

LEGEND FOR LEGENDRIAN KNOT ATLAS

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The table depicts conjectural classifications of Legendrian knots in all prime knot types of arc index up to 9.

- For each knot, the non-destabilizable Legendrian representatives are depicted (modulo the symmetries described below), with their (tb, r) , along with the conjectural mountain range. As usual, rotate 45° counterclockwise to translate from grid diagrams to fronts.
- Legendrian classification is known for torus knots and 4_1 [4], and for twist knots [5]. In the table, torus knots are denoted in the usual way by $T(p, q)$, and the twist knot with n half-twists is denoted by K_n .
- It is interesting to compare this atlas to the table from [6]. It would be interesting to know which of the Legendrian knots can be distinguished via various modern techniques: Massey products, SFT, etc.
- The (tb, r) coordinates for the mountain ranges can be inferred from the other (tb, r) information. Boxes enclose dots with the same (tb, r) . Mountain ranges extend downward from the depicted parts in the usual way; mountain ranges without boxes indicate knot types that are conjecturally Legendrian simple.
- The conjectured mountain ranges include black and red dots, one dot for each Legendrian isotopy type; the black dots represent a “lower-bound” range of Legendrian types that we have been able to distinguish using current techniques, while the red dots are Legendrian knots that we believe to be distinct from the black dots. We believe that the mountain ranges are complete as shown—i.e., that there are no other Legendrian knots besides the ones depicted—but this is only known for the knot types that have been classified.
- Two symmetries on Legendrian knots are: orientation reversal, $L \mapsto -L$, and Legendrian mirror, $L \mapsto \mu(L)$ (reflecting fronts in the x axis). Both of these negate rotation number. Orientation reversal also changes topological type in general, but all of the knots in the atlas are invertible.

Each Legendrian knot L thus produces up to four knots: L , $-L$, $\mu(L)$, $-\mu(L)$. The table depicts one representative from each of these orbits of up to four knots, along with information about which of the four knots in the orbit are isotopic, if any. For knots with nonzero rotation number, we choose a representative L with positive

rotation number, and L is trivially distinct from $-L$ and $\mu(L)$ (this fact depicted by hyphens in the table).

- Legendrian knots that we believe but cannot yet prove to be distinct are labeled with matching letters (see e.g. 6₂). Question marks indicate knots where we believe but cannot prove that L is distinct from $-L$, $\mu(L)$, or $-\mu(L)$. All check marks have been verified by computer; all X marks without question marks have been verified by various techniques (see below for particular knot types).
- Graded ruling invariants and linearized contact homology are calculated using the *Mathematica* notebook `Legendrian_invariants.nb` (the variety found on Josh Sabloff's web page). Knots with no graded rulings/augmentations are denoted in these columns by a hyphen (for nonzero rotation number) or \emptyset (for zero rotation number).
- The atlas was obtained via a computer program that performed moves on grid diagrams (see [9, 11]), in a manner similar to Gridlink [2]. More documentation is forthcoming.
- For 6₂, 6₃, and 7₄, see [7]. In particular, for 7₄, K_1 and K_2 from [7] are the second and third knots in our table; an easy extension of the calculation in [7] shows that K_2 is not isotopic to its Legendrian mirror.
- For $m(10_{132})$ and $m(10_{140})$, see [8]. The same techniques work to distinguish transverse representatives for $m(10_{145})$, $m(10_{161})$, and $12n_{591}$ (see also below).
- For $m(7_2)$, see [5, 10].
- Six knot types that we know of, marked with asterisks, have Legendrian representatives that are non-destabilizable (or conjecturally so) but do not maximize tb . For $m(10_{145})$, $m(10_{161})$, and $12n_{591}$, non-destabilizability can be proven using knot Floer homology [11] (in some cases, by inspection; otherwise via the program of [8]); note that the behavior here had previously only been seen for the $(2, 3)$ cable of the $(2, 3)$ torus knot [3], which is significantly more complicated in various ways than the examples here. Interestingly, each of these three knot types has non-destabilizable Legendrian representatives whose tb is 2 less than maximal, as well as representatives whose tb is 1 less than maximal.

For $m(10_{139})$, 10_{161} , and $m(12n_{242})$, non-destabilizability has not yet been proven, but a new phenomenon conjecturally emerges: knots with non-maximal tb that cannot be destabilized because there is nothing for them to be the stabilization of. In other words, the mountain ranges here have peaks that are not of maximal tb . It would be very interesting to prove this: e.g., that there is no Legendrian $m(10_{139})$ knot with $(tb, r) = (-16, 3)$.

- In some of the above cases, non-destabilizability can also be proven using Legendrian contact homology and the characteristic algebra

- [7]. In particular, see [12] for $m(10_{161})$. However, the characteristic algebra vanishes for the non-max- tb non-destabilizable $m(10_{139})$, 10_{161} , and $m(12n_{242})$ knots.
- Of the knots in the table, $m(7_2)$, $m(10_{132})$, $m(10_{140})$, $m(10_{145})$, $m(10_{161})$, and $12n_{591}$ can be proven to be transversely nonsimple using knot Floer homology [10, 11]. The table suggests that the following knots are also transversely nonsimple: 7_6 , $m(7_6)$, 7_7 , 9_{44} , $m(9_{45})$, 9_{48} , 10_{128} , 10_{136} , and 10_{160} . All other knots with arc index at most 9 may be transversely simple. There is unfortunately no overlap with the Birman–Menasco 3-braid examples [1], all of which have arc index at least 10.

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