Problem 1. Let $P$ be the vector space of all real polynomials and $L : P \to P$ be the linear transformation defined by $L(f) = f + f'$. Prove that $L$ is invertible.

Problem 2. Let $A \in \text{GL}_4(\mathbb{C})$, and suppose $A$ has exactly one eigenvalue $\lambda$. Find all possible Jordan forms of $A$, and prove that $A - \lambda I$ is nilpotent.

Problem 3. Let $k$ be a finite field, and let $M \in \text{GL}_n(k)$. Finally, let $I \in \text{GL}_n(k)$ be the identity matrix. Show that $M^m - I$ is not invertible for some integer $m \geq 1$.

Problem 4. Let $\langle \cdot, \cdot \rangle$ be a positive definite inner product on a finite dimensional real vector space $V$. Let $S = \{v_1, \ldots, v_k\}$ be a set of vectors satisfying $\langle v_i, v_j \rangle < 0$ for all $i \neq j$. Prove that $\dim(\text{span}(S)) \geq k - 1$.

Problem 5. Let $V$ be a finite dimensional real vector space, and let $A : V \to V$ be a linear transformation with $A^2 = A$. Show that $\text{trace}(A) = \text{rank}(A)$.

Problem 6. Let $V$ be a finite dimensional vector space over a field $k$. Show that $V \cong V^*$, where $V^*$ is the vector space of linear transformations $V \to k$.

Problem 7. Let $k$ be a field with $\text{char}(k) \neq 2$, $V$ a finite dimensional vector space over $k$, and $B$ a symmetric bilinear form on $V$.

(a) Prove that if $B \neq 0$, then there exists $v \in V$ such that $B(v, v) \neq 0$.

(b) Prove that for any $v \in V$ with $B(v, v) \neq 0$, there exists a subspace $W \subseteq V$ such that $V = Fv \oplus W$ and $W \perp v$.

(c) Prove that there is a basis $\{v_i\}$ of $V$ such that $B(v_i, v_j) = 0$ for all $i \neq j$.

Problem 8. Let $V$ be a finite dimensional vector space and $T : V \to V$ a nonzero linear transformation. Show that if $\ker(T) = \text{im}(T)$, then $\dim(V)$ is an even integer and the minimal polynomial of $T$ is $m(x) = x^2$.

Problem 9. Let $k$ be a field with characteristic $p$, and let $V$ be a finite dimensional $k$-vector space. Let $T : V \to V$ be a linear transformation with $T^p = I$.

(a) Show that $T$ has an eigenvector in $V$.

(b) Show that $T$ is upper-triangular with respect to a suitable basis of $V$.

Problem 10. Let $V$ and $W$ be finite dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Prove that $\dim(\ker(T)) + \dim(\text{im}(T)) = \dim(V)$. 

Problem 11. Find representatives for the (distinct) conjugacy classes of matrices with characteristic polynomial \( f(\lambda) = (\lambda^2 + 1)^2 \) in

(a) \( \text{GL}_4(\mathbb{Q}) \).

(b) \( \text{GL}_4(\mathbb{C}) \).

Problem 12. Let \( M \) be an \( n \times n \) matrix.

(a) Show that \( M \) is invertible if and only if its characteristic polynomial has a non-zero constant term.

(b) Show that if \( M \) is invertible, then its inverse \( M^{-1} \) may be expressed as a polynomial in \( M \).

Problem 13. If \( A \) is an \( n \times n \) matrix, then show that

\[
A^n = \alpha_0 I + \alpha_1 A + \ldots + \alpha_{n-1} A^{n-1}
\]

for some scalars \( \alpha_0, \ldots, \alpha_{n-1} \).

Problem 14. Let \( V \) be a finite dimensional vector space over a field \( k \). Suppose that \( A : V \to V \) is a \( k \)-linear endomorphism whose minimal polynomial is not equal to its characteristic polynomial. Show that there exist \( k \)-linear endomorphisms \( B, C : V \to V \) with \( AB = BA \) and \( AC = CA \) but \( BC \neq CB \).

Problem 15. Let \( K \) be a degree \( n \) extension of \( \mathbb{Q} \). Let \( \sigma_1, \ldots, \sigma_n : K \hookrightarrow \mathbb{C} \) be the distinct embeddings of \( K \) into \( \mathbb{C} \), and let \( \alpha \in K \). Regarding \( K \) as a vector space over \( \mathbb{Q} \), let \( \varphi : K \to K \) be the linear transformation given by \( \varphi(x) = \alpha x \). Show that the eigenvalues of \( \varphi \) are \( \sigma_1(\alpha), \ldots, \sigma_n(\alpha) \).

Problem 16. Let \( A \) be a matrix over an algebraically closed field \( k \). Show that \( A = A_s + A_n \), where \( A_s \) is a diagonalizable matrix, \( A_n \) is a nilpotent matrix, and \( A_s A_n = A_n A_s \).

Problem 17. Let \( k \) be an algebraically closed field, \( n \in \mathbb{N} \), and \( A \in \text{GL}_n(k) \).

(a) Assume \( \text{char}(k) \neq 2 \). Show that if \( A^2 \) is diagonalizable over \( k \), then \( A \) is also diagonalizable over \( k \).

(b) Given an example with \( \text{char}(k) = 2 \) where \( A^2 \) is diagonalizable and \( A \) is not diagonalizable.

Problem 18. Without using the fact that they are simultaneously triangularizable, show that two commuting square complex matrices share an eigenvector.

Problem 19. Let \( F \) be a field and \( n \in \mathbb{N} \).

(a) For \( F = \mathbb{R} \), classify up to similarity all matrices \( A \in \text{GL}_n(\mathbb{R}) \) with \( A^3 = A \).

(b) For appropriate \( F \) and \( n \), find a matrix \( A \in \text{GL}_n(F) \) that is not diagonalizable that satisfies \( A^3 = A \).

Problem 20. Let \( T \) be a linear operator on a finite dimensional vector space over a field. Prove that \( \text{rank}(T^3) + \text{rank}(T) \geq 2 \cdot \text{rank}(T^2) \).