ON ARC INDEX AND MAXIMAL THURSTON–BENNEQUIN NUMBER

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Abstract. We discuss the relation between arc index, maximal Thurston–Bennequin number, and Khovanov homology for knots. As a consequence, we calculate the arc index and maximal Thurston–Bennequin number for all knots with at most 11 crossings. For some of these knots, the calculation requires a consideration of cables which also allows us to compute the maximal self-linking number for all knots with at most 11 crossings.

1. Introduction and Results

Let K be a knot in S^3. Define a *grid diagram* of K to be an oriented knot diagram for K consisting of a union of horizontal and vertical line segments, such that at every crossing, the vertical segment crosses over the horizontal segment. Any knot has a grid diagram. In the literature, grid diagrams or their equivalents have gone by many alternate names, including "arc presentations", "asterisk presentations", "square-bridge presentations", and "fences". Grid diagrams have been much studied lately, most recently because of their use in the combinatorial definition of knot Floer homology [20]; for background on grid diagrams, see, e.g., [7].

The *arc number* of a grid diagram is the number of horizontal (or, equivalently, vertical) segments in the diagram. The *arc index* of K, written $\alpha(K)$, is the minimal arc number over all grid diagrams for K.

It is well-known that grid diagrams are closely related to Legendrian knots from contact geometry (see, e.g., [9] for an introduction to Legendrian knots and a more geometric description of the invariants tb and sl below). A front for a Legendrian knot can be obtained by rotating any grid diagram slightly counterclockwise and eliminating each corner by either smoothing it out or replacing it by a cusp. Conversely, any Legendrian knot can be represented by a grid diagram.

In this context, the *Thurston–Bennequin number* tb and *self-linking number* sl of a grid diagram G, which are invariants of the associated Legendrian knot, can be defined as follows. Let $w(G)$ denote the writhe of G; let $c(G)$ denote the number of lower-right, "southeast", corners of G (these correspond to the right cusps of the Legendrian front); and let $c_\downarrow(G)$ denote the number of southeast corners oriented down and to the left, plus the number of northwest corners oriented to the left and down (these correspond to the downward-oriented cusps of the Legendrian front). Then

$$\mathrm{tb}(G) = w(G) - c(G)$$
$$\mathrm{sl}(G) = w(G) - c_{\downarrow}(G).$$

We remark that the self-linking number is usually defined for transverse rather than Legendrian knots; sl defined here is the self-linking number of the positive transverse pushoff of the Legendrian knot, and can be expressed as $\mathrm{tb}(G) - \mathrm{r}(G)$, where $\mathrm{r}(G)$ is the rotation number of the Legendrian knot.

The *maximal Thurston–Bennequin number* of a knot K, written $\overline{\mathrm{tb}}(K)$, is the maximal tb over all grid diagrams for K; similarly, the *maximal self-linking number* $\overline{\mathrm{sl}}(K)$ is the maximal sl over all grid diagrams for K. It is not hard to see that $\overline{\mathrm{tb}}(K) \leq \overline{\mathrm{sl}}(K)$ for all K, while it is an important classical result of Bennequin [5] that $\overline{\mathrm{sl}}(K) < \infty$ for any K. Calculating $\overline{\mathrm{tb}}$ and $\overline{\mathrm{sl}}$ is of natural interest to knot theorists, particularly since each provides a lower bound for various topological knot invariants, including the slice genus g_4 [30] and the concordance invariants τ [28] and s [29, 32].

There is a fundamental relation between arc index and the maximal Thurston–Bennequin numbers of a knot K and its mirror $\overline{K}$, first described by Matsuda in [21]:

$$-\alpha(K) \leq \overline{\mathrm{tb}}(K) + \overline{\mathrm{tb}}(\overline{K}). \tag{1}$$

The proof of this inequality is short and we recall it here. Consider a grid diagram for K with arc number $\alpha(K)$. This diagram produces a Legendrian knot of topological type K, as described above, as well as a Legendrian knot of type $\overline{K}$, by rotating the diagram slightly less than 90° clockwise, changing every crossing, and smoothing the corners. Then it is easy to see that the Thurston–Bennequin numbers of these two Legendrian knots sum to $-\alpha(K)$.

Equation (1) leads to an approach to calculate arc index and maximal Thurston–Bennequin number for specific knots, as follows:

(a) find a possibly minimal grid diagram of K;
(b) find upper bounds for $\overline{\mathrm{tb}}(K)$ and $\overline{\mathrm{tb}}(\overline{K})$ individually, or for their sum;
(c) see if equality is forced to hold in (1).

This approach (essentially) has been used to calculate arc index for alternating knots [1] and knots with up to 10 crossings [4]. In both cases, the upper bound in step (b) is provided by the Kauffman polynomial.

In this note, we apply this approach to knots with at most 11 crossings, using grid diagrams provided by Baldwin and Gillam [2] and the Khovanov bound for $\overline{\mathrm{tb}}$ [24]. We compute arc index and maximal Thurston–Bennequin number for all knots with at most 11 crossings. Let min-deg and max-deg denote the minimum and maximum degrees of a Laurent polynomial in the specified variable, let breadth = max-deg − min-deg, and let $Kh_K(q,t)$ denote the two-variable Poincaré polynomial for $\mathfrak{sl}_2$ Khovanov homology.

Proposition 1. *Let K be a knot with 11 or fewer crossings. We have*

$$\alpha(K) = \mathrm{breadth}_q\, Kh_K(q, t/q)$$

with the following exceptions: $\alpha(10_{124}) = 8$, $\alpha(10_{132}) = 9$, $\alpha(11n_{12}) = 10$, $\alpha(11n_{19}) = 9$, $\alpha(11n_{38}) = 9$, $\alpha(11n_{57}) = 10$, $\alpha(11n_{88}) = 10$, *and* $\alpha(11n_{92}) = 10$. *Here the chirality of the knot is irrelevant.*

Proposition 2. *Let K be a knot with 11 or fewer crossings. We have*

$$\overline{\mathrm{tb}}(K) = \text{min-deg}_q\, Kh_K(q, t/q)$$

with the following exceptions:

$$\begin{aligned} \overline{\mathrm{tb}}(\overline{10_{124}}) &= -15 & \qquad \overline{\mathrm{tb}}(\overline{11n_{38}}) &= -4 \\ \overline{\mathrm{tb}}(\overline{10_{132}}) &= -1 & \qquad \overline{\mathrm{tb}}(\overline{11n_{57}}) &= -13 \\ \overline{\mathrm{tb}}(11n_{12}) &= -2 & \qquad \overline{\mathrm{tb}}(\overline{11n_{88}}) &= -13 \\ \overline{\mathrm{tb}}(11n_{19}) &= -8 & \qquad \overline{\mathrm{tb}}(11n_{92}) &= -6. \end{aligned}$$

The $\overline{\mathrm{tb}}$ data from Proposition 2 for knots with up to 11 crossings can be found online at `KnotInfo` [19].

The exceptional cases in Proposition 2 require strengthening previously known upper bounds for $\overline{\mathrm{tb}}$ and are presented in Section 2. The computation of $\overline{\mathrm{tb}}$ for $11n_{19}$ uses a strengthening of the Kauffman bound on $\overline{\mathrm{tb}}$ derived from work of Rutherford [31] and a subsequent observation of Kálmán [15]; the computation of $\overline{\mathrm{tb}}$ for $\overline{10_{132}}$, $11n_{12}$, $\overline{11n_{38}}$, $\overline{11n_{57}}$, $\overline{11n_{88}}$, and $11n_{92}$ uses cable links.

Nutt [26] previously directly computed arc index for all knots with 9 or fewer crossings, and Beltrami [4], as mentioned earlier, extended this computation to knots with 10 crossings. The author [24] previously computed maximal Thurston–Bennequin number for all knots with 10 or fewer crossings except $\overline{10_{132}}$.

Josh Greene [13] has proposed the following very interesting question:

Question 1. *Does a grid diagram realizing the arc index of a knot necessarily realize the maximal Thurston–Bennequin number for the knot? An equivalent statement is that*

$$-\alpha(K) = \overline{\mathrm{tb}}(K) + \overline{\mathrm{tb}}(\overline{K}) \tag{2}$$

for all knots K.

No counterexamples are currently known. In particular, we have the following consequence of Propositions 1 and 2:

Corollary 3. *(2) holds for all knots K with 11 or fewer crossings.*

Greene notes that (2) also holds for alternating knots by [1] and the fact that the Kauffman bound for $\overline{\mathrm{tb}}$ is sharp for alternating knots [24, 31], and for torus knots by Etnyre and Honda's classification of Legendrian torus knots [10].

We conclude this section with a discussion of maximal self-linking number. There is an intriguing analogy between $\overline{\mathrm{tb}}$ and $\overline{\mathrm{sl}}$:

$$\text{arc index} \;:\; \text{braid index} \;::\; \overline{\mathrm{tb}} \;:\; \overline{\mathrm{sl}}.$$

Keiko Kawamuro [16, Conjecture 3.2] has made a conjecture which can be restated as follows to parallel Question 1:

Question 2. *Does a braid whose closure is a particular knot, with a minimal number of strands (the braid index), necessarily realize the maximal self-linking number for the knot? An equivalent statement is that*

$$-2b(K) = \overline{\mathrm{sl}}(K) + \overline{\mathrm{sl}}(\overline{K}) \tag{3}$$

for all knots K, where $b(K)$ is the braid index of K.

Note that (3), like (2), holds if $=$ is replaced by $\leq$. The celebrated MFW inequality [11, 22] gives a lower bound for braid index and an upper bound for $\overline{\mathrm{sl}}$ in terms of the HOMFLY-PT polynomial $P_K(a, z)$:

$$-2b(K) \leq \overline{\mathrm{sl}}(K) + \overline{\mathrm{sl}}(\overline{K}) \leq -\operatorname{breadth}_a P_K(a, z) - 2.$$

Thus the answer to Question 2 is "yes" for all knots for which the "weak" MFW inequality $2b(K) \geq \operatorname{breadth}_a P_K(a, z) + 2$ is sharp.

In fact, more is true. In Section 2.2, we calculate $\overline{\mathrm{sl}}(K)$ for the 5 knots with at most 10 crossings for which MFW is not sharp. This calculation, combined with an analogous calculation by T. Khandhawit [17] for the 14 knots with 11 crossings where MFW is not sharp, yields the following result.

Proposition 4. *Let K be a knot with 11 or fewer crossings. We have*

$$\overline{\mathrm{sl}}(K) = -\operatorname{max-deg}_a P_K(a, z) - 1$$

with the following exceptions:

$\overline{\mathrm{sl}}(\overline{9_{42}}) = -5$	$\overline{\mathrm{sl}}(\overline{11n_{24}}) = -5$	$\overline{\mathrm{sl}}(11n_{86}) = -3$
$\overline{\mathrm{sl}}(\overline{9_{49}}) = -11$	$\overline{\mathrm{sl}}(\overline{11n_{33}}) = -7$	$\overline{\mathrm{sl}}(\overline{11n_{117}}) = -7$
$\overline{\mathrm{sl}}(\overline{10_{132}}) = -1$	$\overline{\mathrm{sl}}(\overline{11n_{37}}) = -3$	$\overline{\mathrm{sl}}(\overline{11n_{124}}) = -7$
$\overline{\mathrm{sl}}(\overline{10_{150}}) = -9$	$\overline{\mathrm{sl}}(\overline{11n_{70}}) = -7$	$\overline{\mathrm{sl}}(\overline{11n_{136}}) = -13$
$\overline{\mathrm{sl}}(10_{156}) = -7$	$\overline{\mathrm{sl}}(\overline{11n_{79}}) = -7$	$\overline{\mathrm{sl}}(\overline{11n_{171}}) = -13$
$\overline{\mathrm{sl}}(11n_{20}) = -7$	$\overline{\mathrm{sl}}(\overline{11n_{82}}) = -5$	$\overline{\mathrm{sl}}(\overline{11n_{180}}) = -13$
		$\overline{\mathrm{sl}}(\overline{11n_{181}}) = -13.$

Corollary 5. *(3) holds for all knots K with 11 or fewer crossings.*

As for (2), no counterexamples to (3) are currently known.

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2. Proofs

In this section, we provide more details for the discussion in Section 1, and prove the main results. Section 2.1 proves Proposition 1, and Proposition 2 for all but six cases; Section 2.2 uses cables to fill in the remaining cases and also prove Proposition 4.

2.1. Arc index and $\overline{\mathrm{tb}}$.

Two very useful bounds for $\overline{\mathrm{tb}}$ are the *Kauffman bound* [12, 30, 34]

$$\overline{\mathrm{tb}}(K) \leq -\,\text{max-deg}_a\, F_K(a,z) - 1, \tag{4}$$

where F_K is the two-variable Kauffman polynomial of K, and the *Khovanov bound* [24]

$$\overline{\mathrm{tb}}(K) \leq \text{min-deg}_q\, Kh_K(q,t/q), \tag{5}$$

where Kh_K is the Poincaré polynomial for $\mathfrak{sl}_2$ Khovanov homology.[1] It was noted in [24] that the Khovanov bound is at least as strong as the Kauffman bound for all knots with 11 or fewer crossings, although the two bounds are incommensurate in general.

Combining (1) and (4) yields

$$\alpha(K) \geq \mathrm{breadth}_a\, F_K(a,z) + 2. \tag{6}$$

The inequality (6) is originally due to Morton and Beltrami [23], and Beltrami [4] used it to compute the arc index of all 10-crossing knots. Bae and Park [1] proved that (6) is sharp (i.e., equality holds) for alternating knots, where both sides are equal to the crossing number plus 2.

Combining (1) and (5) instead yields the following result.

Proposition 6. *If K is a knot, then*

$$\alpha(K) \geq \mathrm{breadth}_q\, Kh_K(q,t/q). \tag{7}$$

If K has a grid diagram with arc number equal to $\mathrm{breadth}_q\, Kh_K(q,t/q)$*, then (7) is sharp, as is the Khovanov bound for both $\overline{\mathrm{tb}}(K)$ and $\overline{\mathrm{tb}}(\overline{K})$.*

We now apply Proposition 6 to prove Proposition 1.

[1]Note: There are many different conventions regarding knot chirality in the literature. These results, and this paper in general, use the conventions that conform to the Knot Atlas [3]. In particular, the Kauffman bound (4) uses the opposite convention for the Kauffman polynomial to the one used in many Legendrian-knot papers, including [12, 25, 34].

Proof of Proposition 1. Because of the behavior of arc index and Khovanov homology under connected sum, it suffices to consider prime knots only. In addition, the result holds for alternating knots K; here $\alpha(K) = \text{breadth}_a F_K(a,z) + 2 = c(K) + 2$, where $c(K)$ is the crossing number of K, and both Kauffman and Khovanov bounds for $\overline{\text{tb}}$ are sharp [24, 31].

Baldwin and Gillam [2], with the help of the program `Gridlink` [8], have constructed grid diagrams for all nonalternating prime knots with 11 or fewer crossings; these presentations, which include a few diagrams constructed by the author, are available at `http://www.math.brown.edu/~wgillam/hfk/`. For most of these diagrams, the arc number is equal to $\text{breadth}_q Kh_K(q, t/q)$, as can easily be checked by computer. (The author used `KnotTheory` [3] for this computation.) The exceptions are 10_{124}, 10_{132}, $11n_{12}$, $11n_{19}$, $11n_{38}$, $11n_{57}$, $11n_{88}$, and $11n_{92}$; for each of these, however, arc index has been computed in [26]. □

Before proving Proposition 2, we introduce a minor strengthening of the Kauffman bound (4), due to Kálmán [15] and based on work of Rutherford [31]. Rutherford's paper relates the Dubrovnik version of the Kauffman polynomial, $D_K(a,z) = F_K(ia, -iz)$, to certain partitions of fronts of Legendrian knots known as rulings [6].

Proposition 7 (Kálmán)**.** *Let K be a knot, and let $p_K(z)$ denote the polynomial in z which is the leading term of $F_K(ia, -iz)$ with respect to a. If $p_K(z)$ does not have all nonnegative coefficients, then*

$$\overline{\text{tb}}(K) \leq -\text{max-deg}_a F_K(a,z) - 2.$$

Proof. Suppose that the Kauffman bound (4) is sharp for K, and consider a Legendrian knot L of type K for which $\text{tb}(L) = -\text{max-deg}_a F_K(a,z) - 1$. By [31], we have $p_K(z) = \sum_{\rho \in \Gamma(L)} z^{j(\rho)}$, where $\Gamma(L)$ is the set of rulings of L and j is an integer-valued function on rulings. In particular, $p_K(z)$ has all nonnegative coefficients. □

Proposition 7 allows us to lower the Kauffman bound by 1 in some cases. Unfortunately, it does not apply to many small knots. The hypotheses of the proposition apply to seven knots with 11 crossings or fewer: $\overline{10_{136}}$, $11n_{19}$, $11n_{20}$, $\overline{11n_{37}}$, $\overline{11n_{50}}$, $11n_{86}$, and $\overline{11n_{126}}$. For six of these, the improved Kauffman bound is only as good as the Khovanov bound (5); for $11n_{19}$, however, it improves on the Khovanov bound as well, to yield $\overline{\text{tb}}(11n_{19}) \leq -8$. For 12-crossing knots, Proposition 7 yields the best known bound on $\overline{\text{tb}}$ for three knots, according to the tabulation from `KnotInfo` [19]: $\overline{\text{tb}}(12n_{25}) \leq -5$, $\overline{\text{tb}}(\overline{12n_{502}}) \leq -17$, $\overline{\text{tb}}(\overline{12n_{603}}) \leq -12$.

One can similarly use Rutherford's work to obtain an improved HOMFLY-PT bound on $\overline{\text{tb}}$, when the leading coefficient of the HOMFLY-PT polynomial does not have all nonnegative coefficients, and a "mixed" improved bound when the HOMFLY-PT and Kauffman bounds agree and the leading coefficient of their difference does not have all nonnegative coefficients.

These seem to be applicable to fewer cases than the improved Kauffman bound, however.

We can now prove Proposition 2.

Proof of Proposition 2. As in the proof of Proposition 1, the result holds unless K is one of the knots 10_{124}, 10_{132}, $11n_{12}$, $11n_{19}$, $11n_{38}$, $11n_{57}$, $11n_{88}$, or $11n_{92}$, with either chirality.

As discussed earlier, the case $10_{124} = T(3,5)$ is covered by [10]; $\overline{\mathrm{tb}}(10_{124}) = 7$ and $\overline{\mathrm{tb}}(\overline{10_{124}}) = -15$, and the Khovanov bound is sharp for the former but not for the latter. For $11n_{19}$, Proposition 7 gives $\overline{\mathrm{tb}}(11n_{19}) \leq -8$, while both Kauffman and Khovanov bounds give $\overline{\mathrm{tb}}(\overline{11n_{19}}) \leq -1$; since $\alpha(11n_{19}) = 9$ by Nutt's table [26], these bounds for $\overline{\mathrm{tb}}(11n_{19})$ and $\overline{\mathrm{tb}}(\overline{11n_{19}})$ are sharp.

The remaining cases, 10_{132}, $11n_{12}$, $11n_{38}$, $11n_{57}$, $11n_{88}$, and $11n_{92}$, are addressed by Corollary 9 in the next section. (In fact, 10_{124} and $11n_{19}$ can also be addressed in the same way.) □

2.2. **Cables, $\overline{\mathrm{tb}}$, and $\overline{\mathrm{sl}}$.** Suppose that we wish to assemble a table of maximal Thurston–Bennequin and self-linking numbers for small knots. There are several knots with 11 or fewer crossings for which all of the known general upper bounds on $\overline{\mathrm{tb}}$ or $\overline{\mathrm{sl}}$ fail to be sharp: 7 for $\overline{\mathrm{tb}}$, 19 for $\overline{\mathrm{sl}}$. What can one do in these cases? One case for $\overline{\mathrm{tb}}$, $\overline{10_{124}}$, is the $(3,-5)$ torus knot, and the classification of Legendrian torus knots due to Etnyre and Honda [10] shows that $\overline{\mathrm{tb}}(\overline{10_{124}}) = -15$; the best general upper bound gives $\overline{\mathrm{tb}}(\overline{10_{124}}) \leq -14$. For the other cases, however, there is no classification result. For these, we turn to cable links.

If K is a knot, let $D_n(K)$ denote the n-framed double (2-cable link) of K, where both components of $D_n(K)$ are oriented the same way as K. Our strategy is to bound $\overline{\mathrm{tb}}$ and $\overline{\mathrm{sl}}$ for $D_n(K)$ from above via one of the standard bounds, and then use these upper bounds to bound $\overline{\mathrm{tb}}$ and $\overline{\mathrm{sl}}$ for K via the following easy result.[2]

Proposition 8. *We have*

$$\overline{\mathrm{tb}}(D_n(K)) \geq \begin{cases} 2\,\overline{\mathrm{tb}}(K) + 2n, & n > \overline{\mathrm{tb}}(K) \\ 4n, & n \leq \overline{\mathrm{tb}}(K) \end{cases} \tag{8}$$

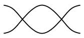
and

$$\overline{\mathrm{sl}}(D_n(K)) \geq 2\,\overline{\mathrm{sl}}(K) + 2n. \tag{9}$$

As a consequence of (8), if $\overline{\mathrm{tb}}(D_n(K)) < 2m+2n$ for some m,n with $m \leq n$, then $\overline{\mathrm{tb}}(K) < m$.

Proof. We first prove (8). Let L be a Legendrian knot of type K. Define the "Legendrian double" $D(L)$ to be the Legendrian link whose front is given by two copies of L offset slightly in the vertical (z) direction; then $D(L)$ is topologically the $\mathrm{tb}(L)$-framed double of K, and $\mathrm{tb}(D(L)) = 4\,\mathrm{tb}(L)$.

[2] The observation that (9) holds for all n, not only $n = 0$, is due to Khandhawit [17].

If $n \leq \overline{\mathrm{tb}}(K)$, then choose L such that $\mathrm{tb}(L) = n$. Since $D(L)$ is topologically $D_n(K)$ and $\mathrm{tb}(D(L)) = 4n$, it follows that $\overline{\mathrm{tb}}(D_n(K)) \geq 4n$. If $n > \overline{\mathrm{tb}}(K)$, then choose L such that $\mathrm{tb}(L) = \overline{\mathrm{tb}}(K)$. Add $n - \overline{\mathrm{tb}}(K)$ positive twists to the framing on $D(L)$ by inserting $n - \overline{\mathrm{tb}}(K)$ pieces of the form  into the front of $D(L)$ to obtain a Legendrian link $D'(L)$ which is topologically $D_n(K)$. Each of the pieces adds 2 to tb, and so $\mathrm{tb}(D'(L)) = 4\,\overline{\mathrm{tb}}(K) + 2(n - \overline{\mathrm{tb}}(K))$; it follows that $\overline{\mathrm{tb}}(D_n(K)) \geq 2\,\overline{\mathrm{tb}}(K) + 2n$.

To prove (9), we use the alternate formulation, first observed by Bennequin [5], for self-linking number in terms of braids. If B is a braid of m strands and writhe (algebraic crossing number) w, then define $\mathrm{sl}(B) = w - m$; $\overline{\mathrm{sl}}(K)$ is the maximum value of $\mathrm{sl}(B)$ over all braids B whose closure is K.

Given K, let B be a braid whose closure is K for which $\mathrm{sl}(B) = \overline{\mathrm{sl}}(K)$. Construct a double B' of B with $2m$ strands consisting of two slightly offset copies of B; in algebraic terms, replace each generator $\sigma_i^{\pm 1}$ in the braid word for B by $(\sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i})^{\pm 1}$ to obtain B'. If w is the writhe of B, then the closure of B' is $D_w(K)$, and $\mathrm{sl}(B') = 4w - 2m$. Add in $n - w$ positive twists to the beginning of B' (i.e., append σ_1^{2n-2w} to the braid word for B') to obtain another braid B'' with $2m$ strands. The closure of B'' is $D_n(K)$, and $\mathrm{sl}(B'') = 2w - 2m + 2n$. It follows that $\overline{\mathrm{sl}}(D_n(K)) \geq 2w - 2m + 2n = 2\,\overline{\mathrm{sl}}(K) + 2n$. □

Corollary 9. *The values of* $\overline{\mathrm{tb}}$ *for* $\overline{10_{132}}$, $11n_{12}$, $\overline{11n_{38}}$, $\overline{11n_{57}}$, $\overline{11n_{88}}$, *and* $11n_{92}$ *(and their mirrors) are as given in Proposition 2.*

Proof. We combine the Khovanov bound for $\overline{\mathrm{tb}}(D_n(K))$ with Proposition 8. For instance, the Khovanov bound yields $\overline{\mathrm{tb}}(D_3(\overline{10_{132}})) \leq 5$, which with Proposition 8 implies that $\overline{\mathrm{tb}}(\overline{10_{132}}) \leq -1$. The Khovanov bound also shows directly that $\overline{\mathrm{tb}}(10_{132}) \leq -8$; from Proposition 6 and (1), we conclude that $\overline{\mathrm{tb}}(10_{132}) = -8$ and $\overline{\mathrm{tb}}(\overline{10_{132}}) = -1$.

Similarly, the Khovanov bound gives $\overline{\mathrm{tb}}(D_3(11n_{12})) \leq 3$, $\overline{\mathrm{tb}}(D_1(\overline{11n_{38}})) \leq -6$, $\overline{\mathrm{tb}}(D_{-7}(\overline{11n_{57}})) \leq -39$, $\overline{\mathrm{tb}}(D_{-7}(\overline{11n_{88}})) \leq -39$, and $\overline{\mathrm{tb}}(D_{-1}(11n_{92}) \leq -13$, and these bounds produce the values of $\overline{\mathrm{tb}}$ for $11n_{12}$, $\overline{11n_{38}}$, $\overline{11n_{57}}$, $\overline{11n_{88}}$, and $11n_{92}$ given in Proposition 2. We remark that these doubles are links with 40+ crossings, and computing their Khovanov homology is not altogether trivial. The particular framings of the doubles were chosen to try to minimize crossings, and each Khovanov homology was computed using the program `JavaKh`, written by Jeremy Green, within `KnotTheory` [3]. □

We next use Proposition 8 to prove Proposition 4.

Proof of Proposition 4. We prove the result for knots with 10 or fewer crossings, and refer the reader to [17] for the 11-crossing case, which is proved by the same technique. There is nothing to prove if the weak MFW inequality $2b(K) \geq \mathrm{breadth}_a\, P_K(a,z) + 2$ is sharp. There are five knots with 10 or fewer crossings for which equality does not hold for MFW: 9_{42}, 9_{49}, 10_{132}, 10_{150}, and 10_{156}. For these, we use the HOMFLY-PT bound on $\overline{\mathrm{sl}}(D_0(K))$

and Proposition 8 to bound $\overline{\mathrm{sl}}$. The HOMFLY-PT bound yields an upper bound on $\overline{\mathrm{sl}}(D_0(K))$ of -8, -20, 0, -16, and -12, respectively. (For some of these computations, the author found the program `K2K` [27] to be useful.) These give the exceptional values for $\overline{\mathrm{sl}}$ in the statement of Proposition 4.

For example, since $\overline{\mathrm{sl}}(D_0(\overline{9_{42}})) \leq -8$, Proposition 8 implies that $\overline{\mathrm{sl}}(\overline{9_{42}}) \leq -4$; since the self-linking number for any knot is odd, it follows that $\overline{\mathrm{sl}}(\overline{9_{42}}) \leq -5$. The usual HOMFLY-PT bound also implies that $\overline{\mathrm{sl}}(9_{42}) \leq -3$. Since $b(9_{42}) = 4$ and $-2b(9_{42}) \leq \overline{\mathrm{sl}}(9_{42}) + \overline{\mathrm{sl}}(\overline{9_{42}})$, equality holds everywhere. □

We close with two remarks. First, using cables along the lines presented here is not entirely new; Stoimenow [33] showed that $\overline{10_{132}}$ is not quasipositive using almost identical methods.

Second, in the situations where the general upper bounds for $\overline{\mathrm{tb}}(K)$ and $\overline{\mathrm{sl}}(K)$ (Kauffman, Khovanov, HOMFLY-PT) fail to be sharp, it seems that one can often apply these bounds to the double or perhaps general m-cable of K to deduce a sharp bound for $\overline{\mathrm{tb}}(K)$ and $\overline{\mathrm{sl}}(K)$. Proposition 8 has a straightforward analogue for m-component cables of K. For instance, if $C_m(K)$ denotes the 0-framed m-component cable of K, then

$$\overline{\mathrm{sl}}(C_m(K)) \geq m\,\overline{\mathrm{sl}}(K).$$

It seems at least within the realm of possibility that $\overline{\mathrm{sl}}(K) = \lim_{m\to\infty} \overline{\mathrm{sl}}(C_m(K))/m$, and that the HOMFLY-PT bound for $\overline{\mathrm{sl}}(C_m(K))$ might in general give a sharp bound for $\overline{\mathrm{sl}}(K)$ for *all* K. A similar but slightly more complicated statement could hold for $\overline{\mathrm{tb}}$.

Thus there might be a way to calculate $\overline{\mathrm{tb}}$ and $\overline{\mathrm{sl}}$ for all knots, by applying the general upper bounds to cables. We note, however, that calculating these upper bounds for cables is generally quite computationally intensive and may be infeasible for "medium-sized" knots of, say, 12 crossings or more.

References

[1] Y. Bae and C.-Y. Park, An upper bound of arc index of links, *Math. Proc. Cambridge Philos. Soc.* **129** (2000), no. 3, 491–500.

[2] J. A. Baldwin and W. D. Gillam, Computations of Heegaard–Floer knot homology, `math/0610167`.

[3] D. Bar-Natan, The Mathematica package `KnotTheory`` , available at The Knot Atlas, `http://katlas.math.toronto.edu/wiki/`.

[4] E. Beltrami, Arc index of non-alternating links, in *Knots 2000 Korea, Vol. 1 (Yongpyong), J. Knot Theory Ramifications* **11** (2002), no. 3, 431–444.

[5] D. Bennequin, Entrelacements et équations de Pfaff, *Astérisque* **107–108** (1983), 87–161.

[6] Yu. V. Chekanov and P. E. Pushkar', Combinatorics of Legendrian links and Arnol'd's 4-conjectures, *Uspekhi Mat. Nauk* **60** (2005), no. 1, 99–154, translated in *Russian Math. Surveys* **60** (2005), no. 1, 95–149.

[7] P. R. Cromwell, *Knots and Links* (Cambridge University Press, Cambridge, 2004).

[8] M. Culler, Gridlink: a tool for knot theorists, `http://www.math.uic.edu/~culler/gridlink/`.

[9] J. B. Etnyre, Legendrian and transversal knots, in *The Handbook of Knot Theory* (Elsevier, Amsterdam, 2005), 105–185; `math/0306256`.

[10] J. B. Etnyre and K. Honda, Knots and contact geometry I, *J. Symplectic Geom.* **1** (2001), 63–120; `math/0006112`.
[11] J. Franks and R. F. Williams, Braids and the Jones polynomial, *Trans. Amer. Math. Soc.* **303** (1987), no. 1, 97–108.
[12] D. Fuchs and S. Tabachnikov, Invariants of Legendrian and transverse knots in the standard contact space, *Topology* **36** (1997), no. 5, 1025–1053.
[13] J. Greene, private communication.
[14] J. Hoste and M. Thistlethwaite, Knotscape, `http://www.math.utk.edu/~morwen/knotscape.html`.
[15] T. Kálmán, private communication.
[16] K. Kawamuro, The algebraic crossing number and the braid index of knots and links, *Algebr. Geom. Topol.* **6** (2006), 2313–2350.
[17] T. Khandhawit, On Legendrian and transverse knots, senior honors thesis, Duke University, 2008.
[18] M. Khovanov, A categorification of the Jones polynomial, *Duke Math. J.* **101** (2000), 359–426; `math/9908171`.
[19] C. Livingston and J.-C. Cha, KnotInfo: Table of knot invariants, `http://www.indiana.edu/~knotinfo/`.
[20] C. Manolescu, P. Ozsváth, and S. Sarkar, A combinatorial description of knot Floer homology, *Ann. of Math. (2)* **169** (2009), no. 2, 633–660; `math/0607691`.
[21] H. Matsuda, Links in an open book decomposition and in the standard contact structure, *Proc. Amer. Math. Soc.* **134** (2006), no. 12, 3697–3702.
[22] H. R. Morton, Seifert circles and knot polynomials, *Math. Proc. Cambridge Philos. Soc.* **99** (1986), no. 1, 107–109.
[23] H. R. Morton and E. Beltrami, Arc index and the Kauffman polynomial, *Math. Proc. Cambridge Philos. Soc.* **123** (1998), no. 1, 41–48.
[24] L. Ng, A Legendrian Thurston–Bennequin bound from Khovanov homology, *Algebr. Geom. Topol.* **5** (2005), 1637–1653; `math/0508649`.
[25] L. Ng, Maximal Thurston–Bennequin number of two-bridge links, *Algebr. Geom. Topol.* **1** (2001), 427–434; `math/0008242`.
[26] I. Nutt, Braid Index of Satellite Links, Ph.D. thesis, Liverpool University, 1995, available at `http://www.liv.ac.uk/~su14/knotprints.html`.
[27] M. Ochiai and N. Imafuji, `Knot2000(K2K)`, available at `http://amadeus.ics.nara-wu.ac.jp/~ochiai/freesoft.html`.
[28] O. Plamenevskaya, Bounds for the Thurston–Bennequin number from Floer homology, *Algebr. Geom. Topol.* **4** (2004), 399–406; `math/0311090`.
[29] O. Plamenevskaya, Transverse knots and Khovanov homology, *Math. Res. Lett.* **13** (2006), no. 4, 571–586; `math/0412184`.
[30] L. Rudolph, A congruence between link polynomials, *Math. Proc. Cambridge Philos. Soc.* **107** (1990), 319–327.
[31] D. Rutherford, Thurston–Bennequin number, Kauffman polynomial, and ruling invariants of a Legendrian link: the Fuchs conjecture and beyond, *Int. Math. Res. Not.* **2006**, Art. ID 78591; `math/0511097`.
[32] A. Shumakovitch, Rasmussen invariant, slice-Bennequin inequality, and sliceness of knots, *J. Knot Theory Ramifications* **16** (2007), no. 10, 1403–1412; `math/0411643`.
[33] A. Stoimenow, On polynomials and surfaces of variously positive links, *J. Eur. Math. Soc. (JEMS)* **7** (2005), no. 4, 477–509; `math/0202226`.
[34] S. Tabachnikov, Estimates for the Bennequin number of Legendrian links from state models for knot polynomials, *Math. Res. Lett.* **4** (1997), no. 1, 143–156.

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