Classical and Modern Formulations of Curvature

Lenny Ng
Mathematics 230a

January 1995

1 Introduction

In this paper, we will look at two different notions of curvature, one from a classical standpoint and one from a modern standpoint. These two notions intersect in the concept of the Gaussian curvature of a two-dimensional surface imbedded in $\Re^3$. After briefly surveying the relevant classical and modern definitions and results, we present our main result, that the sectional curvature of a two-dimensional manifold is nothing more than the Gaussian curvature. We give four proofs of this result from four different standpoints. The first relies on the classical concept of a connection form; the second uses the classical shape operator; the third depends on local formulas for Christoffel symbols and curvature; the fourth applies a computational approach to a classical formula of Gauss.

2 Classical Formulation

Classically, we define the *directional* or *covariant derivative* of a vector field Y in $\Re^{n+1}$ with respect to a vector x_p to be

$$\nabla_{x_p} Y = (Y \circ \alpha)'(0),$$

where α is any curve in $\Re^{n+1}$ with $\alpha(0) = p$ and $\alpha'(0) = x_p$. It is straightforward to check that ∇ satisfies the following properties:

(1) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$
(2) $\nabla_{fX} Y = f\nabla_X Y$
(3) $\nabla_X Y - \nabla_Y X = [X, Y]$
(4) $X(Y \cdot Z) = \nabla_X Y \cdot Z + Y \cdot \nabla_X Z$.
(For instance, (4) is simply the Leibniz rule for dot products; to verify (3), note that both sides obey the Leibniz rule and that (3) is true for $X = \partial_i, Y = \partial_j$.)

Now consider an n-dimensional surface M embedded in $\Re^{n+1}$. We define the *shape operator* or *Weingarten map* $S_p : T_pM \to T_pM$ as follows:

$$S_p(v) = -\nabla_v(U).$$

Note that $S_p(v) \in T_pM$ since $2(U \cdot \nabla_v(U)) = \nabla_v(U \cdot U) = \nabla_v(1) = 0$. Thus S_p is a linear transformation on T_pM, and we may define $\det S_p = K(p)$ to be the *Gauss-Kronecker curvature* of M at p. The *mean curvature* $H(p)$ is similarly defined to be $\frac{1}{n}$ trace S_p.

Since U is only well defined up to a sign, the Gauss-Kronecker and mean curvature are in general not well defined. In the special case when n is even, however, $\det S_p = \det(-S_p)$, so the Gauss-Kronecker curvature is well defined; when $n = 2$, the most important case classically, this is known as the *Gaussian curvature*.

One way to compute the Gaussian curvature is through the connection forms and structural equations (see, e.g., [2, p.312]). Let E_1, E_2 be a *frame field* around a point p on a two-dimensional surface M, that is, a pair of vector fields so that $E_1(q), E_2(q)$ is an orthonormal basis for T_qM for each q in a neighborhood of p. The *connection forms* ω_{ij} are the 1-forms defined by

$$\omega_{ij}(v) = \nabla_v E_i \cdot E_j(p).$$

Note that

$$\omega_{ij}(v) + \omega_{ji}(v) = \nabla_v E_i \cdot E_j + \nabla_v E_j \cdot E_i = v(E_i \cdot E_j) = 0,$$

so that $\omega_{11} = \omega_{22} = 0$ and $\omega_{12} = -\omega_{21}$. Let θ_1, θ_2 be the 1-forms dual to the frame field $\{E_1, E_2\}$, so that $\theta_i(v) = v \cdot E_i(p)$. The *first structural equations* state that

$$d\theta_1 = \omega_{12} \wedge \theta_2,$$

$$d\theta_2 = \omega_{21} \wedge \theta_1;$$

one form of the *second structural equation* gives us a way to compute Gaussian curvature:

$$d\omega_{12} = -K\theta_1 \wedge \theta_2.$$

3 Modern Formulation

Recall several definitions from the modern formulation of curvature.

The Levi-Civita connection on a Riemannian manifold (M, g) is the unique $\Re$-bilinear map $D : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ such that

(1) $D_X(fY) = (Xf)Y + fD_XY$

(2) $D_{fX}Y = fD_XY$

(3) $D_XY - D_YX = [X, Y]$

(4) $Xg(Y, Z) = g(D_XY, Z) + g(Y, D_XZ)$.

Note that we use D instead of ∇ here to avoid confusion with the covariant derivative of Section 2.

The curvature tensor $R \in T^{1,3}M$ is given by

$$R(x, y)z = D_y(D_xz) - D_x(D_yz) + D_{[x,y]}z;$$

this produces a function $R \in T^{0,4}M$ defined by

$$R(x, y, z, w) = g(R(x, y)z, w).$$

The sectional curvature $K(x, y)$ of M with respect to the plane spanned by $x, y \in T_pM$ is defined to be

$$K_p(x, y) = \frac{R(x, y, x, y)}{g(x, x)g(y, y) - g(x, y)^2};$$

this is independent of the choice of basis x, y for this particular 2-plane. Two other notions of curvature are Ricci and scalar curvature; if e_i is an orthonormal basis for T_pM, then

$$Ric_p(x, y) = \sum_i R(x, e_i, y, e_i)$$

$$Scal(p) = \sum_{i,j} R(e_i, e_j, e_i, e_j).$$

Also recall the following local definitions and formulas. If $\{x_1, \ldots, x_n\}$ is a local coordinate system around $p \in M$, we write $\partial_i = \partial/\partial x_i$ as the canonical basis for T_pM. The Christoffel symbols (of the second kind) Γ_{ij}^k are defined by $D_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k$; the Christoffel formulas are

$$\Gamma_{ij}^k = \frac{1}{2}\sum_l g^{kl}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}).$$

Also, if we write $R(\partial_i, \partial_j)\partial_k = \sum_l R_{ijk}^l \partial_l$, then

$$R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \sum_p (\Gamma_{jk}^p \Gamma_{ip}^l - \Gamma_{ik}^p \Gamma_{jp}^l).$$

Now let M be an n-dimensional orientable surface embedded in $\Re^{n+1}$ with unit normal vector field U, and consider the classical case in which the scalar product $g(x, y)$ is the usual dot product $x \cdot y$, i.e., endow M with the metric inherited from $\Re^{n+1}$. We wish to compute the Levi-Civita connection on M. Let ∇ be the covariant derivative defined previously; we define $D : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ to be the tangential component of ∇:

$$D_{x_p}Y = \nabla_{x_p}Y - (\nabla_{x_p}Y \cdot U)U.$$

By taking tangential components of the properties of ∇ from Section 2, it is straightforward to check that D satisfies the conditions to be a Levi-Civita connection. We then define the curvature tensor and the various notions of curvature from this Levi-Civita connection.

4 Gaussian Curvature

In the special case in which M is a 2-dimensional surface, the sectional curvature $K_p(x, y) = K_p$ is the same for all $x, y \in T_pM$. Also, if $\{E_1, E_2\}$ is a frame field (a set of orthonormal vector fields) around p, and $x = x_1E_1 + x_2E_2$, $y = y_1E_1 + y_2E_2$, then

$$\begin{aligned} Ric_p(x, y) &= R(x_2E_2, E_1, y_2E_2, E_1) + R(x_1E_1, E_2, y_1E_1, E_2) \\ &= (x_1y_1 + x_2y_2)K_p(E_1, E_2) \\ &= g(x, y)K_p(E_1, E_2), \end{aligned}$$

and

$$Scal(p) = R(E_1, E_2, E_1, E_2) + R(E_2, E_1, E_2, E_1) = 2K_p(E_1, E_2).$$

Our main result tying together the classical and modern formulations of curvature will be that the sectional curvature on a 2-surface is simply the Gaussian curvature.

Theorem *For a 2-surface M, the sectional curvature $K_p(x, y)$ is equal to the Gaussian curvature $K(p)$.*

In the four subsequent sections, we will present four different proofs of this theorem; they are roughly in order from most global to most local.

Note that the theorem implies that $Ric_p(x, y) = g(x, y)K(p)$ and $Scal(p) = 2K(p)$. Myers's Theorem, which involves Ricci curvature, and Synge's Theorem, which involves sectional curvature, thus reduce to the following results.

Corollary 1 *Let M be a complete two-dimensional manifold, and suppose that there exists an $r > 0$ such that $K(p) \geq r^{-2}$ for all $p \in M$. Then* diam $M \leq$ diam $(S^2(r))$. *In particular, M is compact with finite fundamental group.*

Note that $K \equiv 1/r^2$ on $S^2(r)$.

Corollary 2 *If M is a compact orientable two-dimensional manifold such that $K(p) > 0$ for all $p \in M$, then M has genus zero.*

This last corollary is also implied by the Gauss-Bonnet Theorem.

5 First Proof

I could not find a reference for this proof.

We will use the connection forms and the structural equations from Section 2. Let M be a two-dimensional surface in $\Re^3$ and let E_1, E_2 be a frame field on M, with θ_1, θ_2 its dual basis of 1-forms; define the connection forms ω_{ij} from this frame field. Note that $\omega_{ij}(v) = \nabla_v E_i \cdot E_j = D_v E_i \cdot E_j$ since $U \cdot E_j = 0$. For the rest of this section, we will be using computations made entirely on the surface itself; since we will thus not need covariant derivatives, we will denote the Levi-Civita connection by ∇, as is customary, rather than by D. (In the subsequent proofs, however, we will be using both covariant derivative and Levi-Civita connection; they will be denoted, as before, by ∇ and D, respectively.)

We need the following lemma, which expresses Gaussian curvature explicitly in terms of the connection form ω_{12}.

Lemma 1 $K(p) = E_2(\omega_{12}(E_1)) - E_1(\omega_{12}(E_2)) - (\omega_{12}(E_1))^2 - (\omega_{12}(E_2))^2$.

Proof. A simple calculation shows that we can write ω_{12} as

$$\omega_{12} = \omega_{12}(E_1)\theta_1 + \omega_{12}(E_2)\theta_2.$$

Likewise, we have

$$\begin{aligned} d(\omega_{12}(E_1)) &= d(\omega_{12}(E_1))(E_1)\theta_1 + d(\omega_{12}(E_2))(E_2)\theta_2 \\ &= E_1(\omega_{12}(E_1))\theta_1 + E_2(\omega_{12}(E_1))\theta_2, \end{aligned}$$

with a similar expression for $D(\omega_{12}(E_2))$. Hence

$$\begin{aligned} d\omega_{12} &= d(\omega_{12}(E_1)) \wedge \theta_1 + d(\omega_{12}(E_2)) \wedge \theta_2 + \omega_{12}(E_1)d\theta_1 + \omega_{12}(E_2)d\theta_2 \\ &= E_2(\omega_{12}(E_1))\theta_2 \wedge \theta_1 + E_1(\omega_{12})(E_2))\theta_1 \wedge \theta_2 + \omega_{12}(E_1)\omega_{12} \wedge \theta_2 - \\ &\quad -\omega_{12}(E_2)\omega_{12} \wedge \theta_1 \\ &= (-E_2(\omega_{12}(E_1)) + E_1(\omega_{12}(E_2)) + (\omega_{12}(E_1))^2 + (\omega_{12}(E_2))^2)\theta_1 \wedge \theta_2, \end{aligned}$$

where we have used the first structural equations to evaluate $d\theta_1$ and $d\theta_2$; the lemma then follows from the second structural equation. □

Proof of Theorem. Since $\nabla_{E_1}E_1 \cdot E_1 = \omega_{11}(E_1) = 0$ and $\nabla_{E_2}E_2 \cdot E_2 = \omega_{22}(E_2) = 0$, $\nabla_{E_1}E_1$ is a multiple of E_2 while $\nabla_{E_2}E_2$ is a multiple of E_1, so $(\nabla_{E_1}E_1) \cdot (\nabla_{E_2}E_2) = 0$. Similarly, $(\nabla_{E_2}E_1) \cdot (\nabla_{E_1}E_2) = 0$; also note that

$$\begin{aligned} [E_1, E_2] &= \nabla_{E_1}E_2 - \nabla_{E_2}E_1 \\ &= ((\nabla_{E_1}E_2 - \nabla_{E_2}E_1) \cdot E_1)E_1 + ((\nabla_{E_1}E_2 - \nabla_{E_2}E_1) \cdot E_2)E_2 \\ &= (\omega_{21}(E_1) - \omega_{11}(E_2))E_1 + (\omega_{22}(E_1) - \omega_{12}(E_2))E_2 \\ &= -\omega_{12}(E_1)E_1 - \omega_{12}(E_2)E_2. \end{aligned}$$

Hence the sectional curvature is equal to

$$\begin{aligned}
K_p(E_1, E_2) &= R(E_1, E_2)E_1 \cdot E_2 \\
&= \nabla_{E_2}\nabla_{E_1}E_1 \cdot E_2 - \nabla_{E_1}\nabla_{E_2}E_1 \cdot E_2 + \nabla_{[E_1,E_2]}E_1 \cdot E_2 \\
&= (\nabla_{E_2}\nabla_{E_1}E_1 \cdot E_2 + (\nabla_{E_1}E_1) \cdot (\nabla_{E_2}E_2)) - \\
&\quad -(\nabla_{E_1}\nabla_{E_2}E_1 \cdot E_2 + (\nabla_{E_2}E_1) \cdot (\nabla_{E_1}E_2)) + \nabla_{[E_1,E_2]}E_1 \cdot E_2 \\
&= E_2(\nabla_{E_1}E_1 \cdot E_2) - E_1(\nabla_{E_2}E_1 \cdot E_2) - \omega_{12}(E_1)\nabla_{E_1}E_1 \cdot E_2 - \\
&\quad -\omega_{12}(E_2)\nabla_{E_2}E_1 \cdot E_2 \\
&= E_2(\omega_{12}(E_1)) - E_1(\omega_{12}(E_2)) - (\omega_{12}(E_1))^2 - (\omega_{12}(E_2))^2 \\
&= K(p),
\end{aligned}$$

as desired. □

6 Second Proof

This proof is adapted from the method in [4, pp. 227-230].

Let M be an orientable n-dimensional manifold embedded in $\Re^{n+1}$ with unit normal vector field U; we will specialize later to the case $n = 2$. Again, we will use ∇ and D to represent the covariant derivative and the Levi-Civita connection, respectively. Define $R' \in T^{1,3}(M)$ by

$$R'(x, y)z = (S_p(x) \cdot z)S_p(y) - (S_p(y) \cdot z)S_p(x).$$

Recall that $R \in T^{1,3}(M)$ is defined by

$$R(x, y)z = D_y(D_x z) - D_x(D_y z) + D_{[x,y]}z.$$

Let $\{\partial_1, \ldots, \partial_n\}$ be the canonical basis for T_pM. We will need a lemma; note that this lemma implies that $R(x, y, z) = R'(x, y, z)$ for *all* $x, y, z \in TM$, so that we have, as a bonus, a formula for the curvature tensor in terms of the classical shape operator.

Lemma 2 *If $z \in T_pM$, then $R'(\partial_i, \partial_j, z) = R(\partial_i, \partial_j, z)$.*

Proof. Since $(\nabla_{\partial_j} z) \cdot U + (\nabla_{\partial_j} U) \cdot z = U \cdot z = 0$, we have

$$\begin{aligned}
\nabla_{\partial_i}\nabla_{\partial_j} z &= \nabla_{\partial_i}(D_{\partial_j} z + ((\nabla_{\partial_j} z) \cdot U)U) \\
&= D_{\partial_i} D_{\partial_j} z + ((\nabla_{\partial_i}(D_{\partial_j} z)) \cdot U)U + \nabla_{\partial_i}(((\nabla_{\partial_j} z) \cdot U))U \\
&= D_{\partial_i} D_{\partial_j} z + ((\nabla_{\partial_i}(D_{\partial_j} z)) \cdot U)U - ((\nabla_{\partial_j} U) \cdot z)\nabla_{\partial_i} U \\
&= D_{\partial_i} D_{\partial_j} z + ((\nabla_{\partial_i}(D_{\partial_j} z)) \cdot U)U - ((S_p \partial_j) \cdot z) S_p \partial_i .
\end{aligned}$$

Now the first and third terms of this expression are in T_pM, while the second is normal to the surface, so the tangential component of $\nabla_{\partial_i}\nabla_{\partial_j} z$ is $D_{\partial_i} D_{\partial_j} z - ((S_p\partial_j) \cdot z)S_p\partial_i$. Similarly, the tangential component of $\nabla_{\partial_j}\nabla_{\partial_i} z$ is $D_{\partial_j} D_{\partial_i} z - ((S_p\partial_i) \cdot z)S_p\partial_j$. But by definition, we have that

$$\nabla_{\partial_j} z = (z \circ \alpha)'(0) = \frac{\partial z}{\partial x_j}$$

where $\alpha'(0) = \partial_j$, so

$$\nabla_{\partial_i}\nabla_{\partial_j} z = \frac{\partial^2}{\partial x_i \partial x_j} z = \frac{\partial^2}{\partial x_j \partial x_i} z = \nabla_{\partial_j}\nabla_{\partial_i} z;$$

taking tangential components and rearranging, we conclude that

$$\begin{aligned}
D_{\partial_i} D_{\partial_j} z - D_{\partial_j} D_{\partial_i} z &= ((S_p\partial_i) \cdot z)S_p\partial_j - ((S_p\partial_j) \cdot z)S_p\partial_i \\
&= R'(\partial_i, \partial_j) z.
\end{aligned}$$

Finally, since $[\partial_i, \partial_j] = 0$, we obtain

$$R(\partial_i, \partial_j) z = D_{\partial_i} D_{\partial_j} z - D_{\partial_j} D_{\partial_i} z,$$

whence the lemma. □

Proof of Theorem. Let M have dimension 2. Since $K(x, y)$ is independent of the basis x, y chosen, it suffices to show that $K_p(\partial_1, \partial_2) = K(p)$. But by Lemma 2,

$$\begin{aligned}
(g_{11}g_{22} - g_{12}^2)K_p(\partial_1, \partial_2) &= R(\partial_1, \partial_2)\partial_1 \cdot \partial_2 \\
&= R'(\partial_1, \partial_2)\partial_1 \cdot \partial_2 \\
&= (S_p(\partial_1) \cdot \partial_1)(S_p(\partial_2) \cdot \partial_2) - (S_p(\partial_2) \cdot \partial_1)(S_p(\partial_1) \cdot \partial_2) \\
&= (g_{11}g_{22} - g_{12}^2) \det S_p \\
&= (g_{11}g_{22} - g_{12}^2)K(p);
\end{aligned}$$

the theorem follows. □

7 Third Proof

This proof is essentially a version of the previous proof in local coordinates. In the course of the proof, we obtain and use the so-called Gauss's formulas.

Proof of Theorem. We loosely follow the notation of [1]. As in the previous proof, we first assume that M is n-dimensional embedded in $\Re^{n+1}$; later we will set $n = 2$.

Let $b_{jk} = S_p(\partial_j) \cdot \partial_k$. We wish to express $\nabla_{\partial_j} \partial_k$ in terms of the ∂_l's and of U. Now the tangential component of $\nabla_{\partial_j} \partial_k$ is given by $D_{\partial_j} \partial_k = \sum_l \Gamma^l_{jk} \partial_l$; the normal component is $\nabla_{\partial_j} \partial_k \cdot U = \partial_j(\partial_k \cdot U) - \nabla_{\partial_j} U \cdot \partial_k = S_p(\partial_j) \cdot \partial_k = b_{jk}$. Hence

$$\nabla_{\partial_j} \partial_k = \sum_l \Gamma^l_{jk} \partial_l + b_{jk} U;$$

in [1], these equations are known as Gauss's formulas. Next, from the formulas $S_p(\partial_i) \cdot \partial_p = b_{ip}$ and the fact that $S_p(\partial_i) \in T_pM$, we conclude that

$$-\nabla_{\partial_i} U = S_p(\partial_i) = \sum_{l,p} b_{ip} g^{pl} \partial_l.$$

Now, as in the previous proof, we know that

$$\nabla_{\partial_i} \nabla_{\partial_j} \partial_k = \nabla_{\partial_j} \nabla_{\partial_i} \partial_k.$$

But applying ∇_{∂_i} to Gauss's formulas gives

$$\begin{aligned} \nabla_{\partial_i} \nabla_{\partial_j} \partial_k &= (\partial_i \Gamma^l_{jk}) \partial_l + \Gamma^l_{jk} \nabla_{\partial_i} \partial_l + (\partial_i b_{jk}) U + b_{jk} \nabla_{\partial_i} U \\ &= (\partial_i \Gamma^l_{jk}) \partial_l + \Gamma^l_{jk} \Gamma^p_{il} \partial_p + \Gamma^l_{jk} b_{il} U + (\partial_i b_{jk}) U - b_{jk} b_{ip} g^{pl} \partial_l \\ &= (\partial_i \Gamma^l_{jk} + \Gamma^p_{jk} \Gamma^l_{ip} - b_{jk} b_{ip} g^{pl}) \partial_l + (\text{multiple of } U), \end{aligned}$$

where we have used Einstein summation convention for notational ease. Interchanging i and j gives

$$\nabla_{\partial_j} \nabla_{\partial_i} \partial_k = (\partial_j \Gamma^l_{ik} + \Gamma^p_{ik} \Gamma^l_{jp} - b_{ik} b_{jp} g^{pl}) \partial_l + (\text{multiple of } U).$$

Since $\partial_1, \ldots, \partial_n, U$ are linearly independent, we may equate the components of ∂_l in these last two equations. When we do this and use the local formula for R^l_{ijk} from Section 3, we obtain

$$R^l_{ijk} = \sum_p (b_{ik} b_{jp} - b_{jk} b_{ip}) g^{pl}.$$

Hence if we write $R_{ijkl} = R(\partial_i, \partial_j)\partial_k \cdot \partial_l$, then since $R(\partial_i, \partial_j)\partial_k = \sum_p R^p_{ijk}\partial_p$, we get

$$R_{ijkl} = b_{ik}b_{jl} - b_{jk}b_{il}.$$

In two dimensions, it is customary to write $b_{11} = \ell, b_{12} = b_{21} = m, b_{22} = n$, and $\partial_1 \cdot \partial_1 = g_{11} = E, \partial_1 \cdot \partial_2 = g_{12} = g_{21} = F, \partial_2 \cdot \partial_2 = g_{22} = G$; then the Gaussian curvature is given by $K(p) = \det S_p = \frac{\ell n - m^2}{EG-F^2}$, and

$$R_{1212} = b_{11}b_{22} - b_{12}^2 = \ell n - m^2,$$

so that

$$K(\partial_1, \partial_2) = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = \frac{\ell n - m^2}{EG - F^2} = K(p),$$

as desired. □

8 Fourth Proof (outline)

Since curvature is an intrinsic measurement of the surface, our theorem essentially proves Gauss's *Theorema Egregium*, which in one form states that the Gaussian curvature of a surface is intrinsic, i.e., it can be measured without leaving the surface. In our last proof, we use a formula Gauss derived in his original proof of the *Theorema Egregium*, expressing Gaussian curvature solely in terms of E, F, G, and their derivatives.

Proof of Theorem. We use the approach of [3, pp. 193-195]. Since the sectional curvature is $\frac{R_{1212}}{EG-F^2}$, it suffices to show that

$$4(EG - F^2)R_{1212} = 4(EG - F^2)^2 K.$$

Let E_1 denote $\partial_1 E$, F_{12} denote $\partial_2\partial_1 F$, and so forth. We use the following result of Gauss [3, p. 111]:

$$4(EG-F^2)^2K = \begin{vmatrix} -2G_{11} + 4F_{12} - 2E_{22} & E_1 & 2F_1 - E_2 \\ 2F_2 - G_1 & E & F \\ G_2 & F & G \end{vmatrix} - \begin{vmatrix} 0 & E_2 & G_1 \\ E_2 & E & F \\ G_1 & F & G \end{vmatrix}.$$

Hence we need to show that $4(EG - F^2)R_{1212}$ is equal to this difference of determinants. To do this, we evaluate the Christoffel symbols Γ^k_{ij} using the Christoffel equations (where $g^{11} = \frac{G}{EG-F^2}, g^{12} = g^{21} = -\frac{F}{EG-F^2}$, and $g^{22} = \frac{E}{EG-F^2}$); we find that

$$\Gamma^1_{11} = \frac{GE_1 - 2FF_1 + FE_2}{2(EG - F^2)}, \Gamma^2_{11} = \frac{-FE_1 + 2EF_1 - EE_2}{2(EG - F^2)},$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{GE_2 - FG_1}{2(EG - F^2)}, \Gamma^2_{12} = \Gamma^2_{21} = \frac{-FE_2 + EG_1}{2(EG - F^2)},$$

$$\Gamma^1_{22} = \frac{2GF_2 - GG_1 - FG_2}{2(EG - F^2)}, \Gamma^2_{22} = \frac{-2FF_2 + FG_1 + EG_2}{2(EG - F^2)}.$$

To evaluate R_{1212}, we use the local formulas

$$R^1_{121} = \partial_1\Gamma^1_{21} - \partial_2\Gamma^1_{11} + \Gamma^2_{21}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22},$$

$$R^2_{121} = \partial_1\Gamma^2_{21} - \partial_2\Gamma^2_{11} + \Gamma^1_{21}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{21} + \Gamma^2_{21}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22},$$

$$R_{1212} = (R^1_{121}\partial_1 + R^2_{121}\partial_2) \cdot \partial_2 = FR^1_{121} + GR^2_{121}.$$

We omit this long and tedious calculation; the value which results for $4(EG - F^2)R_{1212}$ is precisely the desired difference of determinants. □

References

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