , we can write

$$C^1(\Gamma) = \mathcal{H}(V_{ij}) \oplus C^1(\Gamma'), \quad d_{\Gamma}(s_{ij}, s') = \rho_{ij,ijk}(s_{ij}) + d_{\Gamma'}(s')$$

where $\rho_{ij,ijk} : \mathcal{H}(V_{ij}) \rightarrow \mathcal{H}(V_{ijk})$ is the corresponding restriction map. Assuming that $d_{\Gamma'} = d_{\Gamma}(0, \cdot)$ is surjective, it follows immediately that d_{Γ} is as well. We can repeat this argument to remove as many leaves as we like.

Now suppose instead that we have removed some leaves along edges $\{e_{i_j i_l}\}$ and all of the adjacent red dots, which are at vertices $\{v_{i_j i_l k_i}\}$. Letting $A^1 = \bigoplus_l \mathcal{H}(V_{i_j i_l})$ and $A^2 = \bigoplus_l \mathcal{H}(V_{i_j i_l k_i})$, we can then write

$$d_{\Gamma} = \begin{pmatrix} f & g \\ 0 & d_{\Gamma'} \end{pmatrix} : A^1 \oplus C^1(\Gamma') \rightarrow A^2 \oplus C^2(\Gamma'),$$

in which the component $A^1 \rightarrow C^2(\Gamma')$ is zero because none of the removed blue squares are adjacent to red dots which remain in Γ' . The map $f : A^1 \rightarrow A^2$ is assumed surjective, so if $d_{\Gamma'}$ is also surjective then d_{Γ} is as well. \square

If we can remove vertices from Γ until no red dots remain, the corresponding $d_{\Gamma'}$ will clearly be surjective since its codomain is zero, and it will follow from Lemma 6.2 that $d^1 : \check{C}^1(\mathcal{V}; \mathcal{H}) \rightarrow \check{C}^2(\mathcal{V}; \mathcal{H})$ is surjective. This will establish that $\check{H}^2(\mathcal{V}; \mathcal{H}) = 0$, as desired.

Lemma 6.3. *Let D_0 be a hexagonal tile containing a cusp of Λ , and D_1 and D_2 the adjacent tiles such that Λ intersects the edges e_{01} and e_{02} . Then the component*

$$\mathcal{H}(V_{01}) \oplus \mathcal{H}(V_{02}) \rightarrow \mathcal{H}(V_{012})$$

of the differential $d^1 : \check{C}^1(\mathcal{V}; \mathcal{H}) \rightarrow \check{C}^2(\mathcal{V}; \mathcal{H})$ is surjective.

Proof. Suppose that the sheaves \mathcal{F} and \mathcal{G} are represented by diagrams

$$A \xrightarrow{i} B \xrightarrow{p} A \quad \text{and} \quad X \xrightarrow{j} Y \xrightarrow{q} X$$

near the cusp, as illustrated in Figure 16. Then we have seen that

$$\begin{aligned} \operatorname{Im}(\mathcal{H}(V_{02}) \rightarrow \mathcal{H}(V_{012})) &= \{\psi : B \rightarrow Y \mid \operatorname{Im}(\psi \circ i) \subset \operatorname{Im}(j)\} \\ \operatorname{Im}(\mathcal{H}(V_{01}) \rightarrow \mathcal{H}(V_{012})) &= \{\varphi : B \rightarrow Y \mid \ker(p) \subset \ker(q \circ \varphi)\}. \end{aligned}$$

We therefore wish to show that every $f : B \rightarrow Y$ can be written as a sum $f = \psi + \varphi$, where ψ and φ are as above.

Since $p \circ i : A \rightarrow A$ is an isomorphism and likewise for $q \circ j : X \rightarrow X$, we have direct sum decompositions

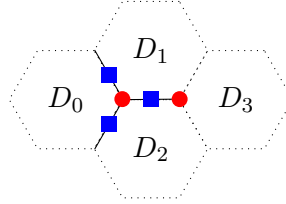
$$B \cong A \oplus K, \quad Y \cong X \oplus L$$

for which $i(a) = (a, 0)$ and $p(a, k) = \alpha(a)$, and $j(x) = (x, 0)$ and $q(x, l) = \beta(x)$ for some automorphisms $\alpha : A \rightarrow A$ and $\beta : X \rightarrow X$. Viewing ψ and φ as maps $A \oplus K \rightarrow X \oplus L$, the conditions $\operatorname{Im}(\psi \circ i) \subset \operatorname{Im}(j)$ and $\ker(p) \subset \ker(q \circ \varphi)$ are equivalent to ψ and φ having block forms

$$\psi = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad \varphi = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix},$$

so it is clear that any $f : B \rightarrow Y$ can be written as such a sum, as desired. \square

Proposition 6.4. *Suppose that Γ contains a subgraph \succ (resembling the letter “Y”, rotated 90 degrees counterclockwise) of the following form:*



consisting of the three blue squares along three edges e_{01} , e_{02} , and e_{12} , and the two red dots at the endpoints v_{012} and v_{123} of the horizontal edge e_{12} . Suppose moreover that the blue squares on e_{01} and e_{02} are both leaves. (We make no claims about whether there are vertices along the dotted edges.) Let $\Gamma' = \Gamma \setminus \succ$ be the subgraph produced by removing these five vertices. If $d_{\Gamma'}$ is surjective, then d_{Γ} is surjective as well.

Proof. Our goal will be to apply Lemma 6.2 to the three vertices on e_{01} , e_{02} , and v_{012} . If we can do this, then the vertex on e_{12} will become a leaf connected only to v_{123} , and the restriction map

$$\rho_{12,123} : \mathcal{H}(V_{12}) \rightarrow \mathcal{H}(V_{123})$$

is automatically an isomorphism since the front projection of Λ does not pass through the horizontal edge e_{12} . In this case, we can apply Lemma 6.2 again to remove the vertices on e_{12} and v_{123} , leaving us with the subgraph Γ' . It will thus suffice to show that the sum of restriction maps

$$\rho_{01,012} + \rho_{02,012} : \mathcal{H}(V_{01}) \oplus \mathcal{H}(V_{02}) \rightarrow \mathcal{H}(V_{012})$$

is surjective.

If the front does not pass through e_{01} , then the restriction map $\rho_{01,012} : \mathcal{H}(V_{01}) \rightarrow \mathcal{H}(V_{012})$ is an isomorphism, and likewise if it avoids e_{02} then $\rho_{02,012} : \mathcal{H}(V_{02}) \rightarrow \mathcal{H}(V_{012})$ is an isomorphism. This establishes the desired surjectivity except when Λ intersects both of the edges e_{01} and e_{02} , in which case D_0 contains either a left cusp of Λ or a crossing.

If D_0 contains a crossing, then since both strands have Maslov potential 0, we have argued that the downward restriction map $\rho_{01,012} : \mathcal{H}(V_{01}) \rightarrow \mathcal{H}(V_{012})$ is surjective. Otherwise D_0 contains a cusp, and then Lemma 6.3 says precisely that $\rho_{01,012} + \rho_{02,012}$ is surjective, as desired. \square

Proposition 6.5. *Let Λ be a rainbow braid closure, and let $\mathcal{F} \in Sh_r(\Lambda)$ and $\mathcal{G} \in Sh_s(\Lambda)$ be sheaves of \mathbb{k} -modules. Then*

$$H^i(\mathbb{R}^2; \mathcal{H}om(\mathcal{F}, \mathcal{G})) = 0$$

for all $i \geq 2$.

Proof. Let $\mathcal{H} = \mathcal{H}om(\mathcal{F}, \mathcal{G})$. As explained at the beginning of this subsection, we have

$$H^*(\mathbb{R}^2; \mathcal{H}om(\mathcal{F}, \mathcal{G})) \cong \varinjlim_{\mathcal{U}} \check{H}^*(\mathcal{U}; \mathcal{H}),$$

and so it suffices to exhibit for any open cover \mathcal{U} of \mathbb{R}^2 a refinement \mathcal{V} such that $\check{H}^i(\mathcal{V}; \mathcal{H}) = 0$ for $i \geq 2$. We produce \mathcal{V} from a tiling of \mathbb{R}^2 by hexagons of sufficiently small side length $\epsilon > 0$, as described above. We have already seen that $\check{C}^i(\mathcal{V}; \mathcal{H}) = 0$ for $i \geq 3$, so we need only show that $d^1 : \check{C}^1(\mathcal{V}; \mathcal{H}) \rightarrow \check{C}^2(\mathcal{V}; \mathcal{H})$ is surjective.

The key observation we need is that in the region $D = \bigcup D_i$ containing the front, every horizontal edge belongs to a unique \succ , and all of these \succ subgraphs are disjoint. We sort the horizontal edges on the interior of D from leftmost to rightmost, breaking ties arbitrarily, and let \succ_1, \dots, \succ_n denote the corresponding \succ subgraphs. (Here we can ignore horizontal edges on the boundary of D , because they lie in the unbounded region of the complement of the front in \mathbb{R}^2 , and hence \mathcal{H} vanishes along them.) We will now apply Proposition 6.4 to remove \succ_1, \dots, \succ_n in order.

Fix $j \leq n$ and suppose that we have removed \succ_i for all $i < j$. Then we claim that the two blue squares on the left end of \succ_j must be leaves. Indeed, labeling the tiles around \succ_j as in Figure 17, suppose that the blue square on e_{13} is not a leaf. Then the red dot at v_{123} must not have been removed yet. It belongs to some \succ_i which must satisfy $i < j$, since e_{12} lies strictly to the left of e_{34} , and which was therefore already removed. But this \succ_i includes the red dot at v_{123} , so we have a contradiction. The blue square on e_{13} is therefore a leaf, and applying the same argument at v_{014} says that the blue square on e_{14} is a leaf as well. We now remove \succ_j and continue by induction.

The end result of this induction is that all of the \succ subgraphs have been removed. Since every red dot lies on a unique horizontal edge, and was thus a vertex of some

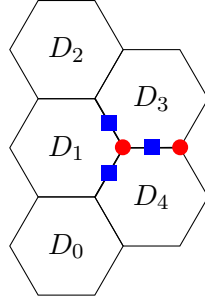


FIGURE 17. A subgraph \succ_j which we wish to remove from the graph Γ .

\succ_i , the resulting graph Γ' has no red vertices, and hence the map $d_{\Gamma'}$ is surjective. This implies by Proposition 6.4 that the original $d_{\Gamma} = d^1 : \check{C}^1(\mathcal{V}; \mathcal{H}) \rightarrow \check{C}^2(\mathcal{V}; \mathcal{H})$ is surjective, as desired. \square

6.2. Vanishing of $R^q \mathcal{H}om(\mathcal{F}, \mathcal{G})$ for $q \geq 1$. The main result of this subsection is the following.

Proposition 6.6. *Let Λ be a rainbow braid closure, and let $\mathcal{F} \in \mathcal{S}h_r(\Lambda, \mathbb{k})$ and $\mathcal{G} \in \mathcal{S}h_s(\Lambda, \mathbb{k})$ be sheaves of \mathbb{k} -modules for some field \mathbb{k} . Then*

$$R^q \mathcal{H}om(\mathcal{F}, \mathcal{G}) = 0$$

for all $q \geq 1$. In other words, $R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \simeq \mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Proposition 6.6 can be checked locally, since for any open $U \subset \mathbb{R}^2$ we have

$$R\mathcal{H}om(\mathcal{F}, \mathcal{G})|_U \simeq R\mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U).$$

Indeed, if $i : U \rightarrow \mathbb{R}^2$ is the inclusion map, and $0 \rightarrow \mathcal{G} \rightarrow \mathcal{I}^\bullet$ is an injective resolution, then i^* is exact and sends injective sheaves to injective sheaves [KS94, Proposition 2.4.1], so $0 \rightarrow \mathcal{G}|_U \rightarrow \mathcal{I}|_U^\bullet$ is also an injective resolution of $\mathcal{G}|_U$. (We use i^* both here and in the sequel for the inverse image functor, as opposed to the notation i^{-1} of [KS94].) Then we have

$$R\mathcal{H}om(\mathcal{F}, \mathcal{G})|_U = \mathcal{H}om(\mathcal{F}, \mathcal{I}^\bullet)|_U \simeq \mathcal{H}om(\mathcal{F}|_U, \mathcal{I}|_U^\bullet) = R\mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U)$$

as claimed.

We thus prove Proposition 6.6 in several steps. We will first prove it for contractible open subsets of 2-dimensional strata in Lemma 6.7, and then for small open balls around points on the interior of strands in Propositions 6.11 and 6.12 (for Maslov potentials 0 and 1 respectively). This reduces the theorem to studying neighborhoods of crossings and cusps, which we then do using a model computation for rainbow braid closures representing a Legendrian unknot. This last computation relies in turn on the vanishing of $H^2(\mathbb{R}^2; \mathcal{H}om(\mathcal{F}, \mathcal{G}))$ established in Proposition 6.5.

We begin with neighborhoods of points in 2-dimensional strata.

Lemma 6.7. *Let \underline{V}_M be the constant sheaf on a connected manifold M whose stalk is a finite-dimensional \mathbb{k} -vector space V , and let \mathcal{F} be any sheaf of \mathbb{k} -modules on M . Then*

$$R\mathcal{H}om(\underline{V}_M, \mathcal{F}) \simeq \mathcal{H}om(\underline{V}_M, \mathcal{F}).$$

In particular, if W is another \mathbb{k} -vector space then $R\mathcal{H}om(\underline{V}_M, \underline{W}_M)$ is the constant sheaf on M with stalk $\text{Hom}(V, W)$.

Proof. If $\dim(V) = 1$ then $\mathcal{H}om(\underline{V}_M, \cdot) \simeq \mathcal{H}om(\underline{\mathbb{k}}_M, \cdot)$ is the identity functor, so its higher derived functors vanish. Otherwise we induct on $\dim(V)$: we pick a 1-dimensional subspace $L \subset V$, with quotient $X = V/L$, and suppose that we have proved the result for \mathbb{k} -vector spaces of dimension less than $\dim(V)$. We apply the contravariant functor $R\mathcal{H}om(\cdot, \mathcal{F})$ to the short exact sequence $0 \rightarrow \underline{L}_M \rightarrow \underline{V}_M \rightarrow \underline{X}_M \rightarrow 0$ to get an exact sequence

$$0 \rightarrow R\mathcal{H}om(\underline{X}_M, \mathcal{F}) \rightarrow R\mathcal{H}om(\underline{V}_M, \mathcal{F}) \rightarrow R\mathcal{H}om(\underline{L}_M, \mathcal{F}) \rightarrow 0.$$

For all $j > 0$ we have $R^j\mathcal{H}om(\underline{X}_M, \mathcal{F}) \cong 0$ and $R^j\mathcal{H}om(\underline{L}_M, \mathcal{F}) \cong 0$ by hypothesis, so by exactness the same is true for $R^j\mathcal{H}om(\underline{V}_M, \mathcal{F})$ and we are done. \square

The following technical lemmas will help us establish the vanishing result near points on 1-dimensional strata, corresponding to strands of Λ .

Lemma 6.8. *Let N be a codimension-0 submanifold of a manifold M , with inclusion map $i : N \hookrightarrow M$, and suppose that $M \setminus N$ is also a manifold of dimension $\dim(M)$. Then $Ri_*\underline{V}_N \simeq i_*\underline{V}_N$ for any \mathbb{k} -vector space V .*

Proof. The higher direct images $R^ji_*\underline{V}_N$ are the sheaves associated to the presheaves $U \mapsto H^j(i^{-1}(U); \underline{V}_N) = H^j(U \cap N; \underline{V}_N)$. If $p \in \overline{N}$, we can take local coordinates (x_1, \dots, x_n) about $p = (0, \dots, 0)$ in which N is either a neighborhood of p or the set $\{x_1 > 0\}$ or its complement; then for any sufficiently small open ball U about p , the intersection $U \cap N$ is contractible and hence $H^j(U \cap N; \underline{V}_N) = 0$ for $j > 0$. Otherwise if $p \notin \overline{N}$ then every sufficiently small neighborhood of p satisfies $U \cap N = \emptyset$ and the same conclusion holds. Thus if $j > 0$ then the sheaf $R^ji_*\underline{V}_N$ must vanish, since its stalks are all zero. \square

Remark 6.9. The hypothesis $\dim(M \setminus N) = \dim(M)$ is necessary in Lemma 6.8: if $M = \mathbb{R}^n$ and $N = \mathbb{R}^n \setminus \{0\}$, and if U is an open ball about the origin, then $H^{n-1}(U \cap N; \underline{\mathbb{k}}_N) \cong H^{n-1}(S^{n-1}; \underline{\mathbb{k}}_N) \cong \mathbb{k}$ and so $R^{n-1}i_*\underline{\mathbb{k}}_N$ is a skyscraper sheaf with stalk \mathbb{k} at 0.

Lemma 6.10. *Let $i : O \hookrightarrow U$ be the inclusion of an open set into U such that $U \setminus O \subset U$ has codimension 0. If \mathcal{G} is a sheaf of \mathbb{k} -modules on U such that the restriction $\mathcal{G}|_O = i^*\mathcal{G}$ is a constant sheaf, then*

$$R\mathcal{H}om(i_*\underline{V}_O, \mathcal{G}) \simeq \mathcal{H}om(i_*\underline{V}_O, \mathcal{G})$$

for any finite-dimensional \mathbb{k} -vector space V .

Proof. We will show more generally that given an open inclusion $i : O \hookrightarrow U$, we have

$$(17) \quad R\mathcal{H}om(i_! \mathcal{F}, \mathcal{G}) \simeq Ri_* R\mathcal{H}om(\mathcal{F}, i^* \mathcal{G})$$

for any sheaves \mathcal{F} on O and \mathcal{G} on U . Since i is an open inclusion, the functor $i_!$ is exact [KS94, Proposition 2.5.4], so that

$$R\mathcal{H}om(i_! \mathcal{F}, \mathcal{G}) \simeq R\mathcal{H}om(Ri_! \mathcal{F}, \mathcal{G}) \simeq Ri_* R\mathcal{H}om(\mathcal{F}, i^! \mathcal{G})$$

where the last adjunction is [KS94, Proposition 3.1.10]. Now we have $i^!(\cdot) \simeq i^* \circ R\Gamma_O(\cdot)$ and $\Gamma_O(\cdot) = i_* \circ i^*$ by [KS94, Proposition 3.1.12] and [KS94, Proposition 2.3.9] respectively; since i^* takes injectives to injectives and is exact, the latter implies that $R\Gamma_O = Ri_* \circ i^*$. Combining all of this and observing that $\mathcal{F} \simeq i^* i_* \mathcal{F}$ gives

$$\begin{aligned} R\mathcal{H}om(i_! \mathcal{F}, \mathcal{G}) &\simeq Ri_* R\mathcal{H}om(i^* i_* \mathcal{F}, i^* \circ R\Gamma_O(\mathcal{G})) \\ &\simeq R\mathcal{H}om(i_* \mathcal{F}, Ri_* \circ i^* \circ R\Gamma_O(\mathcal{G})) \\ &\simeq R\mathcal{H}om(i_* \mathcal{F}, R\Gamma_O \circ R\Gamma_O(\mathcal{G})) \\ &\simeq R\mathcal{H}om(i_* \mathcal{F}, R\Gamma_O(\mathcal{G})) \\ &\simeq R\mathcal{H}om(i_* \mathcal{F}, Ri_* \circ i^* \mathcal{G}) \end{aligned}$$

where the first two rows are related by the adjunction [KS94, Equation (2.6.15)]; and the third and fourth rows can be compared directly or by applying the adjunction [KS94, Equation (2.6.9)] twice, noting that $((i_* \mathcal{F})_O)_O \simeq (i_* \mathcal{F})_O$ by [KS94, Proposition 2.3.6], and then applying the same adjunction once more. Another adjunction gives

$$R\mathcal{H}om(i_! \mathcal{F}, \mathcal{G}) \simeq Ri_* R\mathcal{H}om(i^* i_* \mathcal{F}, i^* \mathcal{G}) \simeq Ri_* R\mathcal{H}om(\mathcal{F}, i^* \mathcal{G})$$

as claimed. Now if $\mathcal{F} = \underline{V}_O$ and $i^* \mathcal{G} \simeq \underline{W}_O$ for some vector space W , then $R\mathcal{H}om(\mathcal{F}, i^* \mathcal{G})$ is the constant sheaf on O with stalk $H = \text{Hom}(V, W)$ by Lemma 6.7, and then Lemma 6.8 says that $R\mathcal{H}om(i_! \underline{V}_O, \mathcal{G}) \simeq Ri_* \underline{H}_O \simeq i_* \underline{H}_O$, as desired. \square

Proposition 6.11. *If U is a sufficiently small neighborhood of a point on the interior of a strand of Λ with Maslov potential 0, then $R\mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U) \simeq \mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U)$.*

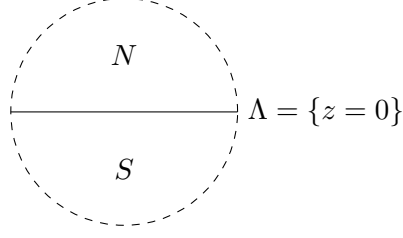
Proof. We identify U with the open unit disk in the xz -plane, consisting of three strata: the open sets $N = \{z > 0\}$ and $S = \{z < 0\}$, and the strand $\{z = 0\}$, as shown in Figure 18. Then $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are represented by quiver representations

$$A \xleftarrow{\sim} A \xrightarrow{f} B \quad \text{and} \quad V \xleftarrow{\sim} V \xrightarrow{g} W,$$

respectively, and f and g are both injective since the Maslov potential is 0. Letting $C = \text{coker}(f)$ and $D = \text{coker}(g)$, we have exact sequences of sheaves on U of the form

$$(18) \quad 0 \rightarrow \underline{A}_U \rightarrow \mathcal{F} \rightarrow i_! \underline{C}_N \rightarrow 0$$

$$(19) \quad 0 \rightarrow \underline{V}_U \rightarrow \mathcal{G} \rightarrow i_! \underline{D}_N \rightarrow 0,$$

FIGURE 18. A neighborhood U of a point on a strand of Λ .

where $i : N \hookrightarrow U$ is the inclusion, and applying $R\mathcal{H}om(\mathcal{F}, \cdot)$ to (19) gives an exact sequence

$$0 \rightarrow R\mathcal{H}om(\mathcal{F}, \underline{V}_U) \rightarrow R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \rightarrow R\mathcal{H}om(\mathcal{F}, i_! \underline{D}_N) \rightarrow 0.$$

We first claim that $R\mathcal{H}om(\mathcal{F}, \underline{V}_U)$ vanishes in positive degree: applying the contravariant functor $R\mathcal{H}om(\cdot, \underline{V}_U)$ to (18) gives an exact sequence

$$0 \rightarrow R\mathcal{H}om(i_! \underline{C}_N, \underline{V}_U) \rightarrow R\mathcal{H}om(\mathcal{F}, \underline{V}_U) \rightarrow R\mathcal{H}om(\underline{A}_U, \underline{V}_U) \rightarrow 0.$$

Applying Lemmas 6.10 and 6.7 to the first and third terms respectively, we see that this is equivalent to

$$0 \rightarrow \mathcal{H}om(i_! \underline{C}_N, \underline{V}_U) \rightarrow R\mathcal{H}om(\mathcal{F}, \underline{V}_U) \rightarrow \mathcal{H}om(\underline{A}_U, \underline{V}_U) \rightarrow 0,$$

so that $R\mathcal{H}om(\mathcal{F}, \underline{V}_U) \simeq \mathcal{H}om(\mathcal{F}, \underline{V}_U)$ as claimed.

We now wish to show that $R^j \mathcal{H}om(\mathcal{F}, i_! \underline{D}_N) = 0$ for $j > 0$, so we apply $R\mathcal{H}om(\cdot, i_! \underline{D}_N)$ to (18) to get

$$0 \rightarrow R\mathcal{H}om(i_! \underline{C}_N, i_! \underline{D}_N) \rightarrow R\mathcal{H}om(\mathcal{F}, i_! \underline{D}_N) \rightarrow R\mathcal{H}om(\underline{A}_U, i_! \underline{D}_N) \rightarrow 0.$$

We can apply Lemma 6.10 to $R\mathcal{H}om(i_! \underline{C}_N, i_! \underline{D}_N)$, since $i_! \underline{D}_N|_N$ is a constant sheaf, and $R\mathcal{H}om(\underline{A}_U, i_! \underline{D}_N)$ vanishes in positive degrees by Lemma 6.7, so the same is true of $R\mathcal{H}om(\mathcal{F}, i_! \underline{D}_N)$ and we are done. \square

Proposition 6.12. *If U is a sufficiently small neighborhood of a point on the interior of a strand of Λ with Maslov potential 1, then $R\mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U) \simeq \mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U)$.*

Proof. Just as in Proposition 6.11, we identify U as the open unit disk with stratification $N \cup \{z = 0\} \cup S$; then \mathcal{F} and \mathcal{G} correspond to quiver representations just as before, except that the Maslov potential is 1 and so f and g are both surjective. Let $K = \ker(f) \subset A$ and $L = \ker(g) \subset V$. Then we have two short exact sequences of sheaves of the form

$$(20) \quad 0 \rightarrow i_* \underline{K}_S \rightarrow \mathcal{F} \rightarrow \underline{B}_U \rightarrow 0$$

$$(21) \quad 0 \rightarrow i_* \underline{L}_S \rightarrow \mathcal{G} \rightarrow \underline{W}_U \rightarrow 0,$$

where $i : S \hookrightarrow U$ is the inclusion map, and applying $R\mathcal{H}om(\mathcal{F}, \cdot)$ to (21) gives

$$0 \rightarrow R\mathcal{H}om(\mathcal{F}, i_* \underline{L}_S) \rightarrow R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \rightarrow R\mathcal{H}om(\mathcal{F}, \underline{W}_U) \rightarrow 0.$$

We have $Ri_*\underline{L}_S \simeq i_*\underline{L}_S$ by Lemma 6.8. We also observe that $i^*\mathcal{F} = \underline{A}_S$, so the adjunction $R\mathcal{H}om(\mathcal{F}, Ri_*\underline{L}_S) \simeq Ri_*R\mathcal{H}om(i^*\mathcal{F}, \underline{L}_S)$ of [KS94, Equation (2.6.15)] becomes

$$R\mathcal{H}om(\mathcal{F}, i_*\underline{L}_S) \simeq Ri_*R\mathcal{H}om(\underline{A}_S, \underline{L}_S) \simeq Ri_*\underline{E}_S \simeq i_*\underline{E}_S,$$

where $E = \text{Hom}(A, L)$, by Lemma 6.7. We will show that $R\mathcal{H}om(\mathcal{F}, \underline{W}_U)$ is supported in degree zero as well, which proves the proposition.

Applying $R\mathcal{H}om(\cdot, \underline{W}_U)$ to (20), we get an exact sequence

$$0 \rightarrow R\mathcal{H}om(\underline{B}_U, \underline{W}_U) \rightarrow R\mathcal{H}om(\mathcal{F}, \underline{W}_U) \rightarrow R\mathcal{H}om(i_*\underline{K}_S, \underline{W}_U) \rightarrow 0.$$

Again $R\mathcal{H}om(\underline{B}_U, \underline{W}_U)$ vanishes in positive degree by Lemma 6.7, so it suffices to show that $R\mathcal{H}om(i_*\underline{K}_S, \underline{W}_U)$ does as well. Letting $j : N \hookrightarrow U$ denote the inclusion map, we apply $R\mathcal{H}om(\cdot, \underline{W}_U)$ to the short exact sequence $0 \rightarrow j_!\underline{K}_N \rightarrow \underline{K}_U \rightarrow i_*\underline{K}_S \rightarrow 0$ to get another short exact sequence

$$0 \rightarrow R\mathcal{H}om(i_*\underline{K}_S, \underline{W}_U) \rightarrow \mathcal{H}om(\underline{K}_U, \underline{W}_U) \rightarrow \mathcal{H}om(j_!\underline{K}_N, \underline{W}_U) \rightarrow 0,$$

where in each of the last two terms we can replace $R\mathcal{H}om$ with $\mathcal{H}om$ by Lemmas 6.7 and 6.10 respectively. It follows that $R^q\mathcal{H}om(i_*\underline{K}_S, \underline{W}_U) = 0$ for $q \geq 2$, and that the sequence

$$\mathcal{H}om(\underline{K}_U, \underline{W}_U) \rightarrow \mathcal{H}om(j_!\underline{K}_N, \underline{W}_U) \rightarrow R^1\mathcal{H}om(i_*\underline{K}_S, \underline{W}_U) \rightarrow 0$$

is exact; but the first map in this last sequence is surjective, as can easily be seen on stalks, so $R^1\mathcal{H}om(i_*\underline{K}_S, \underline{W}_U) = 0$ as well and we are done. \square

Using these preliminary results, we can now prove Proposition 6.6.

Proof of Proposition 6.6. We have seen that the desired result can be checked locally, and Lemma 6.7 and Propositions 6.11 and 6.12 collectively show that for all $q \geq 1$, the sheaf $R^q\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a direct sum of skyscraper sheaves supported on the crossings and cusps of Λ . This implies that in the local-to-global Ext spectral sequence

$$E_2^{p,q} = H^p(\mathbb{R}^2; R^q\mathcal{H}om(\mathcal{F}, \mathcal{G})) \implies \text{Ext}^{p+q}(\mathcal{F}, \mathcal{G})$$

of [God58, Théorème II.7.3.3], all of the groups $E_2^{p,q}$ with $p, q \geq 1$ must vanish. We have also shown that $E_2^{p,0} = H^p(\mathbb{R}^2; \mathcal{H}om(\mathcal{F}, \mathcal{G}))$ vanishes for all $p \geq 2$, by Proposition 6.5. Since the E_2 page is supported on the union of the line $p = 0$ and the point $(p, q) = (1, 0)$, the sequence collapses at this page.

If Λ is the rainbow closure of an n -stranded braid, then the standard Legendrian unknot can be realized as the closure Λ_0 of the n -stranded braid $\sigma_1\sigma_2 \dots \sigma_{n-1}$, where each σ_i exchanges the i th and $(i+1)$ st strands from the top. If U is a small neighborhood of a cusp of Λ , or a crossing of Λ corresponding to the braid generator σ_k , then we let U_0 be the analogous neighborhood of the corresponding left or right cusp or σ_k -crossing of Λ_0 . It is easy to construct sheaves of \mathbb{k} -modules $\mathcal{F}_0 \in \mathcal{S}h_r(\Lambda_0, \mathbb{k})$ and $\mathcal{G}_0 \in \mathcal{S}h_s(\Lambda_0, \mathbb{k})$ such that $\mathcal{F}_0|_{U_0}$ and $\mathcal{G}_0|_{U_0}$ are identical to $\mathcal{F}|_U$ and $\mathcal{G}|_U$ respectively; since $R\mathcal{H}om(\mathcal{F}, \mathcal{G})|_U \simeq R\mathcal{H}om(\mathcal{F}_0, \mathcal{G}_0)|_{U_0}$, we will study the

latter in order to determine the stalks of each $R^q\mathcal{H}om(\mathcal{F}, \mathcal{G})$ at the cusp or crossing in question.

Let Λ_{std} be the standard eye-shaped Legendrian unknot diagram, and note that $\mathcal{S}h_r(\Lambda_0, \mathbb{k}) \cong \mathcal{S}h_r(\Lambda_{\text{std}}, \mathbb{k})$ and $\mathcal{S}h_s(\Lambda_0, \mathbb{k}) \cong \mathcal{S}h_s(\Lambda_{\text{std}}, \mathbb{k})$ since Λ_0 and Λ_{std} are Legendrian isotopic; let $\mathcal{F}_1 \in \mathcal{S}h_r(\Lambda_{\text{std}}, \mathbb{k})$ and $\mathcal{G}_1 \in \mathcal{S}h_s(\Lambda_{\text{std}}, \mathbb{k})$ be sheaves corresponding to \mathcal{F}_0 and \mathcal{G}_0 . The functor $\mathcal{S}h(\Lambda_{\text{std}}, \mathbb{k}) \rightarrow D(\mathbb{k}\text{-mod})$ sending a sheaf to its stalk inside the eye is an equivalence of categories [STZ17, Example 3.5], so we have $\text{Ext}^\bullet(\mathcal{F}_1, \mathcal{G}_1) = \text{Ext}^\bullet(\mathbb{k}^r, \mathbb{k}^s) = \text{Hom}(\mathbb{k}^r, \mathbb{k}^s) \cong \mathbb{k}^{rs}$. We conclude that $\text{Ext}^i(\mathcal{F}_0, \mathcal{G}_0) = 0$ for $i \geq 1$.

We now apply the above discussion of the Ext spectral sequence to the computation of $\text{Ext}^i(\mathcal{F}_0, \mathcal{G}_0)$. Since we know that these groups vanish for $i \geq 1$, examining the $E_2 = E_\infty$ page tells us that

$$H^0(\mathbb{R}^2; R^q\mathcal{H}om(\mathcal{F}_0, \mathcal{G}_0)) = E_2^{0,q} = 0$$

for all $q \geq 1$. But $R^q\mathcal{H}om(\mathcal{F}_0, \mathcal{G}_0)$ is a sum of skyscraper sheaves for $q \geq 1$, so its zeroth cohomology is just the direct sum of its stalks at each cusp and crossing. Thus the stalks of $R^q\mathcal{H}om(\mathcal{F}_0, \mathcal{G}_0)$ all vanish for $q \geq 1$, and this implies as above that

$$R^q\mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U) \simeq R^q\mathcal{H}om(\mathcal{F}_0|_{U_0}, \mathcal{G}_0|_{U_0}) = 0.$$

So we must have $R^q\mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U) = 0$ for all $q \geq 1$, and since this is true for a neighborhood U of any cusp or crossing, we can conclude that $R^q\mathcal{H}om(\mathcal{F}, \mathcal{G}) = 0$ for all $q \geq 1$, as desired. \square

7. CONSTRUCTION OF THE EQUIVALENCE

We are now ready to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. We want to find an equivalence functor

$$F : H^*(\mathcal{R}ep_n(\Lambda_m, \mathbb{k})) \rightarrow H^*(\mathcal{S}h_n(\Lambda_m, \mathbb{k}))$$

which is fully faithful and essentially surjective.

We first define F on objects. To a representation described by (A_1, \dots, A_m) one associates the sheaf where $V = \mathbb{k}^n$ and the maps are given by (A_1^T, \dots, A_m^T) . (Note that by equations (3) and (4) we have

$$P_m(A_1, \dots, A_m)^T = P_m(A_m^T, \dots, A_1^T) = (-1)^m Q_m(A_1, \dots, A_m),$$

and thus Proposition 3.2 implies that this is indeed an object of $\mathcal{S}h_n(\Lambda_m, \mathbb{k})$.) Equation (5) and Proposition 5.1 imply that F is essentially surjective on objects.

Then to each morphism $(u_1, u_2) : (A_1, \dots, A_m) \rightarrow (A'_1, \dots, A'_m)$, we associate the morphism $-(u_2^T, u_1^T)$. The first part of Proposition 3.4 together with Proposition 5.2 implies that this is a well defined isomorphism on Hom^0 .

To a class (w_1, \dots, w_m) in $H^1 \text{Hom}(\rho, \rho')$, we associate the class of extensions given by (w_1^T, \dots, w_m^T) . The second part of Proposition 3.4 and Proposition 5.3 show that

this is a well defined isomorphism on $H^1 \text{Hom}$. By Propositions 3.4 and 6.1, all other $H^i \text{Hom}$ groups ($i \neq 0, 1$) in both categories are zero, so F defines an isomorphism on all of $H^* \text{Hom}$.

It now follows from the computation in Section 5.5 and equations (6), (7) and (8) that the composition rule is preserved. Hence F is a functor, and so it defines an equivalence of categories. \square

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