1 Introduction

In this paper, we will look at two different notions of curvature, one from a classical standpoint and one from a modern standpoint. These two notions intersect in the concept of the Gaussian curvature of a two-dimensional surface imbedded in $\mathbb{R}^3$. After briefly surveying the relevant classical and modern definitions and results, we present our main result, that the sectional curvature of a two-dimensional manifold is nothing more than the Gaussian curvature. We give four proofs of this result from four different standpoints. The first relies on the classical concept of a connection form; the second uses the classical shape operator; the third depends on local formulas for Christoffel symbols and curvature; the fourth applies a computational approach to a classical formula of Gauss.

2 Classical Formulation

Classically, we define the directional or covariant derivative of a vector field $Y$ in $\mathbb{R}^{n+1}$ with respect to a vector $x_p$ to be

$$\nabla_{x_p} Y = (Y \circ \alpha)'(0),$$

where $\alpha$ is any curve in $\mathbb{R}^{n+1}$ with $\alpha(0) = p$ and $\alpha'(0) = x_p$. It is straightforward to check that $\nabla$ satisfies the following properties:
\(\nabla_X(fY) = (Xf)Y + f\nabla_X Y\)
\(\nabla_fX = f\nabla_X Y\)
\(\nabla_X Y - \nabla_Y X = [X, Y]\)
\(X(Y \cdot Z) = \nabla_X Y \cdot Z + Y \cdot \nabla_X Z.\)

(For instance, (4) is simply the Leibniz rule for dot products; to verify (3), note that both sides obey the Leibniz rule and that (3) is true for \(X = \partial_i, Y = \partial_j.\))

Now consider an \(n\)-dimensional surface \(M\) embedded in \(\mathbb{R}^{n+1}\). We define the shape operator or Weingarten map \(S_p : T_p M \to T_p M\) as follows:

\[S_p(v) = -\nabla_v(U).\]

Note that \(S_p(v) \in T_p M\) since \(2(U \cdot \nabla_v(U)) = \nabla_v(U \cdot U) = \nabla_v(1) = 0\). Thus \(S_p\) is a linear transformation on \(T_p M\), and we may define \(\det S_p = K(p)\) to be the Gauss-Kronecker curvature of \(M\) at \(p\). The mean curvature \(H(p)\) is similarly defined to be \(\frac{1}{n} \text{trace } S_p.\)

Since \(U\) is only well defined up to a sign, the Gauss-Kronecker and mean curvature are in general not well defined. In the special case when \(n\) is even, however, \(\det S_p = \det(-S_p)\), so the Gauss-Kronecker curvature is well defined; when \(n = 2\), the most important case classically, this is known as the Gaussian curvature.

One way to compute the Gaussian curvature is through the connection forms and structural equations (see, e.g., [2, p.312]). Let \(E_1, E_2\) be a frame field around a point \(p\) on a two-dimensional surface \(M\), that is, a pair of vector fields so that \(E_1(q), E_2(q)\) is an orthonormal basis for \(T_q M\) for each \(q\) in a neighborhood of \(p\). The connection forms \(\omega_{ij}\) are the 1-forms defined by

\[\omega_{ij}(v) = \nabla_v E_i \cdot E_j(p).\]

Note that

\[\omega_{ij}(v) + \omega_{ji}(v) = \nabla_v E_i \cdot E_j + \nabla_v E_j \cdot E_i = v(E_i \cdot E_j) = 0,\]

so that \(\omega_{11} = \omega_{22} = 0\) and \(\omega_{12} = -\omega_{21}\). Let \(\theta_1, \theta_2\) be the 1-forms dual to the frame field \(\{E_1, E_2\}\), so that \(\theta_i(v) = v \cdot E_i(p)\). The first structural equations state that

\[d\theta_1 = \omega_{12} \land \theta_2,\]
\[d\theta_2 = \omega_{21} \land \theta_1;\]
one form of the second structural equation gives us a way to compute Gaussian curvature:

\[ d\omega_{12} = -K\theta_1 \wedge \theta_2. \]

3 Modern Formulation

Recall several definitions from the modern formulation of curvature.

The Levi-Civita connection on a Riemannian manifold \((M, g)\) is the unique \(\mathbb{R}\)-bilinear map \(D : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)\) such that

1. \(D_X(fY) = (Xf)Y + fD_XY\)
2. \(D_{fX}Y = fD_XY\)
3. \(D_XY - D_YX = [X, Y]\)
4. \(Xg(Y, Z) = g(D_XY, Z) + g(Y, D_XZ)\).

Note that we use \(D\) instead of \(\nabla\) here to avoid confusion with the covariant derivative of Section 2.

The curvature tensor \(R \in T^{1,3}M\) is given by

\[ R(x, y)z = D_y(D_xz) - D_x(D_yz) + D_{[x,y]}z; \]

this produces a function \(R \in T^{0,4}M\) defined by

\[ R(x, y, z, w) = g(R(x, y)z, w). \]

The sectional curvature \(K(x, y)\) of \(M\) with respect to the plane spanned by \(x, y \in T_pM\) is defined to be

\[ K_p(x, y) = \frac{R(x, y, x, y)}{g(x, x)g(y, y) - g(x, y)^2}; \]

this is independent of the choice of basis \(x, y\) for this particular 2-plane.

Two other notions of curvature are Ricci and scalar curvature; if \(e_i\) is an orthonormal basis for \(T_pM\), then

\[ \text{Ric}_p(x, y) = \sum_i R(x, e_i, y, e_i) \]

\[ \text{Scal}(p) = \sum_{i,j} R(e_i, e_j, e_i, e_j). \]
Also recall the following local definitions and formulas. If \( \{x_1, \ldots, x_n\} \) is a local coordinate system around \( p \in M \), we write \( \partial_i = \partial/\partial x_i \) as the canonical basis for \( T_pM \). The Christoffel symbols (of the second kind) \( \Gamma^k_{ij} \) are defined by \( D_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k \); the Christoffel formulas are

\[
\Gamma^k_{ij} = \frac{1}{2} \sum_l g^{kl} \left( \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right).
\]

Also, if we write \( R(\partial_i, \partial_j) \partial_k = \sum_l R^l_{ijk} \partial_l \), then

\[
R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \sum_p (\Gamma^p_{jk} \Gamma^l_{ip} - \Gamma^p_{ik} \Gamma^l_{jp}).
\]

Now let \( M \) be an \( n \)-dimensional orientable surface embedded in \( \mathbb{R}^{n+1} \) with unit normal vector field \( U \), and consider the classical case in which the scalar product \( g(x, y) \) is the usual dot product \( x \cdot y \), i.e., endow \( M \) with the metric inherited from \( \mathbb{R}^{n+1} \). We wish to compute the Levi-Civita connection on \( M \). Let \( \nabla \) be the covariant derivative defined previously; we define \( D : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM) \) to be the tangential component of \( \nabla \):

\[
D_{x_p} Y = \nabla_{x_p} Y - (\nabla_{x_p} Y \cdot U)U.
\]

By taking tangential components of the properties of \( \nabla \) from Section 2, it is straightforward to check that \( D \) satisfies the conditions to be a Levi-Civita connection. We then define the curvature tensor and the various notions of curvature from this Levi-Civita connection.

## 4 Gaussian Curvature

In the special case in which \( M \) is a 2-dimensional surface, the sectional curvature \( K_p(x, y) = K_p \) is the same for all \( x, y \in T_pM \). Also, if \( \{E_1, E_2\} \) is a frame field (a set of orthonormal vector fields) around \( p \), and \( x = x_1 E_1 + x_2 E_2 \), \( y = y_1 E_1 + y_2 E_2 \), then

\[
\text{Ric}_p(x, y) = R(x_2 E_2, E_1, y_2 E_2, E_1) + R(x_1 E_1, E_2, y_1 E_1, E_2)
\]

\[
= (x_1 y_1 + x_2 y_2) K_p(E_1, E_2)
\]

\[
= g(x, y) K_p(E_1, E_2),
\]

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and

\[ \text{Scal}(p) = R(E_1, E_2, E_1, E_2) + R(E_2, E_1, E_2, E_1) = 2K_p(E_1, E_2). \]

Our main result tying together the classical and modern formulations of curvature will be that the sectional curvature on a 2-surface is simply the Gaussian curvature.

**Theorem** For a 2-surface \( M \), the sectional curvature \( K_p(x, y) \) is equal to the Gaussian curvature \( K(p) \).

In the four subsequent sections, we will present four different proofs of this theorem; they are roughly in order from most global to most local.

Note that the theorem implies that \( \text{Ric}_p(x, y) = g(x, y)K(p) \) and \( \text{Scal}(p) = 2K(p) \). Myers’s Theorem, which involves Ricci curvature, and Synge’s Theorem, which involves sectional curvature, thus reduce to the following results.

**Corollary 1** Let \( M \) be a complete two-dimensional manifold, and suppose that there exists an \( r > 0 \) such that \( K(p) \geq r^{-2} \) for all \( p \in M \). Then \( \text{diam } M \leq \text{diam } (S^2(r)) \). In particular, \( M \) is compact with finite fundamental group.

Note that \( K \equiv 1/r^2 \) on \( S^2(r) \).

**Corollary 2** If \( M \) is a compact orientable two-dimensional manifold such that \( K(p) > 0 \) for all \( p \in M \), then \( M \) has genus zero.

This last corollary is also implied by the Gauss-Bonnet Theorem.

## 5 First Proof

I could not find a reference for this proof.

We will use the connection forms and the structural equations from Section 2. Let \( M \) be a two-dimensional surface in \( \mathbb{R}^3 \) and let \( E_1, E_2 \) be a frame field on \( M \), with \( \theta_1, \theta_2 \) its dual basis of 1-forms; define the connection forms \( \omega_{ij} \) from this frame field. Note that \( \omega_{ij}(v) = \nabla_v E_i \cdot E_j = D_v E_i \cdot E_j \) since \( U \cdot E_j = 0 \). For the rest of this section, we will be using computations made
entirely on the surface itself; since we will thus not need covariant derivatives, we will denote the Levi-Civita connection by $\nabla$, as is customary, rather than by $D$. (In the subsequent proofs, however, we will be using both covariant derivative and Levi-Civita connection; they will be denoted, as before, by $\nabla$ and $D$, respectively.)

We need the following lemma, which expresses Gaussian curvature explicitly in terms of the connection form $\omega_{12}$.

**Lemma 1** \( K(p) = E_2(\omega_{12}(E_1)) - E_1(\omega_{12}(E_2)) - (\omega_{12}(E_1))^2 - (\omega_{12}(E_2))^2 \).\n
**Proof.** A simple calculation shows that we can write $\omega_{12}$ as

$$\omega_{12} = \omega_{12}(E_1)\theta_1 + \omega_{12}(E_2)\theta_2.$$\n
Likewise, we have

$$d(\omega_{12}(E_1)) = d(\omega_{12}(E_1))(E_1)\theta_1 + d(\omega_{12}(E_2))(E_2)\theta_2 = E_1(\omega_{12}(E_1))\theta_1 + E_2(\omega_{12}(E_1))\theta_2,$$\n
with a similar expression for $D(\omega_{12}(E_2))$. Hence

$$d\omega_{12} = d(\omega_{12}(E_1)) \land \theta_1 + d(\omega_{12}(E_2)) \land \theta_2 + \omega_{12}(E_1)d\theta_1 + \omega_{12}(E_2)d\theta_2$$

$$= E_2(\omega_{12}(E_1))\theta_2 \land \theta_1 + E_1(\omega_{12}(E_2))\theta_1 \land \theta_2 + \omega_{12}(E_1)\omega_{12} \land \theta_2 - \omega_{12}(E_2)\omega_{12} \land \theta_1$$

$$= (E_2(\omega_{12}(E_1)) + E_1(\omega_{12}(E_2)) + (\omega_{12}(E_1))^2 + (\omega_{12}(E_2))^2)\theta_1 \land \theta_2,$$\n
where we have used the first structural equations to evaluate $d\theta_1$ and $d\theta_2$; the lemma then follows from the second structural equation. \qed

**Proof of Theorem.** Since $\nabla_{E_1}E_1 \cdot E_1 = \omega_{11}(E_1) = 0$ and $\nabla_{E_2}E_2 \cdot E_2 = \omega_{22}(E_2) = 0$, $\nabla_{E_1}E_1$ is a multiple of $E_2$ while $\nabla_{E_2}E_2$ is a multiple of $E_1$, so $\langle \nabla_{E_1}E_1 \rangle \cdot (\nabla_{E_2}E_2) = 0$. Similarly, $\langle \nabla_{E_2}E_1 \rangle \cdot (\nabla_{E_1}E_2) = 0$; also note that

$$[E_1, E_2] = \nabla_{E_1}E_2 - \nabla_{E_2}E_1$$

$$= ((\nabla_{E_1}E_2 - \nabla_{E_2}E_1) \cdot E_1)E_1 + ((\nabla_{E_1}E_2 - \nabla_{E_2}E_1) \cdot E_2)E_2$$

$$= (\omega_{21}(E_1) - \omega_{11}(E_2))E_1 + (\omega_{22}(E_1) - \omega_{12}(E_2))E_2$$

$$= -\omega_{12}(E_1)E_1 - \omega_{12}(E_2)E_2.$$
Hence the sectional curvature is equal to

\[ K_p(E_1, E_2) = R(E_1, E_2)E_1 \cdot E_2 \]
\[ = \nabla_{E_2} \nabla_{E_1} E_1 \cdot E_2 - \nabla_{E_1} \nabla_{E_2} E_1 \cdot E_2 + \nabla_{[E_1, E_2]} E_1 \cdot E_2 \]
\[ = (\nabla_{E_2} \nabla_{E_1} E_1 \cdot E_2 + (\nabla_{E_1} E_1) \cdot (\nabla_{E_2} E_2)) - \]
\[ - (\nabla_{E_1} \nabla_{E_2} E_1 \cdot E_2 + (\nabla_{E_2} E_1) \cdot (\nabla_{E_1} E_2)) + \nabla_{[E_1, E_2]} E_1 \cdot E_2 \]
\[ = E_2(\nabla_{E_1} E_1 \cdot E_2) - E_1(\nabla_{E_2} E_1 \cdot E_2) - \omega_{12}(E_1)\nabla_{E_1} E_1 \cdot E_2 - \]
\[ - \omega_{12}(E_2)\nabla_{E_2} E_1 \cdot E_2 \]
\[ = E_2(\omega_{12}(E_1)) - E_1(\omega_{12}(E_2)) - (\omega_{12}(E_1))^2 - (\omega_{12}(E_2))^2 \]
\[ = K(p), \]

as desired. □

6 Second Proof

This proof is adapted from the method in [4, pp. 227-230].

Let \( M \) be an orientable \( n \)-dimensional manifold embedded in \( \mathbb{R}^{n+1} \) with unit normal vector field \( U \); we will specialize later to the case \( n = 2 \). Again, we will use \( \nabla \) and \( D \) to represent the covariant derivative and the Levi-Civita connection, respectively. Define \( R' \in T^{1,3}(M) \) by

\[ R'(x, y)z = (S_p(x) \cdot z)S_p(y) - (S_p(y) \cdot z)S_p(x). \]

Recall that \( R \in T^{1,3}(M) \) is defined by

\[ R(x, y)z = D_y(D_x z) - D_x(D_y z) + D_{[x,y]} z. \]

Let \( \{\partial_1, \ldots, \partial_n\} \) be the canonical basis for \( T_pM \). We will need a lemma; note that this lemma implies that \( R(x, y, z) = R'(x, y, z) \) for all \( x, y, z \in TM \), so that we have, as a bonus, a formula for the curvature tensor in terms of the classical shape operator.

**Lemma 2** If \( z \in T_pM \), then \( R'(\partial_i, \partial_j, z) = R(\partial_i, \partial_j, z) \).

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Proof. Since \((\nabla_{\partial_j} z) \cdot U + (\nabla_{\partial_j} U) \cdot z = U \cdot z = 0\), we have
\[
\nabla_{\partial_i} \nabla_{\partial_j} z &= \nabla_{\partial_i}(D_{\partial_j} z + ((\nabla_{\partial_j} z) \cdot U) U) \\
&= D_{\partial_i} D_{\partial_j} z + ((\nabla_{\partial_j} (D_{\partial_j} z)) \cdot U) U + \nabla_{\partial_i}(((\nabla_{\partial_j} z) \cdot U)) U \\
&= D_{\partial_i} D_{\partial_j} z + ((\nabla_{\partial_j} (D_{\partial_j} z)) \cdot U) U - ((\nabla_{\partial_j} U) \cdot z) \nabla_{\partial_i} U \\
&= D_{\partial_i} D_{\partial_j} z + ((\nabla_{\partial_j} (D_{\partial_j} z)) \cdot U) U - ((S_p \partial_j) \cdot z) S_p \partial_i.
\]
Now the first and third terms of this expression are in \(T_p M\), while the second is normal to the surface, so the tangential component of \(\nabla_{\partial_j} \nabla_{\partial_i} z\) is \(D_{\partial_j} D_{\partial_i} z - ((S_p \partial_j) \cdot z) S_p \partial_i\). Similarly, the tangential component of \(\nabla_{\partial_j} \nabla_{\partial_i} z\) is \(D_{\partial_j} D_{\partial_i} z - ((S_p \partial_i) \cdot z) S_p \partial_j\). But by definition, we have that
\[
\nabla_{\partial_j} z = (z \circ \alpha)'(0) = \frac{\partial z}{\partial x_j}
\]
where \(\alpha'(0) = \partial_j\), so
\[
\nabla_{\partial_i} \nabla_{\partial_j} z &= \frac{\partial^2}{\partial x_i \partial x_j} z = \frac{\partial^2 z}{\partial x_j \partial x_i} = \nabla_{\partial_j} \nabla_{\partial_i} z;
\]
taking tangential components and rearranging, we conclude that
\[
D_{\partial_i} D_{\partial_j} z - D_{\partial_j} D_{\partial_i} z = ((S_p \partial_j) \cdot z) S_p \partial_j - ((S_p \partial_i) \cdot z) S_p \partial_i \\
&= R'(\partial_i, \partial_j) z.
\]
Finally, since \([\partial_i, \partial_j] = 0\), we obtain
\[
R(\partial_i, \partial_j) z = D_{\partial_i} D_{\partial_j} z - D_{\partial_j} D_{\partial_i} z,
\]
whence the lemma. \(\square\)

Proof of Theorem. Let \(M\) have dimension 2. Since \(K(x, y)\) is independent of the basis \(x, y\) chosen, it suffices to show that \(K_p(\partial_1, \partial_2) = K(p)\). But by Lemma 2,
\[
(g_{11} g_{22} - g_{12}^2) K_p(\partial_1, \partial_2) = R(\partial_1, \partial_2) \partial_1 \cdot \partial_2 \\
&= R'(\partial_1, \partial_2) \partial_1 \cdot \partial_2 \\
&= (S_p(\partial_1) \cdot \partial_1)(S_p(\partial_2) \cdot \partial_2) - (S_p(\partial_2) \cdot \partial_1)(S_p(\partial_1) \cdot \partial_2) \\
&= (g_{11} g_{22} - g_{12}^2) \det S_p \\
&= (g_{11} g_{22} - g_{12}^2) K(p);
\]
the theorem follows. \(\square\)
7 Third Proof

This proof is essentially a version of the previous proof in local coordinates. In the course of the proof, we obtain and use the so-called Gauss's formulas.

**Proof of Theorem.** We loosely follow the notation of [1]. As in the previous proof, we first assume that $M$ is $n$-dimensional embedded in $\mathbb{R}^{n+1}$; later we will set $n = 2$.

Let $b_{jk} = S_p(\partial_j) \cdot \partial_k$. We wish to express $\nabla_{\partial_j} \partial_k$ in terms of the $\partial_l$’s and of $U$. Now the tangential component of $\nabla_{\partial_j} \partial_k$ is given by $D_{\partial_j} \partial_k = \sum_l \Gamma^l_{jk} \partial_l$; the normal component is $\nabla_{\partial_j} \partial_k \cdot U = \partial_j(\partial_k \cdot U) - \nabla_{\partial_j} U \cdot \partial_k = S_p(\partial_j) \cdot \partial_k = b_{jk}$. Hence

$$\nabla_{\partial_j} \partial_k = \sum_l \Gamma^l_{jk} \partial_l + b_{jk} U;$$

in [1], these equations are known as Gauss’s formulas. Next, from the formulas $S_p(\partial_l) \cdot \partial_p = b_{lp}$ and the fact that $S_p(\partial_l) \in T_p M$, we conclude that

$$-\nabla_{\partial_l} U = S_p(\partial_l) = \sum_{l,p} b_{lp} g^{pl} \partial_l.$$

Now, as in the previous proof, we know that

$$\nabla_{\partial_i} \nabla_{\partial_j} \partial_k = \nabla_{\partial_j} \nabla_{\partial_i} \partial_k.$$

But applying $\nabla_{\partial_i}$ to Gauss’s formulas gives

$$\nabla_{\partial_i} \nabla_{\partial_j} \partial_k = (\partial_i \Gamma^l_{jk}) \partial_l + \Gamma^l_{jk} \nabla_{\partial_l} \partial_i + (\partial_i b_{jk}) U + b_{jk} \nabla_{\partial_i} U$$

$$= (\partial_i \Gamma^l_{jk}) \partial_l + \Gamma^l_{jk} \Gamma^p_{il} \partial_p + \Gamma^l_{jk} b_{il} U + (\partial_i b_{jk}) U - b_{jk} b_{lp} g^{pl} \partial_l$$

$$= (\partial_i \Gamma^l_{jk} + \Gamma^p_{jk} \Gamma^l_{ip} - b_{jk} b_{lp} g^{pl}) \partial_l + \text{(multiple of $U$)},$$

where we have used Einstein summation convention for notational ease. Interchanging $i$ and $j$ gives

$$\nabla_{\partial_j} \nabla_{\partial_i} \partial_k = (\partial_j \Gamma^l_{ik} + \Gamma^p_{ik} \Gamma^l_{jp} - b_{ik} b_{jp} g^{pl}) \partial_l + \text{(multiple of $U$)}.$$

Since $\partial_1, \ldots, \partial_n, U$ are linearly independent, we may equate the components of $\partial_l$ in these last two equations. When we do this and use the local formula for $R^l_{ijk}$ from Section 3, we obtain

$$R^l_{ijk} = \sum_p (b_{ik} b_{jp} - b_{jk} b_{ip}) g^{pl}.$$
Hence if we write \( R_{ijkl} = R(\partial_i, \partial_j) \partial_k \cdot \partial_l \), then since \( R(\partial_i, \partial_j) \partial_k = \sum_p R^p_{ijk} \partial_p \), we get
\[
R_{ijkl} = b_{ik} b_{jl} - b_{jk} b_{il}
\]

In two dimensions, it is customary to write \( b_{11} = \ell, b_{12} = b_{21} = m, b_{22} = n \), and \( \partial_1 \cdot \partial_1 = g_{11} = E, \partial_1 \cdot \partial_2 = g_{12} = g_{21} = F, \partial_2 \cdot \partial_2 = g_{22} = G \); then the Gaussian curvature is given by \( K(p) = \det S_p = \frac{\ell n - m^2}{EG - F^2} \), and
\[
R_{1212} = b_{11} b_{22} - b_{12}^2 = \ell n - m^2,
\]
so that
\[
K(\partial_1, \partial_2) = \frac{R_{1212}}{g_{11} g_{22} - g_{12}^2} = \frac{\ell n - m^2}{EG - F^2} = K(p),
\]
as desired. \( \square \)

8 Fourth Proof (outline)

Since curvature is an intrinsic measurement of the surface, our theorem essentially proves Gauss’s \textit{Theorema Egregium}, which in one form states that the Gaussian curvature of a surface is intrinsic, i.e., it can be measured without leaving the surface. In our last proof, we use a formula Gauss derived in his original proof of the \textit{Theorema Egregium}, expressing Gaussian curvature solely in terms of \( E, F, G \), and their derivatives.

Proof of Theorem. We use the approach of \cite[pp. 193-195]{3}. Since the sectional curvature is \( \frac{R_{1212}}{EG - F^2} \), it suffices to show that
\[
4(EG - F^2) R_{1212} = 4(EG - F^2)^2 K.
\]
Let \( E_1 \) denote \( \partial_1 E \), \( F_{12} \) denote \( \partial_2 \partial_1 F \), and so forth. We use the following result of Gauss \cite[p. 111]{3}:
\[
4(EG - F^2)^2 K = \begin{vmatrix}
-2G_{11} + 4F_{12} - 2E_{22} & E_1 & 2F_1 - E_2 \\
2F_2 - G_1 & E & F \\
G_2 & F & G
\end{vmatrix} - \begin{vmatrix}
0 & E_2 & G_1 \\
E_2 & E & F \\
G_1 & F & G
\end{vmatrix}.
\]

Hence we need to show that \( 4(EG - F^2) R_{1212} \) is equal to this difference of determinants. To do this, we evaluate the Christoffel symbols \( \Gamma^k_{ij} \) using
the Christoffel equations (where \( g^{11} = \frac{G}{EG - F^2}, g^{12} = g^{21} = -\frac{F}{EG - F^2}, \) and \( g^{22} = \frac{E}{EG - F^2} \)); we find that

\[
\begin{align*}
\Gamma^1_{11} &= \frac{GE_1 - 2FF_1 + FE_2}{2(EG - F^2)}, \\
\Gamma^2_{11} &= \frac{-FE_1 + 2EF_1 - EE_2}{2(EG - F^2)}, \\
\Gamma^1_{12} &= \Gamma^1_{21} = \frac{GE_2 - FG_1}{2(EG - F^2)}, \\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{-FE_2 + EG_1}{2(EG - F^2)}, \\
\Gamma^1_{22} &= \frac{2GF_2 - GG_1 - FG_2}{2(EG - F^2)}, \\
\Gamma^2_{22} &= \frac{-2FF_2 + FG_1 + EG_2}{2(EG - F^2)}.
\end{align*}
\]

To evaluate \( R_{1212} \), we use the local formulas

\[
R^1_{121} = \partial_1 \Gamma^1_{21} - \partial_2 \Gamma^1_{11} + \Gamma^2_{21} \Gamma^1_{12} - \Gamma^2_{11} \Gamma^1_{22},
\]

\[
R^2_{121} = \partial_1 \Gamma^2_{21} - \partial_2 \Gamma^2_{11} + \Gamma^1_{21} \Gamma^2_{12} - \Gamma^1_{11} \Gamma^2_{22},
\]

\[
R_{1212} = (R^1_{121} \partial_1 + R^2_{121} \partial_2) \cdot \partial_2 = FR^1_{121} + GR^2_{121}.
\]

We omit this long and tedious calculation; the value which results for \( 4(EG - F^2)R_{1212} \) is precisely the desired difference of determinants. □

**References**


