

A COMPLETE KNOT INVARIANT FROM CONTACT HOMOLOGY

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ABSTRACT. We construct an enhanced version of knot contact homology, and show that we can deduce from it the group ring of the knot group together with the peripheral subgroup. In particular, it completely determines a knot up to smooth isotopy. The enhancement consists of the (fully noncommutative) Legendrian contact homology associated to the union of the conormal torus of the knot and a disjoint cotangent fiber sphere, along with a product on a filtered part of this homology. As a corollary, we obtain a new, holomorphic-curve proof of a result of the third author that the Legendrian isotopy class of the conormal torus is a complete knot invariant.

1. INTRODUCTION

1.1. Conormal tori and knot contact homology. A significant thread in recent research in symplectic and contact topology has concerned the study of smooth manifolds through the symplectic structures on their cotangent bundles. In this setting, one can also study a pair of manifolds, one embedded in the other—in particular, a knot in a 3-manifold—via the conormal construction. If $K \subset \mathbb{R}^3$ is a knot, then its unit conormal bundle, the conormal torus Λ_K , is a Legendrian submanifold of the contact cosphere bundle $ST^*\mathbb{R}^3$. Isotopic knots produce conormal tori that are isotopic as Legendrian submanifolds, i.e., the Legendrian isotopy type of the conormal torus is a knot invariant. The fact that this invariant is nontrivial depends essentially on the contact geometry: the conormal tori of any two knots are smoothly isotopic, even if the knots themselves are not isotopic.

Symplectic field theory [EGH10] provides an algebraic knot invariant associated to this geometric invariant: the Legendrian contact homology of Λ_K , also known as the knot contact homology of K . This is the homology of a differential graded algebra generated by Reeb chords of Λ_K with differential given by counting holomorphic disks. In the past few years, there have been indications that knot contact homology and its higher genus generalizations are related via string theory to other knot invariants such as the A-polynomial, HOMFLY-PT polynomial, and possibly various knot homologies: the cotangent bundle equipped with Lagrangian branes along the conormal of the knot and the 0-section is the setting for an open topological string theory that has conjectural relations to all of these invariants. The physical account considers the holomorphic disks that go into knot contact homology, and also crucially takes into account higher genus information; this last part has not yet been fully developed in the mathematical literature, but some beginnings can be found in [AENV14]. In particular, it is explained there how certain quantum invariants should be conjecturally

recovered from a quantization of knot contact homology arising from the consideration of non-exact Lagrangians. In any case, it appears that knot contact homology should be a very strong invariant, in the sense that it encodes a great deal of information about the underlying knot.

Recent work of the third author [She] shows that the Legendrian conormal torus Λ_K is in fact a complete invariant of K : two knots with Legendrian isotopic conormals must in fact be isotopic. Since Λ_K is the starting point for knot contact homology, this can be viewed as evidence for, or in any case is consistent with, the possibility that knot contact homology itself is a complete invariant. Other evidence in this direction is provided by the fact that knot contact homology recovers enough of the knot group (the fundamental group of the knot complement) to detect the unknot [Ng08] and torus knots, among others [GL]. These results use the ring structure on the “fully noncommutative” version of knot contact homology, where the algebra is generated by Reeb chords along with homology classes in $H_1(\Lambda_K)$ that do not commute with Reeb chords; see [Ng14, CELN17]. However, the question of whether knot contact homology is a complete invariant remains open.

1.2. Main results. In this paper, we present an extension of knot contact homology by slightly enlarging the set of holomorphic disks that are counted. We will show that this extension, which we call *enhanced knot contact homology*, contains the knot group along with the peripheral subgroup, and this in turn is enough to completely determine the knot [Wal68, GL89]. As a corollary, we have a new proof of the result from [She], using holomorphic curves rather than constructible sheaves.

For our purposes, we need the Legendrian contact homology of not just Λ_K but the union of Λ_K and a cotangent fiber Λ_p of $ST^*\mathbb{R}^3$; the inclusion of the latter is analogous to choosing a basepoint for the fundamental group. This new invariant, $\text{LCH}_*(\Lambda_K \cup \Lambda_p)$, is a ring that contains the knot contact homology of K as a quotient. Using the “link grading” of Mishachev [Mis03], we can write:

$$\text{LCH}_*(\Lambda_K \cup \Lambda_p) \cong (\text{LCH}_*)_{\Lambda_K, \Lambda_K} \oplus (\text{LCH}_*)_{\Lambda_K, \Lambda_p} \oplus (\text{LCH}_*)_{\Lambda_p, \Lambda_K} \oplus (\text{LCH}_*)_{\Lambda_p, \Lambda_p},$$

where $(\text{LCH}_*)_{\Lambda_i, \Lambda_j}$ denotes the homology of the subcomplex generated by composable words of Reeb chords ending on Λ_i and beginning on Λ_j ; see Section 2 for details.

From this set of data, we pick out what we call the *KCH-triple* (R_{KK}, R_{Kp}, R_{pK}) associated to $\Lambda_K \cup \Lambda_p$, defined by:

$$R_{KK} = (\text{LCH}_0)_{\Lambda_K, \Lambda_K}, \quad R_{Kp} = (\text{LCH}_0)_{\Lambda_K, \Lambda_p}, \quad R_{pK} = (\text{LCH}_1)_{\Lambda_p, \Lambda_K}.$$

Of these, R_{KK} is precisely the degree 0 knot contact homology of K and contains a subring $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ once we equip K with an orientation and framing (which we choose to be the Seifert framing), where l, m denote the longitude and meridian of K ; R_{Kp} and R_{pK} are left and right modules, respectively, over R_{KK} . We remark that $(\text{LCH}_*)_{\Lambda_K, \Lambda_K}$, $(\text{LCH}_*)_{\Lambda_K, \Lambda_p}$,

and $(\text{LCH}_*)_{\Lambda_p, \Lambda_K}$ turn out to be supported in degrees ≥ 0 , 0, and 1, respectively, and so the KCH-triple is comprised of the lowest-degree summand of each.

We need one further piece of data in addition to the KCH-triple: a product $\mu : R_{Kp} \otimes R_{pK} \rightarrow R_{KK}$. While the differential in $\text{LCH}_*(\Lambda_K \cup \Lambda_p)$ counts holomorphic disks in the symplectization $\mathbb{R} \times ST^*\mathbb{R}^3$ with boundary on $\mathbb{R} \times (\Lambda_K \cup \Lambda_p)$ and one positive puncture at a Reeb chord of $\Lambda_K \cup \Lambda_p$, the product μ counts holomorphic disks with *two* positive punctures, at mixed Reeb chords of $\Lambda_K \cup \Lambda_p$. Extending Legendrian contact homology to ‘‘Legendrian Rational Symplectic Field Theory’’ by counting disks with multiple positive punctures has not yet been successfully implemented in general; the difficulty comes from boundary breaking for holomorphic disks, which contributes to the codimension-1 strata of moduli spaces. However, partial results in this direction have been obtained by the first author [Ekh08] in the case of multiple-component Legendrian links when boundary breaking can be avoided for topological reasons, and (with less relevance for our purposes) by the second author [Ng10] in complete generality in the case of Legendrian knots in \mathbb{R}^3 . In particular, the fact that μ is well-defined and invariant follows from [Ekh08].

Our main result is now as follows:

Theorem 1.1. *Let $K \subset \mathbb{R}^3$ be an oriented knot and $p \in \mathbb{R}^3 \setminus K$ be a point, and let Λ_K, Λ_p denote the Legendrian submanifolds of $ST^*\mathbb{R}^3$ given by the unit conormal torus to K and the unit cotangent fiber over p . Then the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) constructed from the Legendrian contact homology of $\Lambda_K \cup \Lambda_p$, equipped with the product $\mu : R_{Kp} \otimes R_{pK} \rightarrow R_{KK}$, is a complete invariant for K .*

More precisely, if there is an isomorphism between the KCH-triples for two oriented knots K_0, K_1 that preserves μ , then:

- (1) K_0 and K_1 are smoothly isotopic up to mirroring and orientation reversal;
- (2) if the isomorphism from $R_{K_0K_0}$ to $R_{K_1K_1}$ restricts to the identity map on the subring $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$, then K_0 and K_1 are smoothly isotopic as oriented knots.

A Legendrian isotopy between $\Lambda_{K_0} \cup \Lambda_p$ and $\Lambda_{K_1} \cup \Lambda_p$ induces an isomorphism between the KCH-triples that respects the product μ . Since \mathbb{R}^3 is noncompact, any Legendrian isotopy between the conormal tori Λ_{K_0} and Λ_{K_1} can be extended to an isotopy between $\Lambda_{K_0} \cup \Lambda_p$ and $\Lambda_{K_1} \cup \Lambda_p$ by pushing p away from the (compact) support of the isotopy. Thus we deduce from Theorem 1.1 a new proof of the following result.

Theorem 1.2 ([She]). *Let K_0, K_1 be smooth knots in \mathbb{R}^3 and let $\Lambda_{K_0}, \Lambda_{K_1}$ denote their conormal tori.*

- (1) If Λ_{K_0} and Λ_{K_1} are Legendrian isotopic, then K_0 and K_1 are smoothly isotopic up to mirroring and orientation reversal.
- (2) If Λ_{K_0} and Λ_{K_1} are parametrized Legendrian isotopic, then K_0 and K_1 are smoothly isotopic as oriented knots.

Here “parametrized Legendrian isotopic” means the following: each conormal torus Λ_K of an oriented knot K has two distinguished classes in $H_1(\Lambda_K)$ given by the meridian and Seifert-framed longitude, and a parametrized Legendrian isotopy between conormal tori is an isotopy that sends meridian and longitude to meridian and longitude.

Our proof of Theorem 1.1 depends crucially on the results of [CELN17], which relates knot contact homology to string topology. It is shown there that one can construct an isomorphism from degree 0 knot contact homology, $\text{LCH}_0(\Lambda_K)$, to a certain string homology constructed from paths (“broken strings”) on the singular Lagrangian given by the union, inside the cotangent bundle, of the zero section and the conormal. This isomorphism is induced by mapping a Reeb chord to the chain of boundaries of all holomorphic disks asymptotic to the Reeb chord with boundary on the singular Lagrangian.

In this paper, we extend the isomorphism from [CELN17] to show that the KCH-triple can also be computed using broken strings. Using this presentation, we prove a ring isomorphism

$$\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)] \cong \mathbb{Z} \oplus (R_{pK} \otimes_{R_{KK}} R_{Kp})$$

where multiplication on the right is induced by the product $\mu : R_{Kp} \otimes R_{pK} \rightarrow R_{KK}$ (see Section 4.4 for details). Knot groups are known to be left orderable [HS85], i.e., they have a total ordering invariant under left multiplication, and left orderable groups are determined by their group ring [Hig40]; it follows that we can recover the knot group itself from the KCH-triple. A further consideration of the subring $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$, which sits naturally in enhanced knot contact homology (more precisely, in R_{KK}), shows that we can also recover the longitude and meridian inside the knot group, and thus by [Wal68] we have a complete knot invariant.

We emphasize that the extra cosphere fiber is critical for our argument. It is shown in [CELN17] that knot contact homology $\text{LCH}_0(\Lambda_K)$ is isomorphic to a certain subring of $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$, and this can be used to prove that $\text{LCH}_0(\Lambda_K)$ detects the unknot and torus knots, as mentioned earlier. It is not clear whether this subring suffices to give a complete invariant. By contrast, the extra cosphere fiber allows the direct recovery of $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$ and thus $\pi_1(\mathbb{R}^3 \setminus K)$.

1.3. Relation to sheaves. We conclude this introduction by sketching a Floer-theoretic path from the arguments of [She] to those of the present work. The body of the paper does not depend on any of the claims below; we include them solely for motivational purposes and conceptual clarity. These claims could be established rigorously by a variant of [BEE12] in the partially wrapped context (a special case of [EL, Conjecture 3]), together with a proof of Kontsevich’s localization conjecture [Kon09]. A significantly more detailed sketch of the following arguments appears in section 6 of the arXiv version of the present paper, arXiv:1606.07050.

In [She] the basic tool is the category of sheaves on \mathbb{R}^3 , constructible with respect to the stratification by the knot K and its complement $\mathbb{R}^3 \setminus K$. This category is identified with

the *infinitesimal Fukaya category* whose objects are, roughly, exact Lagrangians in $T^*\mathbb{R}^3$ asymptotic to the conormal torus $\Lambda_K \subset ST^*\mathbb{R}^3$, and whose morphisms are the intersections between the Lagrangians after perturbing infinitesimally along the Reeb flow at infinity [NZ09, Nad17].

There is another Floer-theoretic category one can associate to the same geometry, the *partially wrapped category with wrapping stopped by Λ_K* . Here the objects are exact Lagrangians asymptotic to Legendrian submanifolds in $ST^*\mathbb{R}^3$, in the complement of the conormal torus Λ_K . The morphisms are computed by wrapping using a Reeb flow which stops at Λ_K in the sense of [Syl]. (A cut and paste model of the Reeb flow is obtained by attaching $T^*([0, \infty) \times \Lambda_K)$ to $ST^*\mathbb{R}^3$ along Λ_K , see [EL, Section B.3].)

The infinitesimally wrapped category embeds into the partially wrapped category: pushing a Lagrangian asymptotic to Λ_K slightly backwards along the Reeb flow gives a Lagrangian with trivial wrapping at infinity. To see this, note that the Reeb flow starting at the shifted Λ_K arrives immediately at the stop Λ_K and hence will flow no further. The image of this embedding is expected to be categorically characterized as the “pseudo-perfect modules”. In particular, the partially wrapped category should know at least as much as the sheaf category.

Two notable objects of the partially wrapped category are the cotangent fiber F at a point not on the knot and the Lagrangian disk C which fills a small ball linking the conormal torus. (In the cut and paste model, C is a cotangent fiber in $T^*([0, \infty) \times \Lambda_K)$.) Taking Hom with these Lagrangians gives functors from the partially wrapped category, hence by restriction followed by the Nadler-Zaslow isomorphism, from the sheaf category, to chain complexes.

In fact, the partially wrapped category also has a conjectural identification with a certain category of sheaves [Kon09, Nad]. Under these identifications, the functor associated to F is computing the stalk at the point away from the knot, and the functor associated to C is computing the microsupport of the sheaf at the knot. These are the main operations used in [She] and having both is crucial to the argument there.

It is also expected that the Lagrangians F and C generate the partially wrapped Fukaya category, i.e., the partially wrapped category can be identified with the category of perfect modules over the endomorphism algebra of $F \cup C$. This means that the partially wrapped Floer cohomology $HW^*(F \cup C, F \cup C)$ of these two disks should contain all the information of the sheaf category, and moreover in a way which makes the information needed in the arguments of [She] immediately accessible.

Both wrapped Floer cohomology and Legendrian contact homology are algebras on Reeb chords; a precise relation between them is established in [BEE12] and generalized to the partially wrapped context in [EL, Conjecture 3]. Specifically, the partially wrapped Floer cohomologies of the disks F and C can be computed from contact homology algebras and

in the notation above, we have:

$$\begin{aligned} HW^0(C, C) &\cong R_{KK}, \\ HW^0(C, F) &\cong R_{pK}, \\ HW^0(F, C) &\cong R_{Kp}, \\ HW^0(F, F) &\cong \mathbb{Z} \oplus (R_{pK} \otimes R_{Kp}). \end{aligned}$$

Moreover, the product $\mu : R_{Kp} \otimes R_{pK} \rightarrow R_{KK}$ is identified with the ordinary pair-of-pants product $HW^0(F, C) \otimes HW^0(C, F) \rightarrow HW^0(C, C)$ in wrapped Floer cohomology. Then the KCH-triple and product determines a ring structure on $HW^0(F, F)$, and our results show that $HW^0(F, F)$ is ring isomorphic to the group ring $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$.

In fact, this ring isomorphism can be induced from moduli spaces of holomorphic disks as follows. Applying Lagrange surgery to $\mathbb{R}^3 \cup L_K$ (i.e., removing the interiors of small disk bundle neighborhoods of K in \mathbb{R}^3 and L_K and joining the resulting boundary fiber circles over K by a family of 1-handles), we obtain a Lagrangian M_K with the topology of $\mathbb{R}^3 \setminus K$. The disk F intersects M_K transversely in one point and the map above is induced from moduli spaces of holomorphic disks with one positive puncture at a Reeb chord of F , two Lagrangian intersection punctures at $M_K \cap F$, and boundary on $F \cup M_K$. This map is then directly analogous to the corresponding map in the cotangent bundle of a closed manifold and, as there, it gives an isomorphism of rings, intertwining the pair of pants product in wrapped Floer cohomology with the Pontryagin product on chains of loops.

1.4. Outline of the paper. In Section 2, we introduce enhanced knot contact homology and the KCH-triple, along with the product map μ . We reformulate these structures in terms of string topology in Section 3 and then in terms of the knot group in Section 4, leading to a proof of Theorem 1.1 in Section 5.

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2. ENHANCED KNOT CONTACT HOMOLOGY

In this section we present the ingredients of enhanced knot contact homology. In Section 2.1 we discuss the structure of the contact homology algebra of a two component Legendrian link, in Section 2.2 we specialize to the case of a link consisting of the conormal of a knot and the fiber sphere over a point. Finally, in Section 2.3 we introduce the product operation on enhanced knot contact homology.

2.1. Legendrian contact homology for a link. Let $V = J^1(M)$ be the 1-jet space of a compact manifold M with the standard contact structure, and let $\Lambda \subset V$ be a connected Legendrian submanifold. The Legendrian contact homology of Λ , which we will write as $\text{LCH}_*(\Lambda)$, is the homology of a differential graded algebra $(\mathcal{A}_\Lambda, \partial)$, where \mathcal{A}_Λ is a noncommutative unital algebra generated by Reeb chords of Λ and homology classes in $H_1(\Lambda)$, with the differential given by a count of certain holomorphic curves in the symplectization $\mathbb{R} \times V$ with boundary on $\mathbb{R} \times \Lambda$.

Remark 2.1. For a Legendrian submanifold Λ of a general contact manifold V the Legendrian algebra \mathcal{A}_Λ is an algebra generated by both Reeb chords and closed Reeb orbits, where the orbits generate a (super)commutative subalgebra. In the case of a 1-jet space there are no closed Reeb orbits and the algebra and its differential involves chords only.

Remark 2.2. Legendrian contact homology is often defined with coefficients in the group ring of $H_2(V, \Lambda)$ rather than $H_1(\Lambda)$, the difference being whether one associates to a holomorphic disk its relative homology class in $H_2(V, \Lambda)$ or the homology class of its boundary in $H_1(\Lambda)$. In the case of knot contact homology, our setup amounts to specializing to $U = 1$ in the language of [EENS13a, Ng11, Ng14] or $Q = 1$ in the language of [AENV14]. Also, as mentioned in the introduction, the version of the DGA that we consider here is the fully noncommutative DGA, in which homology classes in $H_1(\Lambda)$ do not commute with Reeb chords. To get loops rather than paths we fix a base point in each component of Λ and capping paths connecting the base point to each Reeb chord endpoint, see Figure 1.

If $\Lambda \subset V$ is a disconnected Legendrian submanifold, then there is additional structure on the DGA of Λ first described by Mishachev [Mis03]; in modern language this is the “composable algebra”, and we follow the treatment from [BEE12, EENS13b, NRS⁺]. For simplicity we restrict to the case $\Lambda = \Lambda_1 \cup \Lambda_2$. For $i, j = 1, 2$, let \mathcal{R}^{ij} denote the set of Reeb chords that *end* on Λ_i and *begin* on Λ_j . The composable algebra $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ is the noncommutative \mathbb{Z} -algebra generated by Reeb chords of $\Lambda_1 \cup \Lambda_2$, elements of $\mathbb{Z}[H_1(\Lambda_1)]$, elements of $\mathbb{Z}[H_1(\Lambda_2)]$, and two idempotents e_1, e_2 , subject to the relations (where δ_{ij} is the Kronecker delta):

- $e_i e_j = \delta_{ij}$
- $e_i' a = \delta_{i'i'} a$ and $a e_{j'} = \delta_{j'j'} a$ for $a \in \mathcal{R}^{ij}$
- $e_j \alpha = \alpha e_j = \delta_{ij} \alpha$ for $\alpha \in \mathbb{Z}[H_1(\Lambda_i)]$.

Note that $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ is unital with unit $e_1 + e_2$. For $i, j = 1, 2$, define $\mathcal{A}_{\Lambda_i, \Lambda_j} = e_i \mathcal{A}_{\Lambda_1 \cup \Lambda_2} e_j$; then

$$\mathcal{A}_{\Lambda_1 \cup \Lambda_2} = \bigoplus_{i, j \in \{1, 2\}} \mathcal{A}_{\Lambda_i, \Lambda_j}.$$

In more concrete terms, $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ is generated as a \mathbb{Z} -module by monomials of the form

$$\alpha_0 a_1 \alpha_1 a_2 \cdots a_n \alpha_n,$$

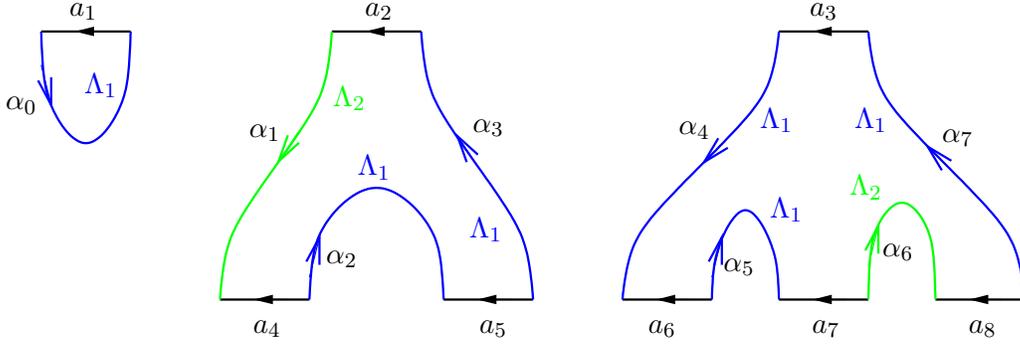


FIGURE 1. Terms contributing to the differential of $a_1 \in \mathcal{R}^{11}$, $a_2 \in \mathcal{R}^{21}$, $a_3 \in \mathcal{R}^{11}$: α_0 , $\alpha_1 a_4 \alpha_2 a_5 \alpha_3$, $\alpha_4 a_6 \alpha_5 a_7 \alpha_6 a_8 \alpha_7$, respectively. Here $a_5, a_6 \in \mathcal{R}^{11}$, $a_4, a_8 \in \mathcal{R}^{21}$, $a_7 \in \mathcal{R}^{12}$, $\alpha_1, \alpha_6 \in H_1(\Lambda_1)$, and $\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7 \in H_1(\Lambda_2)$. The boundaries of the disks lie on $\mathbb{R} \times \Lambda_1$ and $\mathbb{R} \times \Lambda_2$ as shown, and represent the homology classes indicated after closing up with suitably oriented capping paths, see Remark 2.2. Small arrows denote orientations on Reeb chords.

where there is some sequence (i_0, \dots, i_n) with $i_k \in \{1, 2\}$ such that $\alpha_k \in \mathbb{Z}[H_1(\Lambda_{i_k})]$ and $a_k \in \mathcal{R}^{i_{k-1}i_k}$ for all k (and one empty monomial e_i for each component Λ_i). Monomials of this form are the “composable words”. Generators of $\mathcal{A}_{\Lambda_i, \Lambda_j}$ are of the same form but specifically with $i_0 = i$ and $i_n = j$. Note that multiplication $\mathcal{A}_{\Lambda_i, \Lambda_j} \otimes \mathcal{A}_{\Lambda_{j'}, \Lambda_{i'}} \rightarrow \mathcal{A}_{\Lambda_i, \Lambda_{j'}}$ is concatenation if $j = i'$ and 0 otherwise.

The differential ∂ on $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ is defined to be 0 on e_i and on elements of $\mathbb{Z}[H_1(\Lambda_i)]$ and is given by a holomorphic-disk count for Reeb chords of $\Lambda_1 \cup \Lambda_2$. For a Reeb chord $a \in \mathcal{R}^{ij}$ the disks counted in ∂a are maps into $\mathbb{R} \times V$ and have boundary on $(\mathbb{R} \times \Lambda_1) \cup (\mathbb{R} \times \Lambda_2)$, one positive puncture where it is asymptotic to $\mathbb{R} \times a$ and several negative punctures. The contribution to the differential is the composable word of homology classes and Reeb chords in the complement of the positive puncture along the boundary of the disk, see Figure 1. The differential thus respects the direct-sum decomposition $\bigoplus_{i,j \in \{1,2\}} \mathcal{A}_{\Lambda_i, \Lambda_j}$, and this decomposition descends to the homology:

$$\text{LCH}_*(\Lambda_1 \cup \Lambda_2) = H_*(\mathcal{A}_{\Lambda_1 \cup \Lambda_2}, \partial) = \bigoplus_{i,j \in \{1,2\}} (\text{LCH}_*)_{\Lambda_i, \Lambda_j},$$

where $(\text{LCH}_*)_{\Lambda_i, \Lambda_j} = H_*(\mathcal{A}_{\Lambda_i, \Lambda_j}, \partial)$. Recall that Legendrian isotopies induce isomorphisms on Legendrian contact homology via counts of holomorphic disks similar to the differential, see [Ekh08, EES07, EHK]. It follows that a Legendrian isotopy between 2-component Legendrian links induces a quasi-isomorphism between the DGAs that also respects the decomposition.

We can further refine the structure of $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ by considering the filtration

$$\mathcal{A}_{\Lambda_1 \cup \Lambda_2} = \mathcal{F}^0 \mathcal{A}_{\Lambda_1 \cup \Lambda_2} \supset \mathcal{F}^1 \mathcal{A}_{\Lambda_1 \cup \Lambda_2} \supset \mathcal{F}^2 \mathcal{A}_{\Lambda_1 \cup \Lambda_2} \supset \dots$$

where $\mathcal{F}^k \mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ is the subalgebra generated as a \mathbb{Z} -module by words involving at least k mixed chords (Reeb chords either from Λ_1 to Λ_2 or from Λ_2 to Λ_1). This also gives a filtration on the summands of $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$:

$$\begin{aligned} \mathcal{A}_{\Lambda_1, \Lambda_1} &= \mathcal{F}^0 \mathcal{A}_{\Lambda_1, \Lambda_1} \supset \mathcal{F}^2 \mathcal{A}_{\Lambda_1, \Lambda_1} \supset \mathcal{F}^4 \mathcal{A}_{\Lambda_1, \Lambda_1} \supset \dots \\ \mathcal{A}_{\Lambda_1, \Lambda_2} &= \mathcal{F}^1 \mathcal{A}_{\Lambda_1, \Lambda_2} \supset \mathcal{F}^3 \mathcal{A}_{\Lambda_1, \Lambda_2} \supset \mathcal{F}^5 \mathcal{A}_{\Lambda_1, \Lambda_2} \supset \dots \\ \mathcal{A}_{\Lambda_2, \Lambda_1} &= \mathcal{F}^1 \mathcal{A}_{\Lambda_2, \Lambda_1} \supset \mathcal{F}^3 \mathcal{A}_{\Lambda_2, \Lambda_1} \supset \mathcal{F}^5 \mathcal{A}_{\Lambda_2, \Lambda_1} \supset \dots \\ \mathcal{A}_{\Lambda_2, \Lambda_2} &= \mathcal{F}^0 \mathcal{A}_{\Lambda_2, \Lambda_2} \supset \mathcal{F}^2 \mathcal{A}_{\Lambda_2, \Lambda_2} \supset \mathcal{F}^4 \mathcal{A}_{\Lambda_2, \Lambda_2} \supset \dots \end{aligned}$$

We note two properties of the filtration. First, it is compatible with multiplication: the product of elements of \mathcal{F}^{k_1} and \mathcal{F}^{k_2} is an element of $\mathcal{F}^{k_1+k_2}$. Second, the differential ∂ respects the filtration, since the differential of any mixed chord is a sum of words that each includes a mixed chord. As a consequence of this second property, there is an induced filtration on $\text{LCH}_*(\Lambda_1 \cup \Lambda_2)$ as well as its summands $(\text{LCH}_*)_{\Lambda_i, \Lambda_j}(\Lambda_1 \cup \Lambda_2)$.

We abbreviate successive filtered quotients as follows: for k even when $i = j$ and k odd when $i \neq j$, write

$$\mathcal{A}_{\Lambda_i, \Lambda_j}^{(k)} := \mathcal{F}^k \mathcal{A}_{\Lambda_i, \Lambda_j} / \mathcal{F}^{k+2} \mathcal{A}_{\Lambda_i, \Lambda_j}.$$

Then $\mathcal{A}_{\Lambda_i, \Lambda_j}^{(k)}$ is generated as a \mathbb{Z} -module by words with exactly k mixed chords. We will especially be interested in the following filtered quotients with their induced differentials:

- $\mathcal{A}_{\Lambda_i, \Lambda_i}^{(0)}$, which is the DGA of Λ_i itself;
- $\mathcal{A}_{\Lambda_i, \Lambda_j}^{(1)}$ with $i \neq j$, which is generated by words with exactly 1 mixed chord.

Note that for $i \neq j$, the DGAs of Λ_i and of Λ_j act on $\mathcal{A}_{\Lambda_i, \Lambda_j}^{(1)}$ on the left and right, respectively, by multiplication, and this gives $\mathcal{A}_{\Lambda_i, \Lambda_j}^{(1)}$ the structure of a differential bimodule.

2.2. Legendrian contact homology for the conormal and fiber. We now restrict to the case where V is the contact manifold $ST^*\mathbb{R}^3 = J^1(S^2)$. If $K \subset \mathbb{R}^3$ is a knot and $\Lambda_K \subset V$ is the unit conormal bundle of K , then the knot contact homology of K is defined to be the Legendrian contact homology of Λ_K :

$$\text{LCH}_*(\Lambda_K) = H_*(\mathcal{A}_{\Lambda_K}, \partial).$$

The conormal Λ_K is topologically a 2-torus and has trivial Maslov class. The triviality of the Maslov class gives a well-defined integer grading on \mathcal{A}_{Λ_K} , by the Conley–Zehnder index, see [EES07, EENS13b]. A choice of orientation for K gives a distinguished set of generators $l, m \in \pi_1(\Lambda_K) \cong \mathbb{Z}^2$, where m is the meridian and l is the Seifert-framed longitude. The group ring $\mathbb{Z}[H_1(\Lambda_K)] \cong \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ is a subring of \mathcal{A}_{Λ_K} in degree 0, and there is an induced map $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}] \rightarrow HC_0(K)$ that is injective as long as K is not the unknot (see [CELN17]).

Since the Reeb flow on $ST^*\mathbb{R}^3$ is the geodesic flow, Reeb chords correspond under the projection $ST^*\mathbb{R}^3 \rightarrow \mathbb{R}^3$ in a one to one fashion to oriented binormal chords of K : for the flat metric on \mathbb{R}^3 these are simply oriented line segments with endpoints on K that are

perpendicular to K at both endpoints. Furthermore the Conley–Zehnder grading of such a chord agrees with the Morse index for the corresponding critical point of the distance function $K \times K \rightarrow \mathbb{R}$, and hence takes on only the values 0, 1, 2, see [EENS13b, Section 3.3.3].

Next suppose that in addition to the knot K , we choose a point $p \in \mathbb{R}^3 \setminus K$. Then we can form the Legendrian link $\Lambda = \Lambda_K \cup \Lambda_p \subset ST^*\mathbb{R}^3$, where Λ_K is the unit conormal to K as before and Λ_p is the unit cotangent fiber of $ST^*\mathbb{R}^3$ at p .

Let $(\mathcal{A}_{\Lambda_K \cup \Lambda_p}, \partial)$ be the DGA associated to the link $\Lambda_K \cup \Lambda_p$. Then $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$ is generated by Reeb chords of $\Lambda_K \cup \Lambda_p$ (along with homology classes). There are no Reeb chords from Λ_p to itself, and so the Reeb chords of $\Lambda_K \cup \Lambda_p$ come in three types: from Λ_K to itself, to Λ_K from Λ_p , and to Λ_p from Λ_K . These all correspond to binormal chords of $K \cup \{p\}$, where the normality condition is trivial at p .

We now discuss the grading on $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$. Homology classes are graded by 0. The grading on pure Reeb chords from Λ_K to itself is as for \mathcal{A}_{Λ_K} . In order to define the grading of mixed Reeb chords of a two-component Legendrian submanifold of Maslov index 0 such as $\Lambda_K \cup \Lambda_p$, it is customary to choose a path connecting the two Legendrians, along with a continuous field of Legendrian tangent planes along this path (i.e., isotropic 2-planes in the contact hyperplanes along the path) interpolating between the tangent planes to the Legendrians at the two endpoints. There is a \mathbb{Z} 's worth of homotopy classes of such fields of tangent planes, and different choices affect the grading of mixed chords, shifting the grading of chords from Λ_K to Λ_p up by some uniform constant k and shifting the grading of chords from Λ_p to Λ_K down by k . Note here that the usual dimension formulas for holomorphic disks hold and are independent of the path chosen since for any actual disk the path is traversed algebraically zero times.

To assign a specific grading to mixed chords, it is convenient to place K and p in a specific configuration in \mathbb{R}^3 . Let (x, y, z) be linear coordinates on \mathbb{R}^3 . The unit circle in the xy plane is an unknot, and we can braid K around this unknot so that it lies in a small tubular neighborhood of the circle; also, choose p to lie in the xy plane, outside a disk containing the projection of K . If we view $\Lambda_K, \Lambda_p \subset J^1(S^2)$ as fronts in $J^0(S^2) = S^2 \times \mathbb{R}$, then the front of Λ_p is the graph of the function $v \mapsto p \cdot v$ for $v \in S^2$, and in particular the tangent planes to this front over the equator $\{z = 0\} \cap S^2$ are horizontal. On the other hand, if the braid for K has n strands, then the front of Λ_K has $2n$ sheets near the equator, n with positive \mathbb{R} -coordinate and n with negative, and the tangent planes to these sheets are nearly horizontal. We can now take the connecting path between Λ_p and Λ_K as follows: choose a point v in the equator with $p \cdot v < 0$, and over v join the unique point in the front of Λ_p to any of the n points in Λ_K with negative \mathbb{R} -coordinate, see Figure 2. The tangent planes are horizontal at the Λ_p endpoint and nearly horizontal at the Λ_K endpoint; choose the path to consist of nearly horizontal planes over v joining these without rotation.

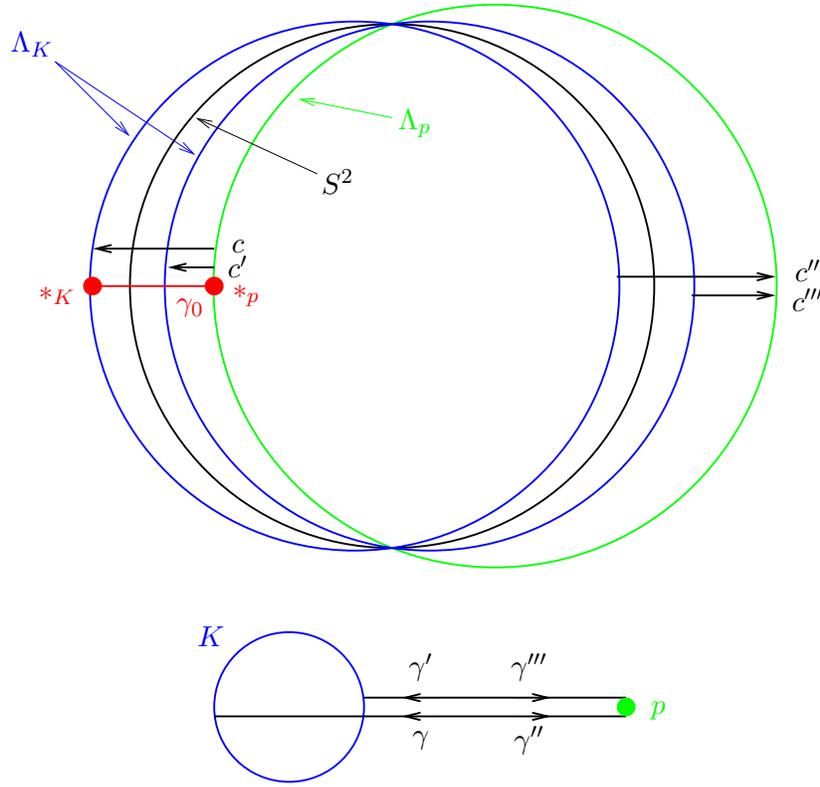


FIGURE 2. Degrees of mixed chords c, c', c'', c''' between the fronts of Λ_K and Λ_p in $S^2 \times \mathbb{R}$, and the corresponding binormal chords $\gamma, \gamma', \gamma'', \gamma'''$ in \mathbb{R}^3 . The connecting path γ_0 between Λ_K and Λ_p is also shown, with endpoints $*_K, *_p$.

Proposition 2.3. *With this choice of configuration, the Reeb chords of $\Lambda_K \cup \Lambda_p$ have grading as follows. Let γ be a binormal chord of $K \cup \{p\}$ corresponding to a Reeb chord c of $\Lambda_K \cup \Lambda_p$. Let “ind” denote the Morse index of the critical point corresponding to γ for the distance function on $K \cup \{p\}$. Then:*

- if $c \in \mathcal{R}^{KK}$ (c goes to Λ_K from Λ_K) then

$$|c| = \text{ind}(\gamma);$$

- if $c \in \mathcal{R}^{Kp}$ (c goes to Λ_K from Λ_p) then

$$|c| = \text{ind}(\gamma);$$

- if $c \in \mathcal{R}^{pK}$ (c goes to Λ_p from Λ_K) then

$$|c| = \text{ind}(\gamma) + 1.$$

Proof. We begin with mixed Reeb chords between Λ_K and Λ_p in either direction. For these, we can use [EENS13b, Lemma 2.5] (cf. [EES05a, Lemma 3.4]), which writes the degree $|c|$ of a Reeb chord c between two sheets of a front projection in terms of the Morse index ind_{loc} of the difference between the functions corresponding to these two sheets, and the difference

$D - U$ between the number of up and down cusps along a capping path for the chord, as

$$|c| = \text{ind}_{\text{loc}} + (D - U) - 1.$$

In our case ind_{loc} is 2 for all mixed Reeb chords: the difference functions between the sheets near the Reeb chords look roughly like the difference function between the front of Λ_p and the 0-section and hence has local maxima, see Figure 2.

To count up and down cusps, we recall the definition of capping path. Let $*_K, *_p$ denote the endpoints of the fixed path γ_0 connecting Λ_K and Λ_p . If c is a mixed chord of $\Lambda_K \cup \Lambda_p$, then the capping path for c is given as follows, cf. [EENS13b, Lemma 2.5]: if c goes to Λ_K (respectively Λ_p) from Λ_p (Λ_K), then take the union of a path in Λ_K (Λ_p) from the endpoint of c to $*_K$ ($*_p$) and a path in Λ_p (Λ_K) from $*_p$ ($*_K$) to the beginning point of c . Any capping path that passes through the north or south pole of S^2 traverses an up cusp if it goes from a negative sheet of Λ_K to a positive sheet, and a down cusp if it goes in the opposite direction; see [EENS13b, section 3.1].

There are four types of mixed chords, which we denote by c, c', c'', c''' as shown in Figure 2. The longer chords c (with corresponding binormal chord γ) from Λ_p to Λ_K begin near $*_p$ and end on the sheet near $*_K$; the capping path for c can be chosen to avoid the poles of S^2 , and so the degree of c is $|c| = 2 - 1 = 1 = \text{ind}(\gamma)$. The shorter chords c' (with corresponding binormal chord γ') from Λ_p to Λ_K end on one of the negative sheets of Λ_K ; the capping path for c' passes from a negative sheet to a positive sheet of Λ_K through one of the poles, traversing one up cusp in the process, and so $|c'| = 2 - 1 - 1 = 0 = \text{ind}(\gamma')$. For the mixed chords c'', c''' from Λ_K to Λ_p with binormal chords γ'', γ''' , similar computations give $|c''| = 2 + 1 - 1 = \text{ind}(\gamma'') + 1$ and $|c'''| = 2 - 1 = \text{ind}(\gamma''') + 1$. This establishes the result for mixed chords.

For pure chords the calculation is similar; we give a brief description and refer to [EENS13b, Lemma 3.7] for details. There are the longer chords corresponding to the chords of the unknot: for the round unknot there is an S^1 Bott-family of chords which after perturbation gives rise to two chords. We write c (with corresponding binormal chord γ) and e (with corresponding binormal chord ϵ) to denote a chord of K corresponding to the shorter and longer chord of the unknot, respectively. The local index at e (respectively c) is 2 (1), and a path connecting the endpoint to the start point has one down cusp. This gives $|e| = 2 + 1 - 1 = 2 = \text{ind}(\epsilon)$, and $|c| = 1 + 1 - 1 = 1 = \text{ind}(\gamma)$. Finally, there are short chords of K that are contained in a tubular neighborhood of the unknot. These are of two types, depending on whether the underlying binormal chord has Morse index 0 or 1. Let a (with corresponding binormal chord α) be of the former type and b (with corresponding binormal chord β) of the latter. Noting that there are paths connecting their start and endpoints without cusps and that the local index is 1 for a and 2 for b , it follows that $|a| = 1 - 1 = 0 = \text{ind}(\alpha)$ and $|b| = 2 - 1 = 1 = \text{ind}(\beta)$. The formulas relating degrees and indices thus hold for all types of chords. \square

From Proposition 2.3, since $\text{ind}(\gamma)$ is in $\{0, 1, 2\}$ if γ joins K to itself and $\{0, 1\}$ otherwise, we find that Reeb chords in \mathcal{R}^{KK} , \mathcal{R}^{Kp} , and \mathcal{R}^{pK} have degrees in $\{0, 1, 2\}$, $\{0, 1\}$, and $\{1, 2\}$, respectively. It follows that $\mathcal{A}_{\Lambda_K, \Lambda_K}$ (respectively $\mathcal{A}_{\Lambda_K, \Lambda_p}$, $\mathcal{A}_{\Lambda_p, \Lambda_K}$) is supported in degree ≥ 0 (respectively ≥ 0 , ≥ 1), and in lowest degree is generated as a \mathbb{Z} -module by only words with the minimal possible number of mixed chords. In particular, $\mathcal{F}^2 \mathcal{A}_{\Lambda_K, \Lambda_K}$, $\mathcal{F}^3 \mathcal{A}_{\Lambda_K, \Lambda_p}$, and $\mathcal{F}^3 \mathcal{A}_{\Lambda_p, \Lambda_K}$ are all zero in degree 0, 0, and 1 respectively, and so:

$$\begin{aligned} H_0(\mathcal{A}_{\Lambda_K, \Lambda_K}) &\cong H_0(\mathcal{A}_{\Lambda_K, \Lambda_K}^{(0)}) \\ H_0(\mathcal{A}_{\Lambda_K, \Lambda_p}) &\cong H_0(\mathcal{A}_{\Lambda_K, \Lambda_p}^{(1)}) \\ H_1(\mathcal{A}_{\Lambda_p, \Lambda_K}) &\cong H_1(\mathcal{A}_{\Lambda_p, \Lambda_K}^{(1)}). \end{aligned}$$

As noted in Section 2.1, the first of these, $H_0(\mathcal{A}_{\Lambda_K, \Lambda_K}^{(0)})$, is exactly the degree 0 Legendrian contact homology of Λ_K , that is, degree 0 knot contact homology. The homology coefficients $\mathbb{Z}[H_1(\Lambda_K)] \cong \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ form a degree 0 subalgebra of the DGA of Λ_K with zero differential, and so we have a map $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ into the degree 0 knot contact homology of K . In addition, $\mathcal{A}_{\Lambda_K, \Lambda_K}$ acts on the left (respectively right) on $\mathcal{A}_{\Lambda_K, \Lambda_p}$ (respectively $\mathcal{A}_{\Lambda_p, \Lambda_K}$), with an induced action on homology.

Definition 2.4. The *KCH-triple* of $\Lambda_K \cup \Lambda_p$ is

$$(R_{KK}, R_{Kp}, R_{pK}) = (H_0(\mathcal{A}_{\Lambda_K, \Lambda_K}), H_0(\mathcal{A}_{\Lambda_K, \Lambda_p}), H_1(\mathcal{A}_{\Lambda_p, \Lambda_K})).$$

Here R_{KK} is viewed as a ring equipped with a map $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}] \rightarrow R_{KK}$, and R_{Kp} and R_{pK} are an (R_{KK}, \mathbb{Z}) -bimodule and a (\mathbb{Z}, R_{KK}) -bimodule, respectively.

Note that although we have chosen a particular placement of K and p above, the KCH-triple of $\Lambda_K \cup \Lambda_p$ is unchanged by isotopy of Λ_K and Λ_p , since it can be defined strictly in terms of graded pieces of the homology of $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$, which is invariant under Legendrian isotopy up to quasi-isomorphism. That is:

Proposition 2.5. *If $\Lambda_{K_0} \cup \Lambda_p$ and $\Lambda_{K_1} \cup \Lambda_p$ are Legendrian isotopic, then they have isomorphic KCH-triples $(R_{K_0 K_0}, R_{K_0 p}, R_{p K_0})$ and $(R_{K_1 K_1}, R_{K_1 p}, R_{p K_1})$, in the sense that there are isomorphisms*

$$\begin{aligned} \psi_{KK} : R_{K_0 K_0} &\xrightarrow{\cong} R_{K_1 K_1} \\ \psi_{Kp} : R_{K_0 p} &\xrightarrow{\cong} R_{K_1 p} \\ \psi_{pK} : R_{p K_0} &\xrightarrow{\cong} R_{p K_1} \end{aligned}$$

compatible with multiplications $R_{K_i K_i} \otimes R_{K_i K_i} \rightarrow R_{K_i K_i}$, $R_{K_i K_i} \otimes R_{K_i p} \rightarrow R_{K_i p}$, and $R_{p K_i} \otimes R_{K_i K_i} \rightarrow R_{p K_i}$. If furthermore the Legendrian isotopy is parametrized in the sense that it sends the basis l_0, m_0 of $H_1(\Lambda_{K_0})$ to the basis l_1, m_1 of $H_1(\Lambda_{K_1})$, then $\psi_{KK}(m_0) = m_1$, $\psi_{KK}(l_0) = l_1$.

Remark 2.6. As mentioned above, the gradings in $\mathcal{A}_{\Lambda_K, \Lambda_p}$ and $\mathcal{A}_{\Lambda_p, \Lambda_K}$ are not canonically defined but rather depend on a choice of homotopy class of a path connecting the tangent planes at base points in the components of the Legendrian link (possible choices are in one to one correspondence with \mathbb{Z}). In general, in Definition 2.4 we would want to set $R_{Kp} = H_d(\mathcal{A}_{\Lambda_K, \Lambda_p})$ and $R_{pK} = H_{1-d}(\mathcal{A}_{\Lambda_p, \Lambda_K})$, where $d \in \mathbb{Z}$ corresponds to the choice of homotopy class of path. (In all cases we still have $R_{KK} = H_0(\mathcal{A}_{\Lambda_K, \Lambda_K})$.)

This indeterminacy would seem to pose problems for Proposition 2.5. However, we can eliminate the ambiguity by stipulating that we have picked the unique choice of grading for which

$$\min\{d \mid H_d(\mathcal{A}_{\Lambda_K, \Lambda_p}) \neq 0\} = 0.$$

This is because with our preferred choice of grading, $H_d(\mathcal{A}_{\Lambda_K, \Lambda_p}) = 0$ for $d < 0$, while we will show that $R_{Kp} = H_0(\mathcal{A}_{\Lambda_K, \Lambda_p})$ is nonzero (see for instance Proposition 4.13).

Remark 2.7. If we choose p sufficiently far away from K , then by action considerations (the action of the Reeb chord at the positive puncture of a holomorphic disk is greater than the sum of the actions at the negative punctures), the differential in $(\mathcal{A}_{\Lambda_K \cup \Lambda_p}, \partial)$ of any word containing exactly k mixed chords must only involve words again containing exactly k mixed chords. In this case, the DGA $(\mathcal{A}_{\Lambda_K \cup \Lambda_p}, \partial)$ is isomorphic to its associated graded DGA under the filtration \mathcal{F}^k , and the homology $\text{LCH}_*(\Lambda_K \cup \Lambda_p)$ decomposes as a direct sum by number of mixed chords.

We will use the KCH-triple of $\Lambda_K \cup \Lambda_p$ to produce a complete knot invariant. More specifically, we have the following object created from the KCH-triple:

Definition 2.8. Let R_{pp} denote the \mathbb{Z} -module

$$R_{pp} = R_{pK} \otimes_{R_{KK}} R_{Kp}.$$

Alternatively, we can write R_{pp} in terms of the homology of $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$:

Proposition 2.9. *We have*

$$R_{pp} \cong H_1(\mathcal{A}_{\Lambda_p, \Lambda_p}) \cong H_1(\mathcal{A}_{\Lambda_p, \Lambda_p}^{(2)}).$$

Proof. The first isomorphism is immediate from the definition of R_{pp} . Since there are no self Reeb chords of Λ_p , and since any mixed Reeb chord to Λ_p from Λ_K has degree ≥ 1 , any degree 1 generator of $\mathcal{A}_{\Lambda_p, \Lambda_p}$ must consist of a mixed chord to Λ_p from Λ_K , followed by some number of Reeb chords of Λ_K , followed by a mixed chord to Λ_K from Λ_p . The result now follows from the definition of the KCH-triple. \square

In fact, we will show (Proposition 4.17) that there is a ring isomorphism

$$\mathbb{Z} \oplus R_{pp} \cong \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$$

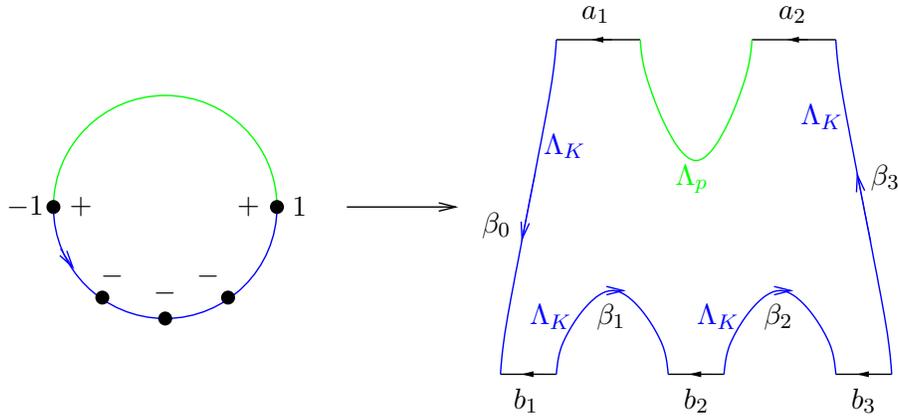


FIGURE 3. A disk in the moduli space $\mathcal{M}(a_1, a_2; \beta_0 b_1 \beta_1 b_2 \beta_2 b_3 \beta_3)$. Here the β_i coefficients record the homology classes of the depicted arcs in Λ_K . The arrows on Reeb chords denote their positive orientations.

and this is the key to proving our main result, Theorem 1.1. To get this, we in particular need a multiplication operation on R_{pp} . In the next subsection, we will define a product map

$$\mu : R_{Kp} \otimes R_{pK} \rightarrow R_{KK}.$$

This will then induce a map

$$\mu : R_{pp} \otimes_{\mathbb{Z}} R_{pp} = R_{pK} \otimes_{R_{KK}} R_{Kp} \otimes_{\mathbb{Z}} R_{pK} \otimes_{R_{KK}} R_{Kp} \rightarrow R_{pK} \otimes_{R_{KK}} R_{KK} \otimes_{R_{KK}} R_{Kp} = R_{pp},$$

which is the desired multiplication on R_{pp} .

2.3. Product. Recall that the differential in the contact homology DGA $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$ that is used to define the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) counts holomorphic disks with one positive puncture in the symplectization $\mathbb{R} \times ST^*\mathbb{R}^3$ with boundary on $\mathbb{R} \times (\Lambda_p \cup \Lambda_K)$. As described in [Ekh08, BEE11], one can also produce invariants by counting holomorphic disks with two positive punctures at mixed Reeb chords, along with an arbitrary number of negative punctures at pure Reeb chords.

For general two-component Legendrian links, the resulting algebraic structure is a bit complicated to describe, but in our case it is simple because Λ_p has no self Reeb chords: reading along the boundary of any of these two-positive-punctured disks, we see a positive puncture from Λ_K to Λ_p , followed by a positive puncture from Λ_p to Λ_K , followed by some number of negative punctures from Λ_K to Λ_K . This allows us to define the product of a Reeb chord from Λ_K to Λ_p with a Reeb chord from Λ_p to Λ_K , or more generally the product of composable words in $\mathcal{A}_{\Lambda_K, \Lambda_p}$ and $\mathcal{A}_{\Lambda_p, \Lambda_K}$, each of which contains exactly one mixed Reeb chord. The result of the product in either case will be an alternating word of pure chords from Λ_K to Λ_K and homotopy classes of loops in Λ_K . (No mixed Reeb chords are involved.)

We now describe this construction in more detail. Let a_1 be a Reeb chord to Λ_K from Λ_p and a_2 a Reeb chord to Λ_p from Λ_K . Let $\mathbf{b} = \beta_0 b_1 \beta_1 \cdots b_m \beta_m$ be a word in Reeb chords

b_i from Λ_K to itself and homology classes β_i in $H_1(\Lambda_K)$. Consider the moduli space of holomorphic disks in the symplectization $\mathbb{R} \times ST^*\mathbb{R}^3$ of the following form. We take the domain of the disks to be the unit disk in the complex plane with punctures and boundary data as follows: there are two positive punctures at 1 and -1 ; the arc in the upper half plane connecting these two punctures maps to $\mathbb{R} \times \Lambda_p$; there are $m \geq 0$ negative punctures along the boundary arc in the lower half plane; and the boundary components in the lower half plane all map to $\mathbb{R} \times \Lambda_K$ according to \mathbf{b} . See Figure 3. We write

$$\mathcal{M}(a_1, a_2; \mathbf{b})$$

for the moduli space of holomorphic disks in the symplectization $\mathbb{R} \times ST^*\mathbb{R}^3$ with punctures and boundary data as described.

The dimension of this moduli space is then the following, where $|c|$ denotes the grading of the Reeb chord c :

$$\dim(\mathcal{M}(a_1, a_2; \mathbf{b})) = |a_1| + |a_2| - |\mathbf{b}|.$$

Remark 2.10. This is a special case of a general dimension formula for holomorphic disks in the symplectization $\mathbb{R} \times ST^*Q$ of the cosphere bundle over an n -manifold, with boundary in $\mathbb{R} \times \Lambda$, where Λ is a Legendrian submanifold of Maslov class 0. If such a disk u has positive punctures at Reeb chords a_1, \dots, a_p and negative punctures at Reeb chords b_1, \dots, b_q then its formal dimension $\dim(u)$ is (see e.g. [CEL10, Theorem A.1] or [Ekh08, Section 3.1]):

$$\dim(u) = (n - 3) + \sum_{j=1}^p (|a_j| - (n - 3)) - \sum_{k=1}^q |b_k|.$$

As in the definition of the differential in Legendrian contact homology, we need to consider orientations of these moduli spaces induced by capping operators and the Fukaya orientation on the space of linearized Cauchy–Riemann operators on the disk with trivialized Lagrangian boundary condition.¹ There is basically only one point where the construction here differs from that used for the differential. The disks in the differential have a unique positive puncture and we write the capped-off linearized problem for a disk with positive puncture at a and negative punctures and boundary data according to $\mathbf{b} = \beta_0 b_1 \cdots b_m \beta_m$ as above as (with C_c^\pm denoting the positive/negative capping operator at the Reeb chord c and L denoting the linearized Cauchy–Riemann operator at the holomorphic disk under consideration)

$$C_a^+ \oplus L \oplus C_{b_1}^- \oplus \cdots \oplus C_{b_m}^- \approx F,$$

where F denotes a trivialized boundary condition on the closed disk and where “ \approx ” means “is related to via a linear gluing exact sequence”, see [EES05b]. For the product, we have disks with two positive punctures and there is no natural way to order the punctures in

¹In fact, for the purposes of this paper and in particular the proof of Theorem 1.1, one could ignore orientations and work over $\mathbb{Z}/2$ rather than \mathbb{Z} . However, for the purposes of the general theory, we will work over \mathbb{Z} throughout.

general. However, in our case the two positive punctures are distinguished since both are mixed and have different endpoint configurations. We choose the following ordering:

$$C_{a_1}^+ \oplus L \oplus C_{b_1}^- \oplus \cdots \oplus C_{b_m}^- \oplus C_{a_2}^+ \approx F.$$

As usual this then induces a linear gluing sequence which in the transverse case orients the moduli space.

With these orientations determined, we can now define μ . Suppose that we have $\mathbf{c}_1 a_1 \in \mathcal{A}_{\Lambda_K, \Lambda_p}^{(1)}$ and $a_2 \mathbf{c}_2 \in \mathcal{A}_{\Lambda_p, \Lambda_K}^{(1)}$, where a_1, a_2 are mixed chords to Λ_K from Λ_p (respectively to Λ_p from Λ_K), and $\mathbf{c}_1, \mathbf{c}_2$ are words in pure Reeb chords on Λ_K and homology classes in Λ_K . Define:

$$\mu(\mathbf{c}_1 a_1, a_2 \mathbf{c}_2) = \sum_{|a_1|+|a_2|-|\mathbf{b}|=1} |\mathcal{M}(a_1, a_2; \mathbf{b})/\mathbb{R}| \mathbf{c}_1 \mathbf{b} \mathbf{c}_2.$$

This produces a map

$$\mu : \mathcal{A}_{\Lambda_K, \Lambda_p}^{(1)} \otimes \mathcal{A}_{\Lambda_p, \Lambda_K}^{(1)} \rightarrow \mathcal{A}_{\Lambda_K, \Lambda_K}^{(0)}.$$

Proposition 2.11 ([Ekh08]). *The product map μ has degree -1 and satisfies the Leibniz rule:*

$$\mu \circ (\partial \otimes 1 + 1 \otimes \partial) = \partial \circ \mu.$$

Thus μ descends to a map on homology.

Here and in the rest of the paper, we use Koszul signs when defining the tensor product of maps: in particular, $(\partial \otimes 1)(a \otimes b) = (\partial a) \otimes b$ while $(1 \otimes \partial)(a \otimes b) = (-1)^{|a|} a \otimes (\partial b)$ if ∂ has odd degree. Although Proposition 2.11 is implicitly contained in [Ekh08], we give the proof for definiteness.

Proof of Proposition 2.11. Once we know that the moduli spaces are transversely cut out for generic data then the fact that μ has degree -1 follows from the dimension formula. The disks with two positive punctures considered here cannot be multiply covered for topological reasons (e.g. only one positive puncture is asymptotic to a chord from Λ_p to Λ_K). Thus the same argument as for disks with one positive puncture can be used to show transversality for generic almost complex structure where the formal dimension then equals the actual dimension, see e.g. [EES07, Proposition 2.3].

To see the displayed equation, we look at the boundary of moduli spaces $\mathcal{M}(a_1, a_2; \mathbf{b})$ of dimension 2. It follows by SFT compactness that the boundary consists of broken curves. We must check that there cannot be any boundary breaking. To see this note that any splitting arc in the domain that separates the positive punctures must connect boundary points that map to distinct components of the Legendrian submanifold. Thus there is no boundary breaking and several-level disks account for the whole boundary. The equation follows from identifying contributing terms with the boundary of an oriented 1-dimensional manifold. \square

We will also need the fact that μ is invariant under Legendrian isotopy. As in [Ekh08] this can be understood by looking at cobordism maps and homotopies of such. We will only need invariance on the level of homology, and this is slightly easier to prove: we need only the statement that the multiplication on homology induced by μ is invariant under Legendrian isotopy and this follows from properties of cobordism maps and analogues of these for the product. To see this first recall that a Legendrian isotopy $\Lambda_{K_t} \cup \Lambda_{p_t}$, $0 \leq t \leq 1$, from $\Lambda_{K_0} \cup \Lambda_{p_0}$ to $\Lambda_{K_1} \cup \Lambda_{p_1}$ gives an exact Lagrangian cobordism $L_K \cup L_p \subset \mathbb{R} \times ST^*\mathbb{R}^3$ that agrees with $(\mathbb{R} \times \Lambda_{K_0}) \cup (\mathbb{R} \times \Lambda_{p_0})$ in the positive end and $(\mathbb{R} \times \Lambda_{K_1}) \cup (\mathbb{R} \times \Lambda_{p_1})$ in the negative. Furthermore there is a cobordism map

$$\Phi: \mathcal{A}_{\Lambda_{K_0} \cup \Lambda_{p_0}} \rightarrow \mathcal{A}_{\Lambda_{K_1} \cup \Lambda_{p_1}}$$

that is a quasi-isomorphism respecting the filtration with respect to the number of mixed chords. This cobordism map counts holomorphic disks in the cobordism as follows. If a is a Reeb chord of $\Lambda_{K_0} \cup \Lambda_{p_0}$ and if \mathbf{b} is an alternating word of Reeb chords and homotopy classes of paths in $\Lambda_{K_1} \cup \Lambda_{p_1}$ then let $\mathcal{M}^{\text{co}}(a, \mathbf{b})$ denote the moduli space of holomorphic disks in $\mathbb{R} \times ST^*\mathbb{R}^3$ with boundary on $L_K \cup L_p$, with one positive puncture where the disk is asymptotic to the Reeb chord a and with several negative punctures which together with the boundary arcs give the word \mathbf{b} . The map Φ is then given by

$$\Phi(a) = \sum_{|a| - |\mathbf{b}| = 0} |\mathcal{M}^{\text{co}}(a, \mathbf{b})| \mathbf{b}.$$

In order to study invariance for the product, we look at moduli spaces in the cobordism analogous to the moduli spaces used in the definition of μ . If a_1 is a Reeb chord from Λ_{K_0} to Λ_{p_0} , a_2 a chord from Λ_{p_0} to Λ_{K_0} , and \mathbf{b} a word of Reeb chords and homotopy classes of paths on Λ_{K_1} as above, then let $\mathcal{M}^{\text{co}}(a_1, a_2; \mathbf{b})$ denote the moduli space of disks with boundary on $L_K \cup L_p$ with two positive punctures asymptotic to a_1 and a_2 , and with negative punctures and boundary arcs mapping according to \mathbf{b} . Then define $\kappa: R_{K_0 p_0} \otimes R_{p_0 K_0} \rightarrow R_{K_1 K_1}$ by:

$$\kappa(\mathbf{c}_1 a_1, a_2 \mathbf{c}_2) = \sum_{|a_1| + |a_2| - |\mathbf{b}| = 0} |\mathcal{M}^{\text{co}}(a_1, a_2; \mathbf{b})| \Phi(\mathbf{c}_1) \mathbf{b} \Phi(\mathbf{c}_2).$$

Proposition 2.12 ([Ekh08]). *Given a Legendrian isotopy $(\Lambda_{K_t}, \Lambda_{p_t})$, $t \leq 0 \leq 1$, and product maps μ_0 and μ_1 for $\Lambda_{K_0} \cup \Lambda_{p_0}$ and $\Lambda_{K_1} \cup \Lambda_{p_1}$ as defined above, we have:*

$$(1) \quad \Phi \circ \mu_0 - \mu_1 \circ \Phi - \kappa \circ (\partial_0 \otimes 1 + 1 \otimes \partial_0) + \partial_1 \circ \kappa = 0.$$

Thus, on the level of homology, Φ sends μ_0 to μ_1 .

Proof. The proof follows from an analysis of the boundary of 1-dimensional moduli spaces of the form $\mathcal{M}^{\text{co}}(a_1, a_2; \mathbf{b})$. By gluing and SFT compactness the boundary of such a moduli space consists of two-level buildings. Hence the terms contributing to (1) are in 1-to-1 correspondence with the boundary of an oriented 1-manifold. The homology statement

follows from (1) together with the fact that Φ is a quasi-isomorphism respecting the filtration. \square

We now connect this general discussion of the product μ with the KCH-triple. Recall from Section 2.2 that we have:

$$R_{KK} \cong H_0(\mathcal{A}_{\Lambda_K, \Lambda_K}^{(0)}) \quad R_{Kp} \cong H_0(\mathcal{A}_{\Lambda_K, \Lambda_p}^{(1)}) \quad R_{pK} \cong H_1(\mathcal{A}_{\Lambda_p, \Lambda_K}^{(1)}).$$

Then the product gives a map

$$\mu : R_{Kp} \otimes R_{pK} \rightarrow R_{KK}.$$

We can now write the invariance property as follows.

Proposition 2.13. *The R_{KK} -bimodule map $\mu : R_{Kp} \otimes_{\mathbb{Z}} R_{pK} \rightarrow R_{KK}$ is invariant under Legendrian isotopy of $\Lambda_K \cup \Lambda_p$: given isotopic $\Lambda_{K_0} \cup \Lambda_p$ and $\Lambda_{K_1} \cup \Lambda_p$ and isomorphisms $\psi_{KK}, \psi_{Kp}, \psi_{pK}$ of KCH-triples as in Proposition 2.5, we have*

$$\psi_{KK} \circ \mu_0 = \mu_1 \circ (\psi_{Kp} \otimes \psi_{pK}).$$

3. STRING TOPOLOGY

In this section we will describe how to extend the results from [CELN17] to enhanced knot contact homology with the product μ . This will allow us to interpret $\text{LCH}_*(\Lambda_K \cup \Lambda_p)$ in low degree in terms of a version of string topology and homotopy data; in particular, we will proceed in Section 4 to use string topology arguments to write the KCH-triple in terms of $\pi_1(\mathbb{R}^3 \setminus K)$.

The main result of [CELN17] is an isomorphism in degree 0 between the Legendrian contact homology of Λ_K and a ‘‘string homology’’ defined using chains of broken strings, where a broken string is a loop in the union $\mathbb{R}^3 \cup L_K$ of the zero section and the conormal bundle of L_K in $T^*\mathbb{R}^3$. Here we will give a modification of this approach that produces an isomorphism between the Legendrian contact homology of $\Lambda_K \cup \Lambda_p$ (in the appropriate degree) and string homology for broken strings in $\mathbb{R}^3 \cup L_K \cup L_p$, where L_p is the conormal of p in $T^*\mathbb{R}^3$, i.e. the fiber $T_p^*\mathbb{R}^3$. We will then prove that the product μ defined in Section 2.3 corresponds to the Pontryagin product on string homology under this isomorphism.

The discussion in this section closely parallels the treatment in [CELN17], as our setup is nearly identical to the one there, differing only in the introduction of L_p . Where convenient, we adopt notation from [CELN17] to make the correspondence clearer.

3.1. Broken strings. Here we recall the definition of broken strings from [CELN17], suitably modified for our purposes. Let $K \subset \mathbb{R}^3$ be a knot and $p \in \mathbb{R}^3$ be a point in the complement of K . Write $Q = \mathbb{R}^3$ and view Q as the zero section in T^*Q , and let $N = L_K \subset T^*Q$ be the conormal bundle to K , while L_p is the cotangent fiber T_p^*Q . We then have three Lagrangians Q , N , and L_p in T^*Q ; N and L_p are disjoint, Q and L_p intersect transversely at p , and Q and N intersect cleanly along K . See Figure 4.

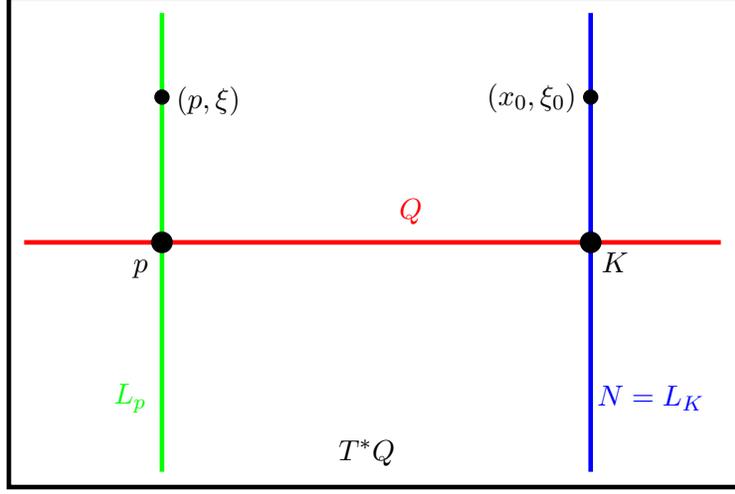


FIGURE 4. The cotangent bundle T^*Q with Lagrangians Q, N, L_p .

Fix base points $(x_0, \xi_0) \in N \setminus K$ and $(p, \xi) \in L_p \setminus \{p\}$. If we use a metric to identify T^*Q and TQ , then these points become (x_0, v_0) with $v_0 \in T_{x_0}N$ and (p, v) with $v \in T_pQ$. This metric also gives a diffeomorphism between a neighborhood of the zero section in N (which in turn is diffeomorphic to all of N) and a tubular neighborhood of K in Q , and we can view $Q \cup N$ as the disjoint union of Q and $N \subset Q$ glued along K . This allows us to identify T_xN with T_xQ for $x \in K$. Similarly we view $Q \cup N \cup L_p$ as the disjoint union of $Q \cup N$ and L_p with $p \in Q$ and $0 \in L_p$ identified, and the metric identifies T_pQ with $T_0L_p = T_p^*Q$.

Now consider a piecewise C^1 path in $Q \cup N \cup L_p$. This path can move between Q and N (in either direction) at a point on $K = Q \cap N$, and between Q and L_p at p ; we call the points where the path changes components *switches*, either at K or at p .

Definition 3.1. A *broken string* is a piecewise C^1 path $s : [a, b] \rightarrow Q \cup N \cup L_p$ such that:

- the endpoints $s(a), s(b)$ are each at one of the two base points $(x_0, v_0) \in N$ or $(p, v) \in L_p$;
- if $s(t_0)$ is a switch at K from N to Q (i.e., for small $\epsilon > 0$, $s((t_0 - \epsilon, t_0]) \subset N$ and $s([t_0, t_0 + \epsilon)) \subset Q$), then:

$$\lim_{t \rightarrow t_0^-} (s'(t))^{\text{normal}} = \lim_{t \rightarrow t_0^+} (s'(t))^{\text{normal}},$$

where we identify $T_{s(t_0)}N$ with $T_{s(t_0)}Q$ and v^{normal} denotes the component of v normal to K with respect to the metric on Q ;

- if $s(t_0)$ is a switch at K from Q to N , then:

$$\lim_{t \rightarrow t_0^-} (s'(t))^{\text{normal}} = - \lim_{t \rightarrow t_0^+} (s'(t))^{\text{normal}};$$

- if $s(t_0)$ is a switch at p from L_p to Q , then:

$$\lim_{t \rightarrow t_0^-} s'(t) = \lim_{t \rightarrow t_0^+} s'(t);$$

- if $s(t_0)$ is a switch at p from Q to L_p , then:

$$\lim_{t \rightarrow t_0^-} s'(t) = - \lim_{t \rightarrow t_0^+} s'(t).$$

The portions of s in Q (respectively N , L_p) are called Q -strings (respectively N -strings, L_p -strings).

Remark 3.2. A broken string models the boundary of a holomorphic disk in T^*Q with boundary on $Q \cup N \cup L_p$ and one positive puncture at infinity at a Reeb chord for $\Lambda_K \cup \Lambda_p$. The condition on the derivatives at a switch follows the behavior of the boundary of such a disk at a point where the boundary switches between Q and N , or between Q and L_p : if v_{in} and v_{out} denote the incoming and outgoing tangent vectors of a broken string at a switch then $v_{\text{out}} = Jv_{\text{in}}$, where J is the almost complex structure along the 0-section induced by the metric.

If we project from T^*Q to Q , then the endpoints of a broken string are each either at p or at the point on K that is the projection of x_0 . With this in mind, we call a broken string s :

- a KK broken string if $s(a) = s(b) = (x_0, v_0)$
- a Kp broken string if $s(a) = (x_0, v_0)$ and $s(b) = (p, v)$
- a pK broken string if $s(a) = (p, v)$ and $s(b) = (x_0, v_0)$
- a pp broken string if $s(a) = s(b) = (p, v)$.

3.2. String homology. We now construct a complex from broken strings whose homology might be called “string homology”; in Section 3.4 below, we will describe an isomorphism between this homology and enhanced knot contact homology.

For $\ell \geq 0$, let Σ_ℓ denote the space of broken strings with ℓ switches at p (note that we do not count switches at K here), equipped with the C^k -topology for some $k \geq 3$. We write

$$\Sigma_\ell = \Sigma_\ell^{KK} \sqcup \Sigma_\ell^{Kp} \sqcup \Sigma_\ell^{pK} \sqcup \Sigma_\ell^{pp}$$

where Σ_ℓ^{ij} denotes the subset of Σ_ℓ corresponding to ij broken strings for $i, j \in \{K, p\}$, and then

$$C_k(\Sigma_\ell) = C_k^{KK}(\Sigma_\ell) \oplus C_k^{Kp}(\Sigma_\ell) \oplus C_k^{pK}(\Sigma_\ell) \oplus C_k^{pp}(\Sigma_\ell)$$

for the free \mathbb{Z} -module generated by generic k -dimensional singular simplices in Σ_ℓ (C_k^{ij} is the summand corresponding to ij broken strings). Here “generic” refers to simplices that satisfy the appropriate transversality conditions at switches and with respect to K and to p ; compare [CELN17, Definition 5.3].

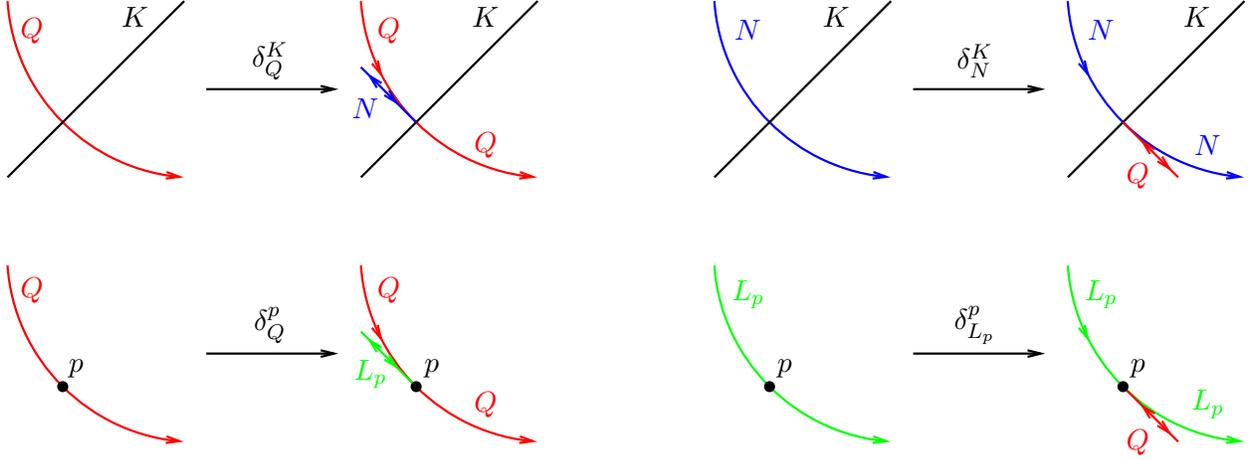


FIGURE 5. The maps δ_Q^K (respectively δ_N^K , δ_Q^p , $\delta_{L_p}^p$) insert an N -string (Q -string, L_p -string, Q -string) at an interior point of a Q -string (N -string, Q -string, L_p -string) that lies on K (K , p , p).

In addition to the usual boundary operator $\partial : C_k(\Sigma_\ell) \rightarrow C_{k-1}(\Sigma_\ell)$ on singular simplices, there are two string operations

$$\delta_Q^K, \delta_N^K : C_k(\Sigma_\ell) \rightarrow C_{k-1}(\Sigma_\ell)$$

defined for $k \leq 2$ in [CELN17, Section 5.3] (where they are called δ_Q , δ_N). We refer to [CELN17] for details, but qualitatively these operations take a generic k -dimensional family of broken strings, identify the subfamily consisting of broken strings where a Q -string or N -string has an interior point in K , and insert a “spike” in N or Q at this point; see Figure 5. We note that this interior intersection condition is codimension 1, and that adding a spike increases the number of switches at K by 2. In our setting, there are two more string operations

$$\delta_Q^p, \delta_{L_p}^p : C_k(\Sigma_\ell) \rightarrow C_{k-2}(\Sigma_{\ell+2})$$

that are defined in the same way as δ_Q^K , δ_N^K , but inserting spikes in L_p or Q where a Q -string or L_p -string has an interior point at p ; see Figure 5 again. Note now that the interior intersection condition is codimension 2, and that adding a spike increases the number of switches at p by 2.

We then have the following result, which is a direct analogue of Proposition 5.8 from [CELN17] and is proved in the same way.

Lemma 3.3. *On generic 2-chains, the operations ∂ , $\delta_Q^K + \delta_N^K$, and $\delta_Q^p + \delta_{L_p}^p$ each have square 0 and pairwise anticommute. In particular, we have*

$$\begin{aligned} (\partial + \delta_Q^K + \delta_N^K)^2 &= 0, \\ (\partial + \delta_Q^K + \delta_N^K)(\delta_Q^p + \delta_{L_p}^p) + (\delta_Q^p + \delta_{L_p}^p)(\partial + \delta_Q^K + \delta_N^K) &= 0, \\ (\delta_Q^p + \delta_{L_p}^p)^2 &= 0. \end{aligned}$$

Lemma 3.3 allows us to construct a complex out of broken strings in the following way. For $m \in \frac{1}{2}\mathbb{Z}$, define

$$C_m = \bigoplus_{k+\ell/2=m} C_k(\Sigma_\ell).$$

By consideration of the parity of the number of switches at p , we can write $C_m = C_m^{KK} \oplus C_m^{pp}$ when m is an integer and $C_m = C_m^{Kp} \oplus C_m^{pK}$ when m is a half-integer. We define a shifted complex \tilde{C}_* , $*$ $\in \mathbb{Z}$, by:

$$\begin{aligned} \tilde{C}_m^{KK} &= C_m^{KK}, & \tilde{C}_m^{Kp} &= C_{m+1/2}^{Kp}, & \tilde{C}_m^{pK} &= C_{m-1/2}^{pK}, & \tilde{C}_m^{pp} &= C_m^{pp}, \\ \tilde{C}_m &= \tilde{C}_m^{KK} \oplus \tilde{C}_m^{Kp} \oplus \tilde{C}_m^{pK} \oplus \tilde{C}_m^{pp}; \end{aligned}$$

that is, we shift the grading up by $1/2$ if the beginning point is p and down by $1/2$ if the endpoint is p . By Lemma 3.3, $\partial + \delta_Q^K + \delta_N^K + \delta_Q^p + \delta_{L_p}^p$ is a differential on \tilde{C}_* that lowers degree by 1.

Remark 3.4. The $\frac{1}{2}$ -grading for strings broken at p has the following geometric counterpart for holomorphic disks with switching Lagrangian boundary conditions on $L_p \cup Q$ and a punctures at the intersection point $p = L_p \cap Q$. Consider a disk $u: (D, \partial D) \rightarrow (T^*Q, L_p \cup Q)$ with m punctures mapping to p and with a positive puncture asymptotic to a Reeb chord a . The formal dimension of u can then be expressed as follows, see [CEL10, Theorem A.1]:

$$(2) \quad \dim(u) = (\dim(Q) - 3) + \mu + m + 1 = \mu + m + 1,$$

where μ is the Maslov index of the loop of Lagrangian tangent planes along the boundary of u . Here we close this loop by the capping operator at a and as follows at the punctures mapping to p : connect the incoming tangent plane (T_p^*Q or T^*L_p) to the outgoing tangent plane (T^*L_p or T_p^*Q) with a negative rotation along the Kähler angle (i.e. act by $e^{-i\frac{\pi}{2}s}$, $0 \leq s \leq 1$). In the case at hand the tangent planes along Q and L_p are stationary with respect to the standard trivialization and the dimension formula reduces to

$$(3) \quad \dim(u) = |a| + m(-\frac{3}{2} + 1).$$

(Note that m is even, i.e., there is an even number of switches, because both the first and the last boundary component map to L_p .) In the dimension formula (3) there is a contribution of $-\frac{1}{2}$. In order to have each puncture contributing with an integer we can for example deform L_p so that the Kähler angles between Q and L_p become $(\epsilon, \frac{\pi}{2}, \frac{\pi}{2})$ instead of the original $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$. This way the contribution to μ in (2) for the punctures at p switching from L_p to Q becomes -2 and the contribution for those switching in the opposite direction -1 , giving total dimension contributions -1 and 0 , respectively. This deformation and the corresponding grading shift are chosen to match our choice of capping path connecting Λ_K to Λ_p : that is, so that both Reeb chords that start at L_p get shifted up by 1 compared to the Morse grading and so that chains of broken strings starting at p are also shifted up.

3.3. Switches at a point in an example. On the complex of broken strings, there are four string operations, δ_Q^K , δ_N^K , δ_Q^p , and $\delta_{L_p}^p$. The two that introduce switches on the knot, δ_Q^K and δ_N^K , have appeared before and are studied at length in [CELN17]. For the other two, δ_Q^p and $\delta_{L_p}^p$, which introduce switches at a point, the only property we need for our main argument is their codimension; in the following section, Section 3.4, we use this to prove an isomorphism to knot contact homology. Here we examine δ_Q^p and $\delta_{L_p}^p$ more closely in a model case. This is a digression from the main argument and can be skipped without loss of continuity, but provides some context for these operations within contact geometry.

Consider $Q = S^n$ and as usual let $L_p \subset T^*S^n$ be the cotangent fiber over p , with $\Lambda_p \subset ST^*S^n$ the Legendrian sphere given by the unit cotangent fiber. By the surgery result from [BEE12, §5.5], we can compute the DGA of Λ_p in ST^*S^n via the DGA for the Legendrian unknot $U \subset S^{2n-1}$, where S^{2n-1} is the standard contact $(2n-1)$ -sphere, i.e., the contact boundary of the standard symplectic $2n$ -ball, with the differential in the latter DGA twisted by a point condition at p .

There is (effectively) only one Reeb chord a of U of grading $|a| = n-1$, see [BEE12], and the differential is $\partial a = p$. To see this one can use the flow tree description of holomorphic disks: it is easy to see that for the standard front of the unknot there is exactly one rigid point constrained Morse flow tree with positive puncture at a . Thus the DGA of Λ_p is generated by chords a^r , $r \geq 1$, of grading $r(n-1) + (n-2)$, and the differential is

$$\partial(a^r) = \sum_{j=2}^r a^{j-1} \cdot a^{r-j}.$$

We claim that this DGA is chain isomorphic to the complex of broken strings in $Q \cup L_p$ with differential given by $\partial + \delta_Q^p + \delta_{L_p}^p$. For the latter, note that L_p is contractible so we simply forget the N -strings and think of the chains of broken strings as the tensor algebra of chains on the based loop space of S^n with differential $\partial + \delta_p$, where δ_p splits a chain over the locus where its evaluation map hits p . By Morse theory, the space of non-constant based loops in S^n is a cell complex with a cell in dimensions

$$(n-1), 2(n-1), 3(n-1), 4(n-1), \dots$$

For degree reasons there are only quadratic terms in the differential $\partial + \delta_p$ and in order to compute δ_p we need to see the unstable manifolds of the cells that correspond to Morse flow in the Bott-manifolds followed by shrinking the loops over half-disks. It is not hard to see that δ_p acts on the Morse cells by splitting the cell of dimension $r(n-1)$ into two cells of dimensions $j(n-1)$ and $k(n-1)$, where $j+k = r-1$, in all possible ways.

We thus conclude that the complex of broken strings in $Q \cup L_p$ is indeed isomorphic to the DGA of the cosphere Λ_p . Furthermore, one can check that this isomorphism is induced by the map that associates to a Reeb chord c of ∂L_p the chain carried by the moduli-space of disks with positive puncture at c and switching boundary condition on $Q \cup L_p$. Note that

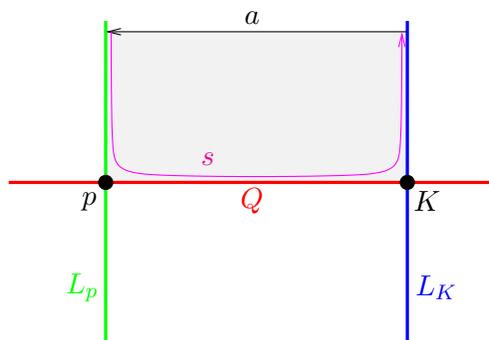


FIGURE 6. A Reeb chord a to Λ_p from Λ_K and a holomorphic disk in $\mathcal{M}^{\text{sw}}(a)$ whose boundary is the depicted broken string s .

each pair of switches in the boundary of such a disk contributes $-(n - 2)$ to the dimension of the moduli space, see Remark 3.4, which explains the difference in grading between the generators of the DGA of Λ_p and generators of the complex of chains of broken strings $(r(n - 1) + (n - 2)$ versus $r(n - 1)$ for $r \geq 1$).

3.4. String homology and enhanced knot contact homology. In [CELN17] the DGA \mathcal{A}_{Λ_K} was related to string homology via a chain map defined through a count of holomorphic disks with switching boundary condition. Here we similarly relate $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$ to string homology. More precisely, if a is a Reeb chord of $\Lambda_K \cup \Lambda_p$ then we let $\mathcal{M}^{\text{sw}}(a)$ denote the moduli space of holomorphic disks in $T^*\mathbb{R}^3$ with one positive puncture asymptotic to the Reeb chord a at infinity, and such that the disk has switching boundary on $Q \cup N \cup L_p$; that is, the boundary of the disk lies on $Q \cup N \cup L_p$, and there are several punctures where the boundary switches between the Lagrangians L_p and Q or between L_K and Q , in either direction.

The boundary of a disk in $\mathcal{M}^{\text{sw}}(a)$, oriented counterclockwise, is a broken string in $Q \cup N \cup L_p$; see Figure 6. More precisely, each endpoint of a Reeb chord of $\Lambda_K \cup \Lambda_p$ is a point in $\Lambda_K \cup \Lambda_p$; fix paths in L_p or N that connect these points to the base points (p, ξ) or (x_0, ξ_0) . Then the union of the boundary of a disk in $\mathcal{M}^{\text{sw}}(a)$ and the paths for the endpoints of a is a broken string.

We can stratify $\mathcal{M}^{\text{sw}}(a)$ by the number of switches at p : for $\ell \geq 0$, let $\mathcal{M}_\ell^{\text{sw}}(a)$ denote the subset of $\mathcal{M}^{\text{sw}}(a)$ of disks with ℓ switches at p . The moduli space $\mathcal{M}_\ell^{\text{sw}}(a)$ is an oriented C^1 -manifold and we let $[\mathcal{M}_\ell^{\text{sw}}(a)]$ denote the chain of broken strings in \tilde{C}_ℓ carried by this moduli space (that is, the chain given by the boundaries of disks in the moduli space). Now define $\Phi: \mathcal{A}_{\Lambda_K \cup \Lambda_p} \rightarrow \tilde{C}_*$ by

$$\Phi(a) = \sum_{\ell} [\mathcal{M}_\ell^{\text{sw}}(a)].$$

Proposition 3.5. *The map*

$$\Phi: (\mathcal{A}_{\Lambda_K \cup \Lambda_p}, \partial) \rightarrow (\tilde{C}_*, \partial + \delta_Q^K + \delta_N^K + \delta_Q^p + \delta_{L_p}^p)$$

is a degree zero chain map of differential graded algebras, where multiplication on \tilde{C}_* is given by chain-level concatenation of broken strings.

Proof. The proof is very similar to [CELN17, Proposition 5.8]. We first check that the map Φ has degree 0. The dimension of $\mathcal{M}_\ell^{\text{sw}}(a)$ is

$$\dim(\mathcal{M}_\ell^{\text{sw}}(a)) = |a| - \ell',$$

where ℓ' is the number switches at p along the boundary where the boundary switches from L_p to Q . To see this, recall from Remark 3.4 that the contribution to the dimension formula is 0 for punctures switching from Q to L_p at p and -1 for the puncture switching from L_p to Q .

We now have three cases. If a joins Λ_K to itself, then $\ell = 2\ell'$ and

$$[\mathcal{M}_\ell^{\text{sw}}(a)] \in C_{|a|-\ell/2}(\Sigma_\ell^{KK}) \subset \tilde{C}_{|a|}^{KK}.$$

If a goes to Λ_K from Λ_p , then if we traverse the boundary of a disk in $\mathcal{M}_\ell^{\text{sw}}(a)$ beginning at the positive puncture, we begin on N , then alternately switch to and from L_p , and end on L_p ; thus $\ell = 2\ell' + 1$ and

$$[\mathcal{M}_\ell^{\text{sw}}(a)] \in C_{|a|-(\ell-1)/2}(\Sigma_\ell^{Kp}) \subset \tilde{C}_{|a|}^{Kp}.$$

Finally, if a goes to Λ_p from Λ_K , then the same argument gives $\ell = 2\ell' - 1$ and

$$[\mathcal{M}_\ell^{\text{sw}}(a)] \in C_{|a|-(\ell+1)/2}(\Sigma_\ell^{pK}) \subset \tilde{C}_{|a|}^{pK}.$$

In all cases we find that Φ preserves degree.

We next study the chain map equation. To this end we must understand the codimension 1 boundary of $\mathcal{M}^{\text{sw}}(a)$ which contributes the singular boundary $\partial\Phi(a)$. This boundary consists of three parts:

- (i) 2-level disks with one level of dimension $\dim(\mathcal{M}^{\text{sw}}(a)) - 1$ and a level of dimension 1 in the symplectization end;
- (ii) 1-level disks in which a boundary arc in Q or in L_K shrinks to a point in K , or equivalently a disk with one boundary arc that hits K in an interior point;
- (iii) 1-level disks in which a boundary arc in Q or in L_p shrinks to a point at p , or equivalently a disk with one boundary arc that hits p in an interior point.

Configurations of type (i) are counted by $\Phi(\partial a)$, configurations of type (ii) by $(\delta_Q^K + \delta_N^K)\Phi(a)$, and configurations of type (iii) by $(\delta_Q^p + \delta_N^p)\Phi(a)$. The chain map equation follows. \square

We will be especially interested in the subcomplexes \tilde{C}_*^{KK} , \tilde{C}_*^{Kp} , \tilde{C}_*^{pK} in the lowest degree. These are given as follows, where the differential is $d = \partial + \delta_Q^K + \delta_N^K$ (the operations δ_Q^p , $\delta_{L_p}^p$

do not appear for degree reasons):

$$\begin{aligned}\tilde{C}_1^{KK} &= C_1^{KK}(\Sigma_0) \oplus C_0^{KK}(\Sigma_2) \xrightarrow{d} \tilde{C}_0^{KK} = C_0^{KK}(\Sigma_0) \\ \tilde{C}_1^{Kp} &= C_1^{Kp}(\Sigma_1) \oplus C_0^{Kp}(\Sigma_3) \xrightarrow{d} \tilde{C}_0^{Kp} = C_0^{Kp}(\Sigma_1) \\ \tilde{C}_2^{pK} &= C_1^{pK}(\Sigma_1) \oplus C_0^{pK}(\Sigma_3) \xrightarrow{d} \tilde{C}_1^{pK} = C_0^{pK}(\Sigma_1).\end{aligned}$$

Note that d acts on the first summand in each case, and is 0 on the second.

A main result from [CELN17] is that Φ induces an isomorphism in degree 0 homology. In our setting, this becomes the following:

Proposition 3.6. *The map Φ induces isomorphisms*

$$\begin{aligned}R_{KK} &= H_0(\mathcal{A}_{\Lambda_K, \Lambda_K}) \xrightarrow{\cong} H_0(\tilde{C}_*^{KK}, d) = \text{coker}(\partial + \delta_Q^K + \delta_N^K : C_1^{KK}(\Sigma_0) \rightarrow C_0^{KK}(\Sigma_0)) \\ R_{Kp} &= H_0(\mathcal{A}_{\Lambda_K, \Lambda_p}) \xrightarrow{\cong} H_0(\tilde{C}_*^{Kp}, d) = \text{coker}(\partial + \delta_Q^K + \delta_N^K : C_1^{Kp}(\Sigma_1) \rightarrow C_0^{Kp}(\Sigma_1)) \\ R_{pK} &= H_1(\mathcal{A}_{\Lambda_p, \Lambda_K}) \xrightarrow{\cong} H_1(\tilde{C}_*^{pK}, d) = \text{coker}(\partial + \delta_Q^K + \delta_N^K : C_1^{pK}(\Sigma_1) \rightarrow C_0^{pK}(\Sigma_1)).\end{aligned}$$

Proof. The isomorphism for R_{KK} is proven in [CELN17, §7] via an action/length filtration argument, and the other isomorphisms use exactly the same argument. A short description of the argument is as follows. A length filtration on chains of broken strings given by the supremum norm of the sum of the lengths of the Q -strings is introduced. On the DGA there is the action filtration and for a suitable choice of almost complex structure on $T^*\mathbb{R}^3$ the chain map Φ respects this filtration. A standard approximation argument shows that the string homology complex is quasi-isomorphic to the string homology complex of piecewise linear broken strings. On the complex of piecewise linear broken strings, a length-decreasing flow (with splittings when the segments cross the knot) then deforms the complex to a complex generated by certain chains associated to binormal chords and using basic holomorphic strips over binormal chords and the action/length filtrations then shows that Φ is a quasi-isomorphism. \square

Remark 3.7. It is likely that Φ is in fact an isomorphism in all degrees. The reason that we restrict to the lowest degree (0 for $\mathcal{A}_{\Lambda_K, \Lambda_K}$ and $\mathcal{A}_{\Lambda_K, \Lambda_p}$ and 1 for $\mathcal{A}_{\Lambda_p, \Lambda_K}$) here, following the same restriction in [CELN17], is that the proof of the isomorphism in [CELN17] involves an explicit examination of moduli spaces of holomorphic disks with switching boundary conditions of dimensions ≤ 2 . To extend the isomorphism to higher degrees would require one to work out the relevant string homology in degree $d+2$, imposing conditions at endpoints of the strings that match degenerations in higher dimensional moduli spaces of holomorphic disks, and this has not been worked out for moduli spaces of dimensions ≥ 3 .

Remark 3.8. In [CELN17], $\text{coker}(\partial + \delta_Q^K + \delta_N^K : C_1^{KK}(\Sigma_0) \rightarrow C_0^{KK}(\Sigma_0))$ is written as “string homology” $H_0^{\text{string}}(K)$, and the first isomorphism in Proposition 3.6 states that $H_0^{\text{string}}(K)$ is isomorphic to knot contact homology in degree 0. A variant of this construction, modified

string homology $\tilde{H}_0^{\text{string}}(K)$, is also considered in [CELN17, §2], and it is observed there that $\tilde{H}_0^{\text{string}}(K) \cong \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$. In our language, modified string homology is defined by

$$\tilde{H}_0^{\text{string}}(K) = \text{coker}(\partial + \delta_Q^K + \delta_N^K : C_1^{pp}(\Sigma_2) \rightarrow C_0^{pp}(\Sigma_2))$$

and the above map is part of the differential $d : \tilde{C}_2^{pp} \rightarrow \tilde{C}_1^{pp}$.

Two things prevent us from using modified string homology to show that $\text{LCH}_*(\Lambda_K \cup \Lambda_p)$ is a complete invariant. First, $\tilde{H}_0^{\text{string}}(K)$ is not directly a summand of the homology of \tilde{C}_* , although it does map to \tilde{C}_1^{pp} . Second, the isomorphism of $\tilde{H}_0^{\text{string}}(K)$ with $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$ is as a \mathbb{Z} -module, without the product structure. One could try to recover the product on $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$, which is crucial to recovering the knot group itself, via the natural concatenation product on \tilde{C}_*^{pp} , but this sends $\tilde{C}_1^{pp} \otimes \tilde{C}_1^{pp}$ to \tilde{C}_2^{pp} rather than to \tilde{C}_1^{pp} .

Instead, we need a product on \tilde{C}_* that (in our grading convention) reduces degree by 1. Intuitively this is given by concatenating a broken string ending at p and a broken string beginning at p , and deleting the two switches at p . More precisely, we use the Pontryagin product; we discuss this product and its holomorphic-curve counterpart next.

3.5. String topology and the product. Having established isomorphisms Φ in low degree between enhanced knot contact homology and string homology, we now examine the behavior of the product map μ under this isomorphism. Recall from Section 2.3 that μ is a map

$$\mu : \mathcal{A}_{\Lambda_K, \Lambda_p}^{(1)} \otimes \mathcal{A}_{\Lambda_p, \Lambda_K}^{(1)} \rightarrow \mathcal{A}_{\Lambda_K, \Lambda_K}^{(0)}.$$

We will show that under the isomorphism Φ , μ maps to the Pontryagin product at the base point $(p, v) \in L_p$, which we now define.

Consider two chains of broken strings in $C_{k_1}^{Kp}(\Sigma_{\ell_1})$ and $C_{k_2}^{pK}(\Sigma_{\ell_2})$. We define their Pontryagin product at (p, v) as the concatenation at (p, v) followed by removing the path between the switches at p that precede and follow this concatenation. This gives a map

$$C_{k_1}^{Kp}(\Sigma_{\ell_1}) \otimes C_{k_2}^{pK}(\Sigma_{\ell_2}) \rightarrow C_{k_1+k_2}^{KK}(\Sigma_{\ell_1+\ell_2-2}).$$

Summing over integers k_1, k_2 and half-integers ℓ_1, ℓ_2 , we get the Pontryagin product at p

$$P : \tilde{C}_*^{Kp} \otimes \tilde{C}_*^{pK} \rightarrow \tilde{C}_*^{KK}$$

which has degree -1 .

We now treat the relation between P and μ . Note that $\Phi : \mathcal{A}_{\Lambda_K \cup \Lambda_p} \rightarrow \tilde{C}_*$ induces maps

$$\mathcal{A}_{\Lambda_K, \Lambda_p}^{(1)} \rightarrow \tilde{C}_*^{Kp}, \quad \mathcal{A}_{\Lambda_p, \Lambda_K}^{(1)} \rightarrow \tilde{C}_*^{pK}.$$

We claim that these Φ maps intertwine P and μ on the level of homology. This is not true on the chain level, but the difference can be measured by a map Ψ that we now define.

If a_1 and a_2 are Reeb chords to Λ_K from Λ_p and to Λ_p from Λ_K , respectively, then we write $\mathcal{M}^{\text{sw}}(a_1, a_2)$ for the moduli space of holomorphic disks $u : D \rightarrow T^*Q$ that send positive punctures at -1 and 1 to a_1 and a_2 at infinity, the arc in the upper half plane connecting these punctures to L_p , and the arc in the lower half plane to $Q \cup L_K \cup L_p$. We can stratify

$\mathcal{M}^{\text{sw}}(a_1, a_2)$ by the number of switches that the boundary of a holomorphic disk has at p ; for $\ell \geq 0$ even, write $\mathcal{M}_\ell^{\text{sw}}(a_1, a_2)$ for the subset of $\mathcal{M}^{\text{sw}}(a_1, a_2)$ corresponding to disks with ℓ switches at p . The formal dimension of this moduli space is, see Remark 2.10,

$$\dim(\mathcal{M}_\ell^{\text{sw}}(a_1, a_2)) = |a_1| + |a_2| - \ell/2.$$

With notation as in Section 2.3, define a map

$$\Psi: \mathcal{A}_{\Lambda_K, \Lambda_p}^{(1)} \otimes \mathcal{A}_{\Lambda_p, \Lambda_K}^{(1)} \rightarrow C_*^{KK}$$

as follows:

$$\Psi(\mathbf{b}_1 a_1 \otimes a_2 \mathbf{b}_2) = \sum_{|a_1|+|a_2|-\ell/2=0} \Phi(\mathbf{b}_1) \cdot [\mathcal{M}_\ell^{\text{sw}}(a_1, a_2)] \cdot \Phi(\mathbf{b}_2),$$

where $[\mathcal{M}_\ell^{\text{sw}}(a_1, a_2)]$ denotes the chain in $C_0^{KK}(\Sigma_\ell) \subset C_{\ell/2}^{KK} = \tilde{C}_{\ell/2}^{KK}$ carried by the moduli space (i.e., the chain of broken strings corresponding to the disks in $\mathcal{M}_\ell^{\text{sw}}(a_1, a_2)$) and where \cdot denotes the concatenation product: given broken strings in $\Phi(\mathbf{b}_1)$, $\mathcal{M}_\ell^{\text{sw}}(a_1, a_2)$, and $\Phi(\mathbf{b}_2)$, we concatenate the three to obtain another broken string.

Remark 3.9. To see that $[\mathcal{M}_\ell^{\text{sw}}(a_1, a_2)]$ is a chain of broken strings we use [CEL10, Theorem 1.2] which implies that there is a uniform bound on the number of switches on the boundary of a disk in any moduli space with two positive punctures.

Proposition 3.10. *On $\mathcal{A}_{\Lambda_K, \Lambda_p}^{(1)} \otimes \mathcal{A}_{\Lambda_p, \Lambda_K}^{(1)}$ we have the following:*

$$\Phi \circ \mu - P \circ (\Phi \otimes \Phi) + \Psi \circ (1 \otimes \partial + \partial \otimes 1) - (\partial + \delta_Q^K + \delta_N^K + \delta_Q^p + \delta_{L_p}^p) \circ \Psi = 0.$$

Proof. To see this we note that the codimension one boundary $\partial \mathcal{M}^{\text{sw}}(a_1, a_2)$ consists of the following breakings:

- Two-level disks with one level in the symplectization of dimension one and one level in the cotangent bundle. These are accounted for by the first and third terms and ∂ in the last term.
- Lagrangian intersection breaking at K , accounted for by the operations $\delta_Q^K + \delta_N^K$ in the last term.
- Lagrangian intersection breaking at p in the upper half disk, accounted for by the second term.
- Lagrangian intersection breaking at p in the lower half disk, accounted for by the operations $\delta_Q^p + \delta_{L_p}^p$ in the last term.

The formula follows. □

We can now assemble our results in low degree into the following result. Recall that $R_{KK} \cong H_0(\mathcal{A}_{\Lambda_K, \Lambda_K}^{(0)})$, $R_{Kp} \cong H_0(\mathcal{A}_{\Lambda_K, \Lambda_p}^{(1)})$, and $R_{pK} \cong H_1(\mathcal{A}_{\Lambda_p, \Lambda_K}^{(1)})$. From Proposition 3.6, we have an isomorphism $\Phi: R_{KK} \rightarrow H_0(\tilde{C}_*^{KK}, d)$. The following is now an immediate consequence of Proposition 3.10.

Proposition 3.11. *The following diagram commutes:*

$$\begin{array}{ccc} R_{Kp} \otimes R_{pK} & \xrightarrow{\mu} & R_{KK} \\ \downarrow \Phi \otimes \Phi & & \downarrow \Phi \\ H_0(\tilde{C}_*^{Kp}, d) \otimes H_1(\tilde{C}_*^{pK}, d) & \xrightarrow{P} & H_0(\tilde{C}_*^{KK}, d). \end{array}$$

4. LEGENDRIAN CONTACT HOMOLOGY AND THE KNOT GROUP

In this section, we use the isomorphism from Section 3 between Legendrian contact homology and string topology to write the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) defined in Section 2.2 in terms of the knot group $\pi_1(\mathbb{R}^3 \setminus K)$. This will allow us to recover the knot group from the KCH-triple along with the product $\mu : R_{Kp} \otimes R_{pK} \rightarrow R_{KK}$. Along the way, we present the KCH-triple in terms of the cord algebra and deduce that enhanced knot contact homology encodes the Alexander module.

4.1. String homology and the cord algebra. From Proposition 3.6, we have isomorphisms between the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) and parts of the homology of the string complex (\tilde{C}_*, d) . As in [CELN17], we can interpret this string homology in terms of the “cord algebra” of K , essentially by considering only the Q -strings. Here we give this cord algebra interpretation of string homology, which will allow us in Section 4.3 to rewrite the KCH-triple in terms of the knot group. The cord-algebra approach has the added benefit of readily yielding the Alexander module of the knot as the homology of a certain linearization of enhanced knot contact homology, as we will see.

We first review the cord algebra as presented in [CELN17, §2.2], adapted to our purposes. Let $K \subset Q$ be an oriented knot and $p \in Q$ be a point in the knot complement, where $Q = \mathbb{R}^3$ as before. Let K' be a parallel copy of K in the Seifert framing, and choose a base point $*$ on K' .

Definition 4.1. A *cord* is a continuous map $\gamma : [0, 1] \rightarrow Q$ with $\gamma(0), \gamma(1) \in (K' \setminus \{*\}) \cup \{p\}$ and $\gamma([0, 1]) \cap K = \emptyset$. A cord is a KK (respectively Kp ; pK ; pp) cord if $\gamma(0), \gamma(1) \in K'$ (respectively $\gamma(0) \in K'$, $\gamma(1) = p$; $\gamma(0) = p$, $\gamma(1) \in K'$; $\gamma(0) = \gamma(1) = p$).

Definition 4.2 ([CELN17]). The *cord algebra* of K , Cord_{KK} , is the noncommutative unital ring freely generated by homotopy classes of KK cords and $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$, modulo the following skein relations, where the cord is drawn in red, K in black, and K' in gray:

$$\begin{array}{l} (1) \quad \begin{array}{c} \text{Red loop on } K' \\ \text{Black line } K \\ \text{Gray line } K' \end{array} = 1 - m \\ (2) \quad \begin{array}{c} \text{Red line } K' \\ \text{Black line } K \\ \text{Gray line } K' \end{array} = m \cdot \begin{array}{c} \text{Red line } K' \\ \text{Black line } K \\ \text{Gray line } K' \end{array} \quad \text{and} \quad \begin{array}{c} \text{Red line } K' \\ \text{Black line } K \\ \text{Gray line } K' \end{array} = \begin{array}{c} \text{Red line } K' \\ \text{Black line } K \\ \text{Gray line } K' \end{array} \cdot m \\ (3) \quad \begin{array}{c} \text{Red line } K' \\ \text{Black line } K \\ \text{Gray line } K' \end{array} = l \cdot \begin{array}{c} \text{Red line } K' \\ \text{Black line } K \\ \text{Gray line } K' \end{array} \quad \text{and} \quad \begin{array}{c} \text{Red line } K' \\ \text{Black line } K \\ \text{Gray line } K' \end{array} = \begin{array}{c} \text{Red line } K' \\ \text{Black line } K \\ \text{Gray line } K' \end{array} \cdot l \end{array}$$

$$(4) \quad \begin{array}{c} \nearrow \nearrow \\ \searrow \searrow \end{array} - \begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} = \begin{array}{c} \nearrow \nearrow \\ \searrow \searrow \end{array} \cdot \begin{array}{c} \nearrow \nearrow \\ \searrow \searrow \end{array} .$$

Note that a typical element of Cord_{KK} is a linear combination of products of cords and elements of $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$, and multiplication in Cord_{KK} is given by formal concatenation of products.

We can extend Definition 4.2 to cover cords with endpoints at p as well, where the relations only apply near K and do not affect the ends at p . Note that the skein relations may still involve KK cords: for example, if the beginning and end points of the cords on the left hand side of (4) lie at K and p respectively, then (4) gives a relation between two Kp cords (the left hand side) and a product of a KK cord and a Kp cord (the right hand side).

Definition 4.3. The Kp cord module of K , Cord_{Kp} , is the left Cord_{KK} -module freely generated by Kp cords, modulo the skein relations (2), (3), and (4) from Definition 4.2. Similarly, the pK cord module of K , Cord_{pK} , is the right Cord_{KK} -module freely generated by pK , modulo the same skein relations.

Now a broken string in $C_0^{KK}(\Sigma_0)$ (respectively $C_0^{Kp}(\Sigma_1)$, $C_0^{pK}(\Sigma_1)$) produces an element of Cord_{KK} (respectively Cord_{Kp} , Cord_{pK}) given by the product of the Q -strings taken in order. This map induces maps from string homology to the cord algebra and modules, and as in [CELN17] we can show that these maps are isomorphisms. Combined with Proposition 3.6, this shows that the cord algebra and modules are isomorphic to the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) , and this is the fact that we will exploit in this section to prove Theorem 1.1.

Proposition 4.4. *There are isomorphisms*

$$\begin{aligned} \text{Cord}_{KK} &\cong H_0(\tilde{C}_*^{KK}, d) \cong R_{KK} \\ \text{Cord}_{Kp} &\cong H_0(\tilde{C}_*^{Kp}, d) \cong R_{Kp} \\ \text{Cord}_{pK} &\cong H_1(\tilde{C}_*^{pK}, d) \cong R_{pK} \end{aligned}$$

where the first line is a ring isomorphism, and the second and third lines send the left and right actions of Cord_{KK} to the left and right actions of R_{KK} . Under these isomorphisms, the map $\mu : R_{Kp} \otimes R_{pK} \rightarrow R_{KK}$ is the concatenation map

$$\text{Cord}_{Kp} \otimes \text{Cord}_{pK} \rightarrow \text{Cord}_{KK} .$$

Proof. The first line is proved in Proposition 2.9 of [CELN17], and the other two lines have the same proof. The fact that these isomorphisms preserve multiplication follows formally from the construction of the cord algebra and modules. The description of μ as a concatenation product is a direct consequence of Proposition 3.11. \square

4.2. Enhanced knot contact homology and the Alexander module. Here we digress from the main argument to observe that we can use the cord modules to recover the Alexander module $H_1(\tilde{X}_K)$ of K , where \tilde{X}_K is the infinite cyclic cover of $\mathbb{R}^3 \setminus K$ and $H_1(\tilde{X}_K)$ is viewed as a $\mathbb{Z}[m^{\pm 1}]$ -module as usual by deck transformations. As a consequence, we show that a certain canonical linearization of enhanced knot contact homology contains the Alexander module and thus the Alexander polynomial.

It was previously known [Ng08] that the Alexander module can be extracted from the same linearization of usual knot contact homology $\text{LCH}_*(\Lambda_K)$, but in a somewhat obscure way—essentially, the degree 1 linearized homology is the second tensor product of $H_1(\tilde{X}_K) \oplus \mathbb{Z}[m^{\pm 1}]$, with the proof involving an examination of the combinatorial form of the DGA of Λ_K in terms of a braid representative for K , and a relation to the Burau representation. Here we will see that with the introduction of the fiber Λ_p alongside Λ_K , we can instead deduce the Alexander module in a significantly simpler way. In particular, we will use linearized homology not in degree 1 but in degree 0, which is more geometrically natural (for instance, it relates more easily to the cord algebra).

We first present a variant of the cord algebra and modules, following [Ng08] and especially the discussion in [CELN17, §2.2]. Choose a base point $*$ on K corresponding to the base point $*$ on K' . Let an *unframed cord* of K be a path whose endpoints are in $(K \setminus \{*\}) \cup \{p\}$ and which is disjoint from K in its interior; we can divide these into KK , Kp , pK , pp cords depending on where the endpoints lie.

Definition 4.5 ([Ng08]). The *unframed cord algebra* of K , Cord'_{KK} , is the noncommutative algebra over $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ generated by homotopy classes of unframed KK cords, modulo the following skein relations:

$$\begin{aligned}
 (1) \quad & \text{A cord with a red loop} = 1 - m \\
 (2) \quad & \text{A cord with a red dot} = l \cdot \text{A cord with a black dot} \quad \text{and} \quad \text{A cord with a red dot} = \text{A cord with a black dot} \cdot l \\
 (3) \quad & \text{A cord with a red dot} - m \cdot \text{A cord with a red dot} = \text{A cord with a red dot} \cdot \text{A cord with a red dot}
 \end{aligned}$$

The *unframed Kp (respectively pK) cord module* of K , Cord'_{Kp} (respectively Cord'_{pK}), is the right (respectively left) Cord'_{KK} -module generated by unframed Kp (respectively pK) cords, modulo the skein relations (2) and (3).

Note that Cord'_{KK} , Cord'_{Kp} , and Cord'_{pK} are all $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ -modules, unlike their framed counterparts Cord_{KK} , Cord_{Kp} , Cord_{pK} , where elements of $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ do not necessarily commute with cords. However, we have the following.

Proposition 4.6. *The unframed cord algebra and modules Cord'_{KK} , Cord'_{Kp} , Cord'_{pK} are isomorphic to the quotients of the cord algebra and modules Cord_{KK} , Cord_{Kp} , Cord_{pK} obtained by imposing the relations that elements of $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ commute with cords.*

Proof. This is essentially laid out in [CELN17, §2.2]. Fix a cord γ_0 from p to a point $x_0 \in K' \setminus \{*\}$. Given any cord γ , we can produce a loop $\tilde{\gamma}$ in $\mathbb{R}^3 \setminus K$ based at p , by joining any endpoint of γ on K' to x_0 along (any path in) K' , and appending γ_0 or $-\gamma_0$ as necessary. Let γ' be the unframed cord obtained from γ by joining any endpoint of γ on K' to the corresponding point on K by a straight line segment normal to K . Then the map

$$\gamma \mapsto m^{-\text{lk}(\tilde{\gamma}, K)} \gamma'$$

gives the desired isomorphisms from the quotients of Cord_{KK} , Cord_{Kp} , Cord_{pK} to Cord'_{KK} , Cord'_{Kp} , Cord'_{pK} . (For the inverse maps from Cord' to Cord , homotope any cord with a beginning or end point on K so that it begins or ends with γ_0 or $-\gamma_0$, and then remove $\pm\gamma_0$.) Note that the displayed map from Cord to Cord' sends the skein relations (1), (3), (4) in Definition 4.2 to (1), (2), (3) in Definition 4.5, and the normalization by powers of m means that (2) from Definition 4.2 becomes trivial under this map. \square

Now from [Ng08], there is a canonical augmentation of the DGA for K ,

$$\epsilon : (\mathcal{A}_{\Lambda_K}, \partial) \rightarrow (\mathbb{Z}[m^{\pm 1}], 0),$$

whose definition we recall here. Since \mathcal{A}_{Λ_K} is supported in nonnegative degree, the graded map ϵ is determined by its action on the degree 0 part of \mathcal{A}_{Λ_K} , or equivalently (since $\epsilon \circ \partial = 0$) by the induced action on $H_0(\mathcal{A}_{\Lambda_K}, \partial)$. This in turn is determined by the induced action on Cord'_{KK} , which by Proposition 4.6 is the quotient of $H_0(\mathcal{A}_{\Lambda_K}, \partial)$ by setting l, m to commute with everything. On Cord'_{KK} , ϵ is defined as follows:

$$\epsilon(l) = 1$$

$$\epsilon(m) = m$$

$$\epsilon(\gamma) = 1 - m$$

for any unframed KK cord γ . (Note that ϵ preserves the skein relations for Cord'_{KK} and is thus well-defined.) We can extend ϵ from an augmentation of \mathcal{A}_{Λ_K} to an augmentation of $\mathcal{A}_{\Lambda_K \cup \Lambda_p}$ by setting ϵ to be 0 for any mixed chord between Λ_K and Λ_p .

Remark 4.7. Applying [AENV14, Theorem 6.15] to the holomorphic strips over binormal chords shows that the augmentation ϵ is induced by an exact Lagrangian filling M_K diffeomorphic to the knot complement, obtained by joining the conormal L_K and the zero-section Q via Lagrange surgery along the knot K .

Linearizing with respect to this augmentation gives the linearized contact homology

$$\text{LCH}_*^\epsilon(\Lambda_K \cup \Lambda_p) = (\text{LCH}_*^\epsilon)_{\Lambda_K, \Lambda_K} \oplus (\text{LCH}_*^\epsilon)_{\Lambda_K, \Lambda_p} \oplus (\text{LCH}_*^\epsilon)_{\Lambda_p, \Lambda_K} \oplus (\text{LCH}_*^\epsilon)_{\Lambda_p, \Lambda_p}.$$

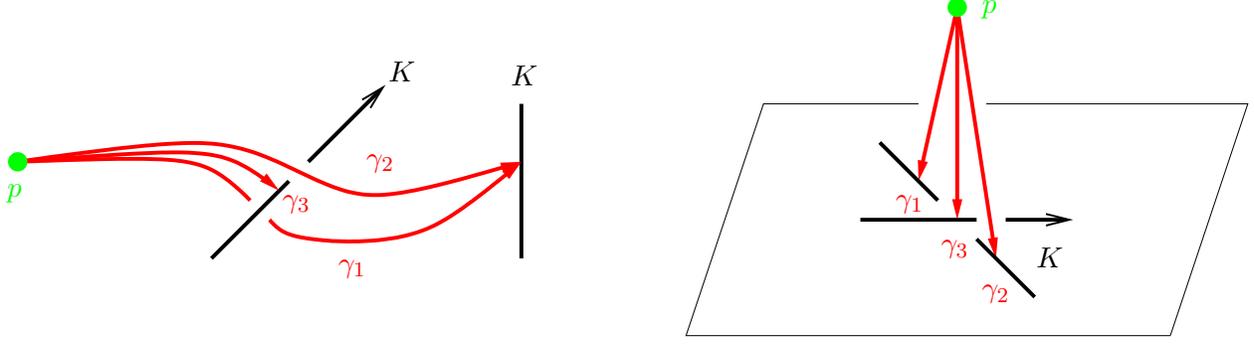


FIGURE 7. Three pK cords $\gamma_1, \gamma_2, \gamma_3$ related by $\gamma_1 - m\gamma_2 = (1 - m)\gamma_3$.

As discussed previously, in [Ng08] it is shown that $(\text{LCH}_1^\epsilon)_{\Lambda_K, \Lambda_K}$ recovers the Alexander module $H_1(\tilde{X}_K)$. Here instead we have the following.

Proposition 4.8. *We have isomorphisms of $\mathbb{Z}[m^{\pm 1}]$ -modules*

$$(\text{LCH}_0^\epsilon)_{\Lambda_K, \Lambda_p} \cong (\text{LCH}_1^\epsilon)_{\Lambda_p, \Lambda_K} \cong H_1(\tilde{X}_K) \oplus \mathbb{Z}[m^{\pm 1}].$$

Proof. We will prove the isomorphism for $(\text{LCH}_1^\epsilon)_{\Lambda_p, \Lambda_K}$; the isomorphism for $(\text{LCH}_0^\epsilon)_{\Lambda_K, \Lambda_p}$ follows by symmetry between Cord'_{Kp} and Cord'_{pK} . The complex whose homology computes $(\text{LCH}_*^\epsilon)_{\Lambda_p, \Lambda_K}$ is the free $\mathbb{Z}[m^{\pm 1}]$ -module generated by Reeb chords to Λ_p from Λ_K , with the differential ∂^{lin} given by applying the augmentation ϵ to all pure Reeb chords from Λ_K to itself to the usual differential ∂ . In particular, since the degree 1 homology $(\text{LCH}_1^\epsilon)_{\Lambda_p, \Lambda_K}$ is the quotient of the $\mathbb{Z}[m^{\pm 1}]$ -module generated by degree 1 Reeb chords to Λ_p from Λ_K by the image of ∂^{lin} , we have:

$$(\text{LCH}_1^\epsilon)_{\Lambda_p, \Lambda_K} \cong R_{pK} \otimes_\epsilon \mathbb{Z}[m^{\pm 1}] \cong \text{Cord}'_{pK} \otimes_\epsilon \mathbb{Z}[m^{\pm 1}].$$

Here by “ \otimes_ϵ ” we mean $\otimes_{R_{KK}}$ (or $\otimes_{\text{Cord}'_{KK}}$) where we use ϵ to give $\mathbb{Z}[m^{\pm 1}]$ the structure of an R_{KK} -module (or Cord'_{KK} -module), and implicitly we are setting l, m to commute with everything in $(\text{LCH}_1^\epsilon)_{\Lambda_p, \Lambda_K}$ and R_{pK} .

Now $\text{Cord}'_{pK} \otimes_\epsilon \mathbb{Z}[m^{\pm 1}]$ is the quotient of the free $\mathbb{Z}[m^{\pm 1}]$ -module generated by unframed pK cords by the skein relations (2) and (3) from Definition 4.5, where l is sent to 1 and all KK cords are sent to $1 - m$. Relation (2) then says that pK cords are unchanged if we move their K endpoint over $*$, while relation (3) becomes:

$$\begin{array}{c} \nearrow \\ \searrow \end{array} - m \begin{array}{c} \nearrow \\ \searrow \end{array} = (1 - m) \begin{array}{c} \nearrow \\ \searrow \end{array} .$$

That is, if $\gamma_1, \gamma_2, \gamma_3$ are unframed pK cords that are related as shown in the left side of Figure 7, then we impose the relation:

$$\gamma_1 - m\gamma_2 = (1 - m)\gamma_3.$$

Thus we can describe $\text{Cord}'_{pK} \otimes_{\epsilon} \mathbb{Z}[m^{\pm 1}]$ in terms of a knot diagram for K as follows. Use the diagram to place K in a neighborhood of the xy plane in \mathbb{R}^3 , and place p high above the xy plane along the z axis. If the diagram has n crossings, then it divides K into n strands from undercrossing to undercrossing. Then $\text{Cord}'_{pK} \otimes_{\epsilon} \mathbb{Z}[m^{\pm 1}]$ is generated by n unframed pK cords, namely straight line segments from p to any point on each of these strands, and each crossing gives a relation $\gamma_1 - m\gamma_2 = (1 - m)\gamma_3$ if $\gamma_1, \gamma_2, \gamma_3$ are as shown in the right side of Figure 7. But this is the well-known presentation of $H_1(\tilde{X}_K) \oplus \mathbb{Z}[m^{\pm 1}]$ from knot colorings. In particular, what we have just described is the Alexander quandle of K , see [Joy82]. \square

Remark 4.9. The description we have given in this section for the unframed cord modules is highly reminiscent of the construction of the knot quandle from [Joy82], which is known to be a complete invariant. However, we do not know how to extract the entire knot quandle, rather than just the Alexander quandle (which is a quotient), from the unframed cord module.

4.3. String homology in terms of the knot group. Having expressed the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) in terms of cords in Section 4.1, our next step en route to proving Theorem 1.1 is to rewrite the KCH-triple further, in terms of the knot group and the peripheral subgroup of the knot K . For R_{KK} , which is the degree 0 knot contact homology of K , this was done in [CELN17, §2.3–2.4], and we follow the treatment there.

Write $\pi = \pi_1(\mathbb{R}^3 \setminus K)$ for the knot group and $\hat{\pi} = \pi_1(\Lambda_K)$ for the peripheral subgroup. A framing and orientation on K gives meridian and longitude classes $m, l \in \hat{\pi}$, which we can then view as classes in π as well. In what follows, we place square brackets around elements of π and curly brackets around elements of $\hat{\pi}$.

Define S to be the \mathbb{Z} -module freely generated by words that are formal products of nontrivial words whose letters are alternately in π and $\hat{\pi}$, divided by the following string relations, where we use x and α to denote elements of π and $\hat{\pi}$ respectively:

- (1) $\cdots_1 [x\alpha_1] \{\alpha_2\} \cdots_2 = \cdots_1 [x] \{\alpha_1\alpha_2\} \cdots_2$
- (2) $\cdots_1 \{\alpha_1\} [\alpha_2x] \cdots_2 = \cdots_1 \{\alpha_1\alpha_2\} [x] \cdots_2$
- (3) $(\cdots_1 [x_1x_2] \cdots_2) - (\cdots_1 [x_1mx_2] \cdots_2) = \cdots_1 [x_1] \{1\} [x_2] \cdots_2$
- (4) $(\cdots_1 \{\alpha_1\alpha_2\} \cdots_2) - (\cdots_1 \{\alpha_1m\alpha_2\} \cdots_2) = \cdots_1 \{\alpha_1\} [1] \{\alpha_2\} \cdots_2$.

Note that there is no restriction on generators of S as to whether the first or last letters are in π or $\hat{\pi}$. We can define a product on S as follows: multiplication of two words w_1, w_2 generating S is zero unless the last letter of w_1 and the first letter of w_2 are both in π or both in $\hat{\pi}$, in which case it is concatenation combined with the product in π or $\hat{\pi}$; that is,

$$\begin{aligned} (\cdots_1 \{\alpha_1\}) \cdot (\{\alpha_2\} \cdots_2) &= \cdots_1 \{\alpha_1\alpha_2\} \cdots_2 \\ (\cdots_1 [x_1]) \cdot ([x_2] \cdots_2) &= \cdots_1 [x_1x_2] \cdots_2. \end{aligned}$$

We now have the following result identifying R_{KK}, R_{Kp}, R_{pK} from Section 3.1 with summands of S .

Proposition 4.10. R_{KK} , R_{Kp} , and R_{pK} are isomorphic to the \mathbb{Z} -submodules of S generated by the following sets:

- for R_{KK} , words beginning and ending in $\hat{\pi}$;
- for R_{Kp} , words beginning in $\hat{\pi}$ and ending in π ;
- for R_{pK} , words beginning in π and ending in $\hat{\pi}$.

Multiplication in S induces maps $R_{KK} \otimes R_{KK} \rightarrow R_{KK}$, $R_{KK} \otimes R_{Kp} \rightarrow R_{Kp}$, $R_{pK} \otimes R_{KK} \rightarrow R_{pK}$ that agree with, respectively, the ring structure on R_{KK} and the R_{KK} -module structure on R_{Kp} and R_{pK} .

Proof. Same as the proof of [CELN17, Proposition 2.14]. Briefly, by Proposition 4.4, the KCH-triple is isomorphic to $(\text{Cord}_{KK}, \text{Cord}_{Kp}, \text{Cord}_{pK})$. Given a cord, we can produce a closed loop in $\mathbb{R}^3 \setminus K$ based at p , and hence an element of π , as in the proof of Proposition 4.6. Thus products of cords, with elements of $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ in between, correspond to alternating products of elements of π and $\hat{\pi}$. The string relations on S come from the skein relations on cords. Note that the distinct behaviors of R_{KK} , R_{Kp} , and R_{pK} in the statement of Proposition 4.10 come from the construction of the cord algebra and modules: an element of Cord_{KK} begins and ends with an element of $\hat{\pi}$ (possibly 1), while an element of Cord_{Kp} begins with an element of $\hat{\pi}$ and ends with a cord (which maps to π), and similarly for Cord_{pK} . \square

To clarify:

- R_{KK} is generated by $\{\alpha_1\}, \{\alpha_1\}[x_1]\{\alpha_2\}, \{\alpha_1\}[x_1]\{\alpha_2\}[x_2]\{\alpha_3\}, \dots$
- R_{Kp} is generated by $\{\alpha_1\}[x_1], \{\alpha_1\}[x_1]\{\alpha_2\}[x_2], \dots$
- R_{pK} is generated by $[x_1]\{\alpha_1\}, [x_1]\{\alpha_2\}[x_2]\{\alpha_2\}, \dots$

where $\alpha_i \in \hat{\pi}$ and $x_i \in \pi$. To this, we can then add:

- $R_{pp} = R_{pK} \otimes_{R_{KK}} R_{Kp}$ is generated by $[x_1]\{\alpha_1\}[x_2], [x_1]\{\alpha_1\}[x_2]\{\alpha_2\}[x_3], \dots$

Finally, the product $\mu : R_{Kp} \otimes R_{pK} \rightarrow R_{KK}$ has a simple interpretation in terms of S , since by Proposition 4.4 it is the concatenation product:

$$\mu(\{\alpha_0\} \cdots \{\alpha_1\}[x_1], [x_2]\{\alpha_2\} \cdots \{\alpha_3\}) = \{\alpha_0\} \cdots \{\alpha_1\}[x_1x_2]\{\alpha_2\} \cdots \{\alpha_3\}.$$

The product on $R_{pp} = R_{pK} \otimes_{R_{KK}} R_{Kp}$ induced by μ is then also given by concatenation.

4.4. The KCH-triple within $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$. Although the notation from Section 4.3 using square and curly brackets is natural from the viewpoint of broken strings, it will be convenient for our purposes to reinterpret the KCH-triple (R_{KK}, R_{Kp}, R_{pK}) directly in terms of the group ring of the knot group, which we henceforth denote by

$$R := \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)].$$

This is the content of Proposition 4.13 below.

To prepare for this result, extend the notation $\cdots [x_1]\{\alpha_1\}[x_2]\cdots \in S$, where up to now we have $x_i \in \pi$ and $\alpha_i \in \hat{\pi}$, by linearity to allow for arbitrary $x_i \in R = \mathbb{Z}\pi$. Given any element of S of the form $\cdots [x_1]\{\alpha_1\}[x_2]\cdots$ with $\alpha_i \in \hat{\pi}$ and $x_i \in \mathbb{Z}\pi$, the string relations on S allow us to get rid of any internal part in curly braces, where “internal” means not at the far left or far right. More precisely, by (1) and (3) from the defining relations for S in Section 4.3, we can write:

$$\cdots [x_1]\{\alpha_1\}[x_2]\cdots = \cdots [x_1\alpha_1x_2]\cdots - \cdots [x_1\alpha_1mx_2]\cdots .$$

This allows us to inductively reduce the number of internal curly braces until none are left.

Thus for instance we can write any element of R_{Kp} as a linear combination of elements of the form $\{\alpha_1\}[x_1]$, where $x_1 \in \mathbb{Z}\pi$, and this in turn is equal to $\{1\}[\alpha_1x_1]$ by string relation (2). Similar results hold for R_{pK} and R_{KK} , as well as for R_{pp} , and we conclude the following:

Proposition 4.11. *As \mathbb{Z} -submodules of S , we have:*

- R_{KK} is generated by the elements of the form $\{\alpha\}$ and $\{1\}[x]\{1\}$ for $\alpha \in \hat{\pi}$ and $x \in \pi$;
- R_{Kp} is generated by $\{1\}[x]$ for $x \in \pi$;
- R_{pK} is generated by $[x]\{1\}$ for $x \in \pi$;
- R_{pp} is generated by $[x_1]\{1\}[x_2]$ for $x_1, x_2 \in \pi$.

Write $\hat{R} = \mathbb{Z}[\hat{\pi}] = \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$, and view \hat{R} as a subring of R . In [CELN17, Proposition 2.20], it is shown that the map $\{\alpha\} \mapsto \alpha$, $\{1\}[x]\{1\} \mapsto x(1-m)$ induces an isomorphism from R_{KK} to $\hat{R} + R(1-m)$, where the latter is viewed as a subring of R (and $R(1-m)$ is the left ideal generated by $1-m$).

Remark 4.12. To be precise, the map in [CELN17, Proposition 2.20] is from R_{KK} to $\hat{R} + (1-m)R$ rather than $\hat{R} + R(1-m)$, and sends $\{1\}[x]\{1\}$ to $(1-m)x$ rather than $x(1-m)$. This however is just a choice of where to place the $(1-m)$ factors. Our convention can be derived from the convention in [CELN17] by the symmetry that reverses the order of words in S .

We can now generalize this isomorphism to the entire KCH-triple.

Proposition 4.13. *We have \mathbb{Z} -module isomorphisms between the KCH-triple and the following \mathbb{Z} -submodules of $R = \mathbb{Z}[\pi]$:*

$$\begin{array}{ll} R_{KK} \xrightarrow{\cong} \hat{R} + R(1-m) & \{\alpha\} \mapsto \alpha, \{1\}[x]\{1\} \mapsto x(1-m); \\ R_{Kp} \xrightarrow{\cong} R & \{1\}[x] \mapsto x; \\ R_{pK} \xrightarrow{\cong} R(1-m) & [x]\{1\} \mapsto x(1-m), \end{array}$$

where the second and third isomorphisms hold for any knot K and the first isomorphism holds as long as K is not the unknot. We use ϕ to denote all of these isomorphisms; then it

is furthermore the case that ϕ sends all multiplications $R_{KK} \otimes R_{KK} \rightarrow R_{KK}$, $R_{KK} \otimes R_{Kp} \rightarrow R_{Kp}$, $R_{pK} \otimes R_{KK} \rightarrow R_{pK}$, as well as the product $R_{Kp} \otimes R_{pK} \rightarrow R_{KK}$, to multiplication in R .

Proof. This follows the proof of [CELN17, Proposition 2.20]. To see that ϕ is well-defined, extend the definition of ϕ to all generators of S (ignoring Proposition 4.11 for the moment) by:

$$\begin{aligned} \{\alpha_1\}[x_1]\{\alpha_2\} \cdots \{\alpha_{k-1}\}[x_k]\{\alpha_k\} &\mapsto \alpha_1(1-m)x_1\alpha_2(1-m) \cdots \alpha_{k-1}(1-m)x_k\alpha_k \\ \{\alpha_1\}[x_1]\{\alpha_2\} \cdots \{\alpha_{k-1}\}[x_k] &\mapsto \alpha_1x_1\alpha_2(1-m) \cdots \alpha_{k-1}(1-m)x_k \\ [x_1]\{\alpha_1\}[x_2] \cdots [x_k]\{\alpha_k\} &\mapsto x_1\alpha_1(1-m)x_2 \cdots x_k\alpha_k(1-m) \\ [x_1]\{\alpha_1\}[x_2] \cdots [x_k] &\mapsto x_1\alpha_1(1-m)x_2 \cdots x_k; \end{aligned}$$

that is, replace each term $\{\alpha\}$ in curly braces by $\alpha(1-m)$ unless $\{\alpha\}$ is at the end of a word, in which case replace it by α . It is easily checked that this map preserves the string relations on S , and so it gives a well-defined map $\phi : S \rightarrow R$. Restricted to generators of the form $\{\alpha\}$, $\{1\}[x]\{1\}$, $\{1\}[x]$, $[x]\{1\}$, ϕ is as given in the statement of the proposition; note now by Proposition 4.11 that these suffice to determine ϕ .

We next check bijectivity. The maps ϕ are clearly surjective. It is proved in [CELN17, Proposition 2.20] that ϕ on R_{KK} is injective as long as K is knotted. The fact that ϕ is injective on R_{Kp} is trivial: by Proposition 4.11, any element of R_{Kp} can be written as $\{1\}[x]$ for some $x \in R$, and then $\phi(\{1\}[x]) = 0$ implies $x = 0$. To prove that ϕ is injective on R_{pK} , note that if $\phi([x]\{1\}) = 0$, then $x(1-m) = 0$ in R ; then since knot groups are left orderable, $R = \mathbb{Z}\pi$ has no zero divisors [Hig40], and so $x = 0$.

Finally, the fact that ϕ respects multiplication and μ follows readily from the definition of ϕ : for example,

$$\begin{aligned} \phi(\mu(\{\alpha_0\} \cdots \{\alpha_1\}[x_1], [x_2]\{\alpha_2\} \cdots \{\alpha_3\})) &= \phi(\{\alpha_0\} \cdots \{\alpha_1\}[x_1x_2]\{\alpha_2\} \cdots \{\alpha_3\}) \\ &= \alpha_0(1-m) \cdots \alpha_1(1-m)x_1x_2\alpha_2(1-m) \cdots \alpha_3 \\ &= \phi(\{\alpha_0\} \cdots \{\alpha_1\}[x_1]) \cdot \phi([x_2]\{\alpha_2\} \cdots \{\alpha_3\}). \end{aligned}$$

□

Remark 4.14. In Section 6.5 of the arXiv version of this paper, we give an interpretation of the maps in Proposition 4.13 in terms of moduli spaces of holomorphic disks inducing maps from partially wrapped Floer homology into chains on spaces of paths and loops in $\mathbb{R}^3 \setminus K$.

Recall that the product μ gives a ring structure on $R_{pp} = R_{pK} \otimes R_{Kp}$. Similarly to Proposition 4.13, we then have the following.

Proposition 4.15. *The isomorphisms $\phi : R_{pK} \xrightarrow{\cong} R(1-m)$ and $R_{Kp} \xrightarrow{\cong} R$ induce a ring isomorphism*

$$\phi : R_{pp} \xrightarrow{\cong} R(1-m)R$$

where $R(1 - m)R$ denotes the two-sided ideal of R generated by $1 - m$.

Proof. Same as the proof of Proposition 4.13, but now use the fact that ϕ is defined on R_{pp} by

$$[x_1]\{\alpha_1\}[x_2] \cdots \{\alpha_{k-1}\}[x_k] \mapsto x_1\alpha_1(1 - m)x_2 \cdots \alpha_{k-1}(1 - m)x_k$$

and that any element of R_{pp} can be written as $[x]$ for some $x \in R$. \square

Although $R(1 - m)R$ is not all of R , we observe the following.

Proposition 4.16. *The submodules $\mathbb{Z} = \mathbb{Z} \cdot 1$ and $R(1 - m)R$ of R are complementary: $R \cong \mathbb{Z} \oplus R(1 - m)R$ as \mathbb{Z} -modules.*

Proof. The homomorphism from π to the trivial group induces a map $R = \mathbb{Z}[\pi] \rightarrow \mathbb{Z}$, which restricts to the identity on \mathbb{Z} and to the zero map on $R(1 - m)R$; thus $\mathbb{Z} \cap R(1 - m)R = 0$. On the other hand, $\pi = \pi_1(\mathbb{R}^3 \setminus K)$ is generated by a finite collection of meridians, each of which is an element of π of the form $\gamma m \gamma^{-1}$ for some $\gamma \in \pi$, and $\gamma m \gamma^{-1} = 1 - \gamma(1 - m)\gamma^{-1} \in \mathbb{Z} + R(1 - m)R$; thus $\mathbb{Z} + R(1 - m)R = R$. \square

We can restate the combination of Propositions 4.15 and 4.16 as follows. Consider the direct sum $\mathbb{Z} \oplus R_{pp}$, and give this a ring structure by defining the generator 1 of \mathbb{Z} to be the multiplicative identity and setting multiplication on the factor R_{pp} to be as usual. That is:

$$(n_1, r_1) \cdot (n_2, r_2) = (n_1 n_2, n_1 r_2 + n_2 r_1 + r_1 r_2).$$

Then:

Proposition 4.17. *We have a ring isomorphism $\phi : \mathbb{Z} \oplus R_{pp} \xrightarrow{\cong} R$ defined by*

$$\phi(n, [x_1]\{\alpha_1\}[x_2]) = n + x_1\alpha_1(1 - m)x_2.$$

Remark 4.18. The ring $\mathbb{Z} \oplus R_{pp}$ may seem like an odd candidate to be isomorphic to $R = \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$. In fact, it is the correct object to consider from at least two perspectives. One is through the cord algebra and modules: we have $R_{pp} \cong \text{Cord}_{pK} \otimes \text{Cord}_{Kp}$, and we can use the skein relation for cords (Definition 4.2 (4)) to rewrite the product of a pK cord with a Kp cord as the difference of two pp cords. This gives a map

$$\text{Cord}_{pK} \otimes \text{Cord}_{Kp} \rightarrow \text{Cord}_{pp}.$$

This map is not surjective, since it maps to the ideal generated by differences of cords, but it becomes an isomorphism if we add the \mathbb{Z} -module generated by the trivial pp cord to the left hand side to obtain $\mathbb{Z} \oplus R_{pp}$. On the other hand, any pp cord is a loop in $\mathbb{R}^3 \setminus K$, and this induces an isomorphism between Cord_{pp} and $\mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$.

The other perspective is through partially wrapped Floer homology; see the discussion in the Introduction, and the arXiv version of this paper for a fuller discussion.

5. PROOF OF THEOREM 1.1

Here we present the proof of our main result. Briefly, by Proposition 4.17, we can recover $R = \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K)]$ from the ring structure on R_{pp} , which itself is determined by the Legendrian contact homology of $\Lambda_K \cup \Lambda_p$ along with the product μ . This allows us to recover the knot K itself. The details of the proof are broken into two subsections corresponding to statements (1) and (2) from Theorem 1.1, and at the end of this section we make some remarks about the difference between the two.

5.1. Proof of Theorem 1.1 (2). Suppose that K_0, K_1 are knots such that there are isomorphisms between the KCH-triples of K_0 and K_1 ; these isomorphisms are compatible with the products $\mu : R_{K_i p} \otimes R_{p K_i} \rightarrow R_{K_i K_i}$; and the isomorphism $R_{K_0 K_0} \xrightarrow{\cong} R_{K_1 K_1}$ sends the meridian and longitude m_0, l_0 of K_0 to the meridian and longitude m_1, l_1 of K_1 , respectively. Under these assumptions, we want to conclude that K_0, K_1 are isotopic in \mathbb{R}^3 as oriented knots. Since R_{KK} detects the unknot (see e.g. [CELN17, Corollary 1.5]), we will assume that K_0, K_1 are both knotted.

For $i = 0, 1$, write $R_i = \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K_i)]$ and $\hat{R}_i = \mathbb{Z}[m_i^{\pm 1}, l_i^{\pm 1}] \subset R_i$. By Proposition 4.13, we can write the isomorphisms between the KCH-triples $(R_{K_0 K_0}, R_{K_0 p}, R_{p K_0})$ and $(R_{K_1 K_1}, R_{K_1 p}, R_{p K_1})$ as a triple of maps

$$(\psi_{KK}, \psi_{Kp}, \psi_{pK}) : (\hat{R}_0 + R_0(1 - m_0), R_0, R_0(1 - m_0)) \xrightarrow{\cong} (\hat{R}_1 + R_1(1 - m_1), R_1, R_1(1 - m_1)).$$

These maps are compatible with multiplication and the product μ in the way described in Proposition 4.13.

Write $R_{pp}^i = R_{p K_i} \otimes_{R_{K_i K_i}} R_{K_i p}$; then the map $\psi_{pK} \otimes \psi_{Kp}$ is an isomorphism from R_{pp}^0 to R_{pp}^1 , and it preserves the ring structure on R_{pp}^i determined by μ . By Proposition 4.17, we can then view

$$\psi_{pp} := \text{id} \oplus (\psi_{pK} \otimes \psi_{Kp}) : \mathbb{Z} \oplus R_{pp}^0 \xrightarrow{\cong} \mathbb{Z} \oplus R_{pp}^1$$

as a ring isomorphism $\psi_{pp} : R_0 \xrightarrow{\cong} R_1$.

Now $R_i = \mathbb{Z}[\pi_1(\mathbb{R}^3 \setminus K_i)]$ and knot groups are left-orderable, and so any isomorphism between R_0 and R_1 must come from a group isomorphism between the knot groups [Hig40]. More precisely, the set of units in the group ring $\mathbb{Z}[G]$ of a left-orderable group G are exactly the elements $\pm g$ for $g \in G$, and so there exists an isomorphism

$$\psi : \pi_1(\mathbb{R}^3 \setminus K_0) \xrightarrow{\cong} \pi_1(\mathbb{R}^3 \setminus K_1)$$

such that $\psi_{pp}(\gamma) = \pm \psi(\gamma)$ for all $\gamma \in \pi_1(\mathbb{R}^3 \setminus K_0)$.

Now that we know that the knot groups of K_0 and K_1 are isomorphic, it remains to show that the isomorphism preserves the peripheral structure. We will show that there is some $\gamma \in \pi_1(\mathbb{R}^3 \setminus K_1)$ such that $\psi(m_0) = \gamma^{-1} m_1 \gamma$ and $\psi(l_0) = \gamma^{-1} l_1 \gamma$, whence the composition of ψ and conjugation by γ gives an isomorphism $\pi_1(\mathbb{R}^3 \setminus K_0) \xrightarrow{\cong} \pi_1(\mathbb{R}^3 \setminus K_1)$ sending m_0, l_0 to m_1, l_1 as desired.

For this, we use the assumption that $\psi_{KK}(m_0) = m_1$ and $\psi_{KK}(l_0) = l_1$. The elements $1 \in R_0 \cong R_{K_0p}$ and $1 - m_0 \in R_0(1 - m_0) \cong R_{pK_0}$ have images under ψ_{Kp} and ψ_{pK}

$$\psi_{Kp}(1) = x \qquad \psi_{pK}(1 - m_0) = x'(1 - m_1)$$

for some $x, x' \in R_1$. In R_1 , we have

$$\begin{aligned} 1 - m_1 &= \psi_{KK}(1 - m_0) = \psi_{KK}\mu(1, 1 - m_0) \\ &= \mu(\psi_{Kp}(1), \psi_{pK}(1 - m_0)) = \mu(x, x'(1 - m_1)) = xx'(1 - m_1). \end{aligned}$$

Now the group ring of a left-orderable group has no zero divisors, and so it follows that $xx' = 1 \in R_1$ and hence there exists some $\gamma \in \pi_1(\mathbb{R}^3 \setminus K_1)$ such that $x = \pm\gamma$, $x' = \pm\gamma^{-1}$.

Next, for $\alpha \in \mathbb{Z}$, view $l_0^\alpha(1 - m_0) = (1 - m_0) \cdot l_0^\alpha \cdot 1$ as an element of $R_{pK_0} \otimes_{R_{K_0K_0}} R_{K_0K_0} \otimes_{R_{K_0K_0}} R_{K_0p} = R_{pp}^0$. Then we have

$$\psi_{pp}(l_0^\alpha(1 - m_0)) = \psi_{pK}(1 - m_0) \cdot \psi_{KK}(l_0^\alpha) \cdot \psi_{Kp}(1) = \gamma^{-1}(1 - m_1) \cdot l_1^\alpha \cdot \gamma.$$

Now since $\psi_{pp} = \pm\psi$ on elements of $\pi_1(\mathbb{R}^3 \setminus K_0)$, both of $\psi_{pp}(l_0^\alpha(1 - m_0))$ and $\gamma^{-1}(1 - m_1) \cdot l_1^\alpha \cdot \gamma$ are binomials (i.e., sums of the form $n_1\gamma_1 + n_2\gamma_2$ with $n_i \in \mathbb{Z}$ and $\gamma_i \in \pi_1(\mathbb{R}^3 \setminus K_1)$), and equating terms gives

$$\{\psi(l_0^\alpha), \psi(m_0 l_0^\alpha)\} = \{\gamma^{-1} l_1^\alpha \gamma, \gamma^{-1} m_1 l_1^\alpha \gamma\}.$$

Plugging in $\alpha = 0$ and $\alpha = 1$ in succession gives $\psi(m_0) = \gamma^{-1} m_1 \gamma$ and $\psi(l_0) = \gamma^{-1} l_1 \gamma$, as desired.

5.2. Proof of Theorem 1.1 (1). We now consider the case where there is an isomorphism between the KCH-triples of K_0 and K_1 preserving the product μ , but without the assumption that longitude and meridian classes are mapped to themselves. We will prove that there is an isomorphism $\pi_1(\mathbb{R}^3 \setminus K_0) \cong \pi_1(\mathbb{R}^3 \setminus K_1)$ sending m_0 to $m_1^{\pm 1}$ and l_0 to $l_1^{\pm 1}$. It then follows from Waldhausen [Wal68] that K_0 is smoothly isotopic to either K_1 or the mirror of K_1 , as unoriented knots. (Note that our argument identifies not only the peripheral subgroups but also the meridians in each, and so we do not need to appeal to Gordon and Luecke [GL89].) As in the previous proof, we can assume that K_0 and K_1 are both knotted.

The setup is as in Section 5.1, except that $\psi_{KK}(m_0) = m_1^{n_1} l_1^{n_2}$ and $\psi_{KK}(l_0) = m_1^{n_3} l_1^{n_4}$ for some $\begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} \in GL_2(\mathbb{Z})$ not necessarily the identity matrix. As in the previous proof, ψ_{pp} is induced by a group isomorphism $\psi : \pi_1(\mathbb{R}^3 \setminus K_0) \rightarrow \pi_1(\mathbb{R}^3 \setminus K_1)$. Now however when we write $\psi_{Kp}(1) = x$, $\psi_{pK}(1 - m_0) = x'(1 - m_1)$ for $x, x' \in R_1$, we have

$$(4) \qquad \qquad \qquad xx'(1 - m_1) = 1 - m_1^{n_1} l_1^{n_2}.$$

We can then appeal to the following algebraic result.

Lemma 5.1. *Suppose G is left orderable and $m, g \in G$, $z \in \mathbb{Z}[G]$ satisfy $z(1 - m) = 1 - g$ in $\mathbb{Z}[G]$. Then $g = m^n$ for some $n \in \mathbb{Z}$.*

Proof. Let $<$ be the left-invariant ordering on G . Without loss of generality, we may assume $m > 1$; the lemma is trivial if $m = 1$, and we can replace m by m^{-1} if $m < 1$. Write $z = \sum_{i=1}^k a_i g_i$ for $a_i \in \mathbb{Z}$ and $g_i \in G$; we may assume that $a_i \neq 0$ for all i and $g_1 < g_2 < \dots < g_k$. Then $g_i < g_i m$ for all i , and so among the subset $S = \{g_1, \dots, g_k, g_1 m, \dots, g_k m\}$ of G (which may contain repeated elements), g_1 is strictly lowest. Furthermore, let $g_j m$ be the largest among the k distinct elements $g_1 m, \dots, g_k m$; then $g_j m$ is strictly largest among the elements of S . It follows that the expansion of $z(1 - m) = \sum_{i=1}^k (a_i g_i - a_i g_i m)$ involves g_1 and $g_j m$ at least. Since $1 - g$ is a binomial, it follows that $\{g_1, g_j m\} = \{1, g\}$, and furthermore that all terms in the expansion not involving g_1 or $g_j m$ must cancel. Thus $g_1 m$ must be canceled by g_{i_1} for some i_1 , whence $g_{i_1} m$ must be canceled by g_{i_2} for some i_2 , and so forth. We conclude that there is a sequence $i_0 = 1, i_1, \dots, i_\ell = j$ such that $g_{i_r} = g_{i_{r-1}} m$ for all $r = 1, \dots, \ell$, and so $g_j = g_1 m^{\ell-1}$. The lemma follows. \square

We now continue with the proof of Theorem 1.1 (1). By Equation (4) and Lemma 5.1, $m_1^{n_1} l_1^{n_2} = m_1^n$ for some n . By the Loop Theorem, since K_2 is knotted, l_1 is not a power of m_1 , and so $n_2 = 0$. Since $\begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}$ is invertible, it follows that $n_1 = \pm 1$ and $n_4 = \pm 1$.

We treat the cases $n_1 = 1$ and $n_1 = -1$ separately. If $n_1 = 1$, then $\psi_{KK}(m_0) = m_1$ and $\psi_{KK}(l_0) = m_1^{n_3} l_1^{\pm 1}$. As in Section 5.1, we have $xx'(1 - m_1) = 1 - m_1$ and so $x = \pm\gamma$, $x' = \pm\gamma^{-1}$ for some $\gamma \in \pi_1(\mathbb{R}^3 \setminus K_1)$. Now for any $\alpha \in \mathbb{Z}$, we compute

$$\psi_{pp}(l_0^\alpha(1 - m_0)) = \psi_{pK}(1 - m_0) \cdot \psi_{KK}(l_0^\alpha) \cdot \psi_{Kp}(1) = \gamma^{-1} m_1^{\alpha n_3} l_1^{\pm \alpha} (1 - m_1) \gamma.$$

Identifying terms as in Section 5.1 gives

$$\{\psi(l_0)^\alpha, \psi(m_0)\psi(l_0)^\alpha\} = \{\gamma^{-1} m_1^{\alpha n_3} l_1^{\pm \alpha} \gamma, \gamma^{-1} m_1^{\alpha n_3 + 1} l_1^{\pm \alpha} \gamma\}.$$

Plugging in $\alpha = 0$ and $\alpha = 1$ in succession gives $\psi(m_0) = \gamma^{-1} m_1 \gamma$ and $\psi(l_0) = \gamma^{-1} m_1^{n_3} l_1^{\pm 1} \gamma$. Conjugating ψ by γ gives a group isomorphism $\pi_1(\mathbb{R}^3 \setminus K_1) \xrightarrow{\cong} \pi_1(\mathbb{R}^3 \setminus K_2)$ sending m_0 to m_1 and l_0 to $m_1^{n_3} l_1^{\pm 1}$. Since the longitude is the identity in (and the meridian generates) the abelianization of π_1 , we must have $n_3 = 0$: thus m_0 is sent to m_1 and l_0 to $l_1^{\pm 1}$.

If instead $n_1 = -1$, we can run the same argument to conclude $x = \pm\gamma$, $x' = \mp\gamma^{-1} m_1^{-1}$; $\psi_{pp}(l_0^\alpha(1 - m_0)) = -\gamma^{-1} m_1^{-1} (1 - m_1) m_1^{\alpha n_3} l_1^{\pm \alpha} \gamma$; $\psi(m_0) = \gamma^{-1} m_1^{-1} \gamma$ and $\psi(l_0) = \gamma^{-1} m_1^{n_3} l_1^{\pm 1} \gamma$; and finally there is an isomorphism $\pi_1(\mathbb{R}^3 \setminus K_0) \xrightarrow{\cong} \pi_1(\mathbb{R}^3 \setminus K_1)$ sending m_0 to m_1^{-1} and l_0 to $l_1^{\pm 1}$.

This completes the proof of Theorem 1.1 (1).

5.3. A note on different types of Legendrian isotopy. The difference between statements (1) and (2) in Theorem 1.1 is in the strength of the assumption about the Legendrian isotopy relating two conormal tori Λ_{K_0} and Λ_{K_1} . One might ask if the weaker assumption— an unparametrized Legendrian isotopy between Λ_{K_0} and Λ_{K_1} —might still imply the stronger result—a smooth isotopy between K_0 and K_1 as oriented knots. This appears to be possible, but our invariants do not show this.

The issue is a symmetry of $ST^*\mathbb{R}^3$: the diffeomorphism of \mathbb{R}^3 given by $(x, y, z) \mapsto (x, y, -z)$ induces a coorientation-preserving contactomorphism of $ST^*\mathbb{R}^3$ that preserves cotangent fibers and sends the conormal torus of K to the conormal torus of the mirror $m(K)$ of K . It follows that there is an isomorphism

$$\mathrm{LCH}_*(\Lambda_p \cup \Lambda_K) \cong \mathrm{LCH}_*(\Lambda_p \cup \Lambda_{m(K)}),$$

and indeed between the DGAs for $\Lambda_p \cup \Lambda_K$ and $\Lambda_p \cup \Lambda_{m(K)}$. Further, this isomorphism preserves the product μ . (On the homology of the conormal torus, the mirroring map preserves l but sends m to m^{-1} , and so this symmetry does not contradict Theorem 1.1 (2).)

There is a result that is somewhere in between statements (1) and (2) from Theorem 1.2: if $\Lambda_{K_0}, \Lambda_{K_1}$ are Legendrian isotopic in an orientation-preserving manner, then K_0, K_1 are smoothly isotopic as unoriented knots. Indeed, from the proof of Theorem 1.1 (1) in Section 5.2, the Legendrian isotopy must send (m_0, l_0) to $(m_1^{\pm 1}, l_1^{\pm 1})$, and the orientation-preserving condition implies that the two signs agree. It follows that there is an isomorphism between $\pi_1(\mathbb{R}^3 \setminus K_0)$ and $\pi_1(\mathbb{R}^3 \setminus K_1)$ that preserves the peripheral subgroup, possibly after changing the orientation of K_1 .

REFERENCES

- [AENV14] Mina Aganagic, Tobias Ekholm, Lenhard Ng, and Cumrun Vafa. Topological strings, D-model, and knot contact homology. *Adv. Theor. Math. Phys.*, 18(4):827–956, 2014.
- [BEE11] Frédéric Bourgeois, Tobias Ekholm, and Yakov Eliashberg. Symplectic homology product via Legendrian surgery. *Proc. Natl. Acad. Sci. USA*, 108(20):8114–8121, 2011.
- [BEE12] Frédéric Bourgeois, Tobias Ekholm, and Yasha Eliashberg. Effect of Legendrian surgery. *Geom. Topol.*, 16(1):301–389, 2012. With an appendix by Sheel Ganatra and Maksim Maydanskiy.
- [CEL10] K. Cieliebak, T. Ekholm, and J. Latschev. Compactness for holomorphic curves with switching Lagrangian boundary conditions. *J. Symplectic Geom.*, 8(3):267–298, 2010.
- [CELN17] Kai Cieliebak, Tobias Ekholm, Janko Latschev, and Lenhard Ng. Knot contact homology, string topology, and the cord algebra. *J. Éc. polytech. Math.*, 4:661–780, 2017.
- [EENS13a] Tobias Ekholm, John Etnyre, Lenhard Ng, and Michael Sullivan. Filtrations on the knot contact homology of transverse knots. *Math. Ann.*, 355(4):1561–1591, 2013.
- [EENS13b] Tobias Ekholm, John B. Etnyre, Lenhard Ng, and Michael G. Sullivan. Knot contact homology. *Geom. Topol.*, 17(2):975–1112, 2013.
- [EES05a] Tobias Ekholm, John Etnyre, and Michael Sullivan. Non-isotopic Legendrian submanifolds in \mathbb{R}^{2n+1} . *J. Differential Geom.*, 71(1):85–128, 2005.
- [EES05b] Tobias Ekholm, John Etnyre, and Michael Sullivan. Orientations in Legendrian contact homology and exact Lagrangian immersions. *Internat. J. Math.*, 16(5):453–532, 2005.
- [EES07] Tobias Ekholm, John Etnyre, and Michael Sullivan. Legendrian contact homology in $P \times \mathbb{R}$. *Trans. Amer. Math. Soc.*, 359(7):3301–3335 (electronic), 2007.
- [EGH10] Y. Eliashberg, A. Glvental, and H. Hofer. Introduction to symplectic field theory. In N. Alon, J. Bourgain, A. Connes, M. Gromov, and V. Milman, editors, *Visions in Mathematics*, Modern Birkhauser Classics, pages 560–673. Birkhauser Basel, 2010.

- [EHK] Tobias Ekholm, Ko Honda, and Tamas Kálmán. Legendrian knots and exact Lagrangian cobordisms. arXiv:1212.1519, to appear in *J. Eur. Math. Soc. (JEMS)*.
- [Ekh08] Tobias Ekholm. Rational symplectic field theory over \mathbb{Z}_2 for exact Lagrangian cobordisms. *J. Eur. Math. Soc. (JEMS)*, 10(3):641–704, 2008.
- [EL] Tobias Ekholm and Yanki Lekili. Duality between Lagrangian and Legendrian invariants. arXiv:1701.01284.
- [GL] Cameron Gordon and Tye Lidman. Knot contact homology detects cabled, composite, and torus knots. arXiv:1509.01642.
- [GL89] C. McA. Gordon and J. Luecke. Knots are determined by their complements. *J. Amer. Math. Soc.*, 2(2):371–415, 1989.
- [Hig40] Graham Higman. The units of group-rings. *Proc. London Math. Soc. (2)*, 46:231–248, 1940.
- [HS85] James Howie and Hamish Short. The band-sum problem. *J. London Math. Soc. (2)*, 31(3):571–576, 1985.
- [Joy82] David Joyce. A classifying invariant of knots, the knot quandle. *J. Pure Appl. Algebra*, 23(1):37–65, 1982.
- [Kon09] Maxim Kontsevich. Symplectic geometry of homological algebra. available at the author’s webpage, 2009.
- [Mis03] K. Mishachev. The N -copy of a topologically trivial Legendrian knot. *J. Symplectic Geom.*, 1(4):659–682, 2003.
- [Nad] David Nadler. Wrapped microlocal sheaves on pairs of pants. arXiv:1604.00114.
- [Nad17] David Nadler. Arboreal singularities. *Geom. Topol.*, 21(2):1231–1274, 2017.
- [Ng08] Lenhard Ng. Framed knot contact homology. *Duke Math. J.*, 141(2):365–406, 2008.
- [Ng10] Lenhard Ng. Rational symplectic field theory for Legendrian knots. *Invent. Math.*, 182(3):451–512, 2010.
- [Ng11] Lenhard Ng. Combinatorial knot contact homology and transverse knots. *Adv. Math.*, 227(6):2189–2219, 2011.
- [Ng14] Lenhard Ng. A topological introduction to knot contact homology. In *Contact and symplectic topology*, volume 26 of *Bolyai Soc. Math. Stud.*, pages 485–530. János Bolyai Math. Soc., Budapest, 2014.
- [NRS⁺] Lenhard Ng, Dan Rutherford, Vivek Shende, Steven Sivek, and Eric Zaslow. Augmentations are sheaves. arXiv:1502.04939.
- [NZ09] David Nadler and Eric Zaslow. Constructible sheaves and the Fukaya category. *Journal of the American Mathematical Society*, 22(1):233–286, 2009.
- [She] Vivek Shende. The conormal torus is a complete knot invariant. arXiv:1604.03520.
- [Syl] Zachary Sylvan. On partially wrapped Fukaya categories. arXiv:1604.02540.
- [Wal68] Friedhelm Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Annals of Mathematics (2)*, 87(1):56–88, 1968.