Problem 1. Find the Galois group of the splitting field for \( f(x) = x^3 - 7 \) over \( K = \mathbb{Q}(\sqrt{-3}) \).

Problem 2. Let \( \zeta \) be a primitive 37th root of unity, and let \( \eta = \zeta + \zeta^{10} + \zeta^{26} \). Determine the Galois group of \( \mathbb{Q}(\eta) \) over \( \mathbb{Q} \).

Problem 3. Let \( f(x) = x^6 + x^3 + 1 = (x^9 - 1)/(x^3 - 1) \).
(a) Prove that \( f(x) \) is irreducible over \( \mathbb{Q} \).
(b) Find the factorization of \( f(x) \) over \( \mathbb{F}_{19} \).

Problem 4. Let \( K \) be the splitting field over \( \mathbb{Q} \) of an irreducible polynomial of degree 3. What are the possibilities for \( [K : \mathbb{Q}] \)? Give an example to show that each possibility does occur.

Problem 5. Let \( f(x) \) be a polynomial of degree \( n \) that is irreducible over \( \mathbb{Q} \).
(a) If \( n \) is prime, prove that the Galois group of \( f(x) \) over \( \mathbb{Q} \) contains an \( n \) cycle.
(b) If \( n \) is not prime, show that the Galois group of \( f(x) \) over \( \mathbb{Q} \) need not contain an \( n \) cycle. (Hint: consider the cyclotomic polynomial \( \Phi_8(x) \)).

Problem 6. Give an example of two field extensions \( F/\mathbb{Q} \) and \( K/\mathbb{Q} \) with \( [F : \mathbb{Q}] = [K : \mathbb{Q}] = 6 \) such that \( \text{Gal}(F/\mathbb{Q}) \) is abelian and \( \text{Gal}(K/\mathbb{Q}) \) is non-abelian.

Problem 7. Show that for any field \( F \) and any integer \( d \geq 1 \), there exists at most one finite multiplicative subgroup \( G \subseteq F^\times \) of order \( d \).

Problem 8. Let \( F = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \). List all intermediate fields \( \mathbb{Q} \subset K \subset F \), and find all elements \( \alpha \in F \) such that \( F = \mathbb{Q}(\alpha) \).

Problem 9. Let \( p \) be a prime and \( q = p^n \) for some positive integer \( n \). Show that the map \( x \mapsto x^p \) is an automorphism of \( \mathbb{F}_q \). Determine all automorphisms of \( \mathbb{F}_q \).

Problem 10. Let \( K/F \) be a Galois extension with \( \text{Gal}(K/F) \cong S_3 \). Is it true that \( K \) is the splitting field of an irreducible cubic polynomial over \( F \)?
Problem 11. Consider the polynomial \( f(x) = \frac{x^{23} - 1}{x - 1} = \sum_{i=0}^{22} x^i \). Determine the number of irreducible factors of \( f(x) \) over \( \mathbb{Q}, \mathbb{F}_2 \), and \( \mathbb{F}_{2048} \).

Problem 12. Find the Galois group of the splitting field of \( f(x) = x^3 - x + 1 \) over each of the following fields:
(a) \( \mathbb{F}_2 \)
(b) \( \mathbb{R} \)
(c) \( \mathbb{Q} \)

Problem 13. Find a factorization of \( f(x) = 6x^4 - 4x^3 + 24x^2 - 4x - 8 \) into prime elements in \( \mathbb{Z}[x] \).

Problem 14. Show that \( x^3 - 3x - 1 \) is an irreducible element of \( \mathbb{Z}[x] \). Compute the Galois group of the splitting field of \( f(x) \) over \( \mathbb{Q} \) and over \( \mathbb{R} \).

Problem 15. Compute the Galois group of \( x^4 - x^2 - 6 \) over \( \mathbb{Q} \).

Problem 16. Let \( F \subseteq E \) be an algebraic field extension. Show that \( F \subseteq E \) is primitive if and only if the set of intermediate fields \( F \subseteq L \subseteq E \) is finite.

Problem 17. Prove that \( f(x) = x^4 + 1 \) is reducible modulo every prime \( p \) but is irreducible in \( \mathbb{Q}[x] \).

Problem 18. Using the fact that there are infinitely many primes congruent to 1 modulo \( m \) for all \( m \in \mathbb{N} \), prove that every finite abelian group appears as the Galois group of some finite Galois extension of \( \mathbb{Q} \).

Problem 19. Let \( K \) be a finite extension of a field \( F \), and let \( P \) be a monic irreducible polynomial in \( K[x] \). Prove that there is a non-zero \( Q \in K[x] \) such that \( PQ \in F[x] \).

Problem 20. Is there an injective field homomorphism from \( \mathbb{F}_4 \to \mathbb{F}_{16} \)? Is there an injective field homomorphism from \( \mathbb{F}_9 \to \mathbb{F}_{27} \)? Justify your answer.