LINES ON SMOOTH CUBIC SURFACES OVER Q

STEPHEN MCKEAN

Abstract. We provide a complete list of all possible counts of lines on smooth cubic surfaces over the rational numbers. In particular, a smooth cubic surface over the rational numbers contains 0, 1, 2, 3, 5, 7, 9, 15, or 27 lines.

1. Introduction

In 1849, Cayley and Salmon proved that every smooth cubic surface over $\mathbb{C}$ contains exactly 27 complex lines [Cay49]. By 1858, Schläfli had proved that every smooth cubic surface over $\mathbb{R}$ contains exactly 3, 7, 15, or 27 real lines [Sch58]. Following this theme, we classify all possible rational line counts for smooth cubic surfaces over $\mathbb{Q}$.

Theorem 1.1. *Every smooth cubic surface over $\mathbb{Q}$ contains 0, 1, 2, 3, 5, 7, 9, 15, or 27 lines defined over $\mathbb{Q}$.*

In order to prove this theorem, we need to show that no smooth cubic surface over $\mathbb{Q}$ contains exactly n lines defined over $\mathbb{Q}$ for $n = 4$, $n = 6$, $n = 8$, $10 \leq n \leq 14$, and $16 \leq n \leq 26$. We do this by studying the intersection graphs of smooth cubic surfaces over $\mathbb{C}$, over $\mathbb{R}$, and over $\mathbb{Q}$. In a few cases, we use ideas of Pannekoek [Pan09] (who builds on the work of Swinnerton-Dyer [SD69]) to study Galois-invariant collections of lines. In addition, we make frequent use of equations for lines on smooth cubic surfaces that we derived in joint work with Minahan and Zhang [MMZ20]. We also need to show that there exist smooth cubic surfaces over $\mathbb{Q}$ that contain exactly n lines defined over $\mathbb{Q}$ for n as specified in Theorem 1.1. We prove that the desired examples exist using blow-ups in Section 3. To conclude, we use the aforementioned equations for lines on cubic surfaces to write down some explicit examples in Section 5. Appendix A contains relevant code for these computations.

Throughout this article, we will only consider smooth cubic surfaces. We may therefore simply refer to smooth cubic surfaces as *cubic surfaces*. We will also use the terms *rational, real,* or *complex cubic surface* to refer to a cubic surface defined over $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, respectively. In particular, we do not use the term *rational cubic surface* when discussing cubic surfaces that are birational to $\mathbb{P}^2$. Likewise, we will use the terms *rational, real,* or *complex lines* to refer to lines defined over $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, respectively.

When a smooth cubic surface contains three skew lines, we say that the cubic surface contains a *skew triple*. When a rational cubic surface does not contain a skew triple, we

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obtain a constraint on the total number of lines it can contain. This allows us to treat the case of rational cubic surfaces with no skew triples in Section 4. In joint work with Minahan and Zhang [MMZ20], we give explicit equations for all lines on any complex cubic surface in terms of a skew triple. We use these equations to study rational cubic surfaces with a skew triple in Section 5.

1.1. **Related work.** Let $I = \{0, 1, 2, 3, 5, 7, 9, 15, 27\}$, and let k be any field. In forthcoming work, Jean-Pierre Serre provides a combinatorial proof that the number of lines on a smooth cubic surface over k must be an element of I. While this full list of line counts is not realized over every field, one can modify the blow-up argument used in Lemma 3.2 to give examples of fields over which every line count is realized. For example, if k is a finitely-generated extension of $\mathbb{Q}$ and $n \in I$, then there is a smooth cubic surface over k with exactly n lines.

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2. Preliminaries

We state a few classical results that we will use throughout this article.

Definition 2.1. [Dol16, Proposition 1.2] Let k be a perfect field, and let $k \subseteq F \subseteq E$ be a tower of fields. We say that a closed subscheme $X \subseteq \mathbb{P}^n_E$ is *defined over* F or *has field of definition* F if the following equivalent conditions are satisfied.

(a) The defining ideal of X is generated by homogeneous polynomials in $F[x_0, \ldots, x_n]$.

(b) There exists a closed subscheme $Y \subseteq \mathbb{P}^n_F$ such that $X = Y \times_F \operatorname{Spec} E$.

Any closed subscheme of projective space has a minimal field of definition by [DG67, IV_2, Corollaire (4.8.11)]. If a scheme X has field of definition k, we may also say that X is *k-rational*.

Proposition 2.2. *Let k'/k be a field extension, X a k-scheme, and $X_{k'} = X \times_k \operatorname{Spec} k'$ the base change of X. Then X is smooth over k if and only if $X_{k'}$ is smooth over k'.*

Proof. See [DG67, IV_4, Proposition (17.3.3) (iii) and Corollaire (17.7.3) (ii)]. □

Given a smooth cubic surface S over $\mathbb{Q}$ and an intermediate field $\mathbb{Q} \subseteq k \subseteq \mathbb{C}$, Proposition 2.2 enables us to base change and consider the smooth cubic surface S_k over k. We remark that given a $\mathbb{Q}$-rational line $L \subset S$, the line $L_k \subset S_k$ remains $\mathbb{Q}$-rational. Moreover, if S_k contains a $\mathbb{Q}$-rational line L_k, then S contains L as well. In particular, we can enumerate $\mathbb{Q}$-rational lines on S by base changing to $\mathbb{C}$ or $\overline{\mathbb{Q}}$ and studying the $\mathbb{Q}$-rationality of lines on $S_{\mathbb{C}}$ or $S_{\overline{\mathbb{Q}}}$.

We can study the field of definition of lines on cubic surfaces by acting on the relevant varieties by the absolute Galois group. This was done classically for lines on cubic surfaces over $\mathbb{R}$, as well as by Pannekoek [Pan09] for studying Galois orbits of lines on cubic surfaces over number fields.

Proposition 2.3. *Let k be a perfect field, and fix an algebraic closure $\overline{k}$ of k. A geometrically reduced closed subscheme $X \subseteq \mathbb{P}^n_{\overline{k}}$ is defined over k if and only if $\sigma \cdot X = X$ for all $\sigma \in \mathrm{Gal}(\overline{k}/k)$.*

Proof. The group $\mathrm{Gal}(\overline{k}/k)$ acts on the defining ideal $\mathcal{I} \subseteq \overline{k}[x_0, \ldots, x_n]$ of X by acting on the coefficients of each $f \in \mathcal{I}$. If X is defined over k, then the coefficients of any generating set of $\mathcal{I}$ are fixed under $\mathrm{Gal}(\overline{k}/k)$-action and hence so is X.

Now suppose X is fixed under $\mathrm{Gal}(\overline{k}/k)$-action. By Hilbert's Basis Theorem, X is defined by a finite set $\{f_1, \ldots, f_r\}$ of polynomials over some finite extension k' of k. Given $f \in \mathcal{I}$ and $\sigma \in \mathrm{Gal}(k'/k)$, denote the image of f under σ-action by f^σ. Since $\sigma \cdot X = X$, we have that $f^\sigma(p) = 0$ for all $p \in X$. In particular, $f^\sigma \in \mathcal{I}$ for all $f \in \mathcal{I}$. The desired result follows from [HRC12, Lemma 1 (b)]. We describe the relevant ideas here. Fix a k-basis $\{e_1, \ldots, e_m\}$ of k', and let $\mathrm{Tr}_{k'/k} : k'[x_0, \ldots, x_n] \to k[x_0, \ldots, x_n]$ be given by taking the Galois trace of each coefficient of a given polynomial. Then $\{\mathrm{Tr}_{k'/k}(e_i f_j)\}_{i,j}$ generates the ideal $\mathcal{I}$. Moreover, since $\mathrm{Tr}_{k'/k}(e_i f_j)^\sigma = \mathrm{Tr}_{k'/k}(e_i f_j)$ for all $\sigma \in \mathrm{Gal}(k'/k)$, it follows that $\mathrm{Tr}_{k'/k}(e_i f_j) \in k[x_0, \ldots, x_n]$. Thus $\mathcal{I}$ is generated by polynomials over k, as desired. □

A cubic surface S defined over k is fixed by $\mathrm{Gal}(\overline{k}/k)$-action, so Galois action preserves the set of 27 lines on $S_{\overline{k}}$. Moreover, Galois action preserves the incidence relations of the 27 lines:

Proposition 2.4. *Let k be a perfect field with $\overline{k}$ a fixed algebraic closure, S be a smooth cubic surface defined over k, and $\sigma \in \mathrm{Gal}(\overline{k}/k)$. Two lines L and L' in $S_{\overline{k}}$ intersect if and only if $\sigma \cdot L$ and $\sigma \cdot L'$ intersect.*

Proof. The σ-action is defined pointwise. In particular, if L and L' intersect in the point p, then $\sigma \cdot L$ and $\sigma \cdot L'$ intersect in the point $\sigma \cdot p$. Conversely, if $\sigma \cdot L$ and $\sigma \cdot L'$ intersect in the point q, then L and L' intersect in the point $\sigma^{-1} \cdot q$. □

Proposition 2.5. *Let $k \subseteq \mathbb{C}$ be a field, and let S_k be a smooth cubic surface defined over k. If $L_1, L_2, L_3 \subseteq S_{\mathbb{C}}$ are three coplanar lines, and if L_1 and L_2 are defined over k, then L_3 is also defined over k.*

Proof. Let $\overline{k}$ be an algebraic closure of k in $\mathbb{C}$. Since L_1 and L_2 are defined over k, the plane $H \subset \mathbb{P}^3_k$ that contains them is also defined over k. By Bézout's Theorem, we have $H_{\mathbb{C}} \cap S_{\mathbb{C}} = L_1 \cup L_2 \cup L_3$. The varieties L_1, L_2, H, and S are each fixed by all $\mathrm{Gal}(\overline{k}/k)$-actions since they are defined over k. We now act on the configuration $H_{\mathbb{C}} \cap S_{\mathbb{C}}$ by each $\sigma \in \mathrm{Gal}(\overline{k}/k)$. Since H and S are defined over k, we have $\sigma \cdot (H_{\mathbb{C}} \cap S_{\mathbb{C}}) = H_{\mathbb{C}} \cap S_{\mathbb{C}}$. That is, $L_1 \cup L_2 \cup L_3 = (\sigma \cdot L_1) \cup (\sigma \cdot L_2) \cup (\sigma \cdot L_3)$. Since L_1 and L_2 are defined over k, we have $\sigma \cdot L_1 = L_1$ and $\sigma \cdot L_2 = L_2$, so $L_1 \cup L_2 \cup L_3 = L_1 \cup L_2 \cup (\sigma \cdot L_3)$. It follows that $L_3 = \sigma \cdot L_3$ for all $\sigma \in \mathrm{Gal}(\overline{k}/k)$, so L_3 is defined over k. □

Corollary 2.6. *If a smooth cubic surface S over $\mathbb{Q}$ contains two lines L_1, L_2 that intersect each other, then S contains a third line L_3 that intersects L_1 and L_2.*

Proof. Let H be the plane containing L_1 and L_2. By Bézout's Theorem, $S_{\mathbb{C}} \cap H_{\mathbb{C}}$ consists of L_1, L_2, and a third line L_3. Since L_1 and L_2 are defined over $\mathbb{Q}$, Proposition 2.5 implies that L_3 is also defined over $\mathbb{Q}$, so S contains L_3. □

3. Blow-ups

Every smooth cubic surface over $\mathbb{C}$ is the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at six general points. On the other hand, not every smooth cubic surface over $\mathbb{R}$ is the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at six general points. There are five isotopy classes of smooth cubic surfaces over $\mathbb{R}$:

(i) Real smooth cubic surfaces that contain 27 real lines. These are all a blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at six real points.

(ii) Real smooth cubic surfaces that contain 15 real lines. These are all a blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at four real points and a complex conjugate pair.

(iii) Real smooth cubic surfaces that contain 7 real lines. These are all a blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at two real points and two complex conjugate pairs.

(iv) Real smooth cubic surfaces that contain 3 real lines that are birational to $\mathbb{P}^2_{\mathbb{C}}(\mathbb{R})$. These are all a blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at three pairs of complex conjugate points.

(v) Real smooth cubic surfaces that contain 3 real lines that are not birational to $\mathbb{P}^2_{\mathbb{C}}(\mathbb{R})$.

See e.g. [Seg42, PBT08] for more details. We note that although some real cubic surfaces do not arise as a blow-up of $\mathbb{P}^2_{\mathbb{C}}$, every possible count and configuration of real lines on a cubic surface is realized by a real cubic surface that is the blow-up of $\mathbb{P}^2_{\mathbb{C}}$.

As in the real case, not every smooth cubic surface over $\mathbb{Q}$ is the blow-up of $\mathbb{P}^2_{\overline{\mathbb{Q}}}$ at six general points (see e.g. [KSC04]). However, studying cubic surfaces over $\mathbb{Q}$ that do arise as the blow-up of $\mathbb{P}^2_{\overline{\mathbb{Q}}}$ give us a wealth of examples of rational lines on smooth cubic surfaces. In fact, it will turn out that every possible count and configuration of rational lines on a smooth cubic surface is realized by a rational cubic surface that is the blow-up of $\mathbb{P}^2_{\overline{\mathbb{Q}}}$. We will describe the examples here, and the remainder of this article will obstruct any other cases from occurring.

Let $\{p_1, \ldots, p_6\} \subset \mathbb{P}^2_{\mathbb{C}}$ be a collection of six points that do not lie on a conic, and such that no three lie on a line. Then the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at $\{p_1, \ldots, p_6\}$ is a smooth cubic surface defined over $\mathbb{C}$. By the Lefschetz principle, we may also conclude that the blow-up of $\mathbb{P}^2_{\overline{\mathbb{Q}}}$ at such a collection of six points is a smooth cubic surface defined over $\overline{\mathbb{Q}}$. We denote the blow-up of $\mathbb{P}^2_{\overline{\mathbb{Q}}}$ at $\{p_1, \ldots, p_6\}$ by $S(p_1, \ldots, p_6)$. The 27 lines on $S(p_1, \ldots, p_6)$ are obtained as follows:

(i) E_i is the exceptional divisor corresponding to p_i;

(ii) C_i is the strict transform of the conic passing through $\{p_1, \ldots, p_6\} - \{p_i\}$;

(iii) L_{ij} is the strict transform of the line passing through p_i and p_j.

The cubic surface $S(p_1, \ldots, p_6)$ is defined over $\mathbb{Q}$ if and only if it is invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. In particular, $S(p_1, \ldots, p_6)$ is defined over $\mathbb{Q}$ if and only if $\{p_1, \ldots, p_6\}$ is invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. Likewise, a line $L \subset S(p_1, \ldots, p_6)$ is defined over $\mathbb{Q}$ if and only if it is invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. In particular:

(i) E_i is defined over $\mathbb{Q}$ if and only if p_i is defined over $\mathbb{Q}$;

(ii) C_i is defined over $\mathbb{Q}$ if and only if $\{p_1, \ldots, p_6\} - \{p_i\}$ is invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action;

(iii) L_{ij} is defined over $\mathbb{Q}$ if and only if $\{p_i, p_j\}$ is invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action.

Example 3.1. We may now construct various examples by partitioning $\{p_1, \ldots, p_6\}$ into Galois-invariant subsets. In each of these cases, the entire set $\{p_1, \ldots, p_6\}$ is Galois-invariant, so that $S(p_1, \ldots, p_6)$ is defined over $\mathbb{Q}$.

(1) If $p_1, \ldots, p_6$ are all defined over $\mathbb{Q}$, then all 27 lines on $S(p_1, \ldots, p_6)$ are rational. This also forces all 27 lines on $S(p_1, \ldots, p_6)$ to be real.

(2) If $p_1, \ldots, p_4$ are defined over $\mathbb{Q}$ and p_5, p_6 are Galois conjugates, then precisely the lines L_{56} and E_i, C_j, L_{ij} for $1 \leq i, j \leq 4$ are defined over $\mathbb{Q}$. Thus $S(p_1, \ldots, p_6)$ contains 15 rational lines. Moreover, $\{p_1, \ldots, p_6\}$ can contain zero or one complex conjugate pair, so $S(p_1, \ldots, p_6)$ contains 15 or 27 real lines.

(3) If p_1, p_2, p_3 are defined over $\mathbb{Q}$ and p_4, p_5, p_6 are Galois conjugates, then precisely the lines E_i, C_j, L_{ij} for $1 \leq i, j \leq 3$ are defined over $\mathbb{Q}$. Thus $S(p_1, \ldots, p_6)$ contains 9 rational lines. Moreover, $\{p_1, \ldots, p_6\}$ can contain zero or one complex conjugate pair, so $S(p_1, \ldots, p_6)$ contains 15 or 27 real lines.

(4) If p_1, p_2 are defined over $\mathbb{Q}$, and if p_3, p_4 and p_5, p_6 are two pairs of Galois conjugates, then precisely the lines $E_1, E_2, C_1, C_2, L_{12}, L_{34}$, and L_{56} are defined over $\mathbb{Q}$. Thus $S(p_1, \ldots, p_6)$ contains 7 rational lines. Moreover, $\{p_1, \ldots, p_6\}$ can contain zero, one, or two complex conjugate pairs, so $S(p_1, \ldots, p_6)$ contains 7, 15, or 27 real lines.

(5) If p_1, p_2 are defined over $\mathbb{Q}$ and $p_3, \ldots, p_6$ are Galois conjugates, then precisely the lines E_1, E_2, C_1, C_2, and L_{12} are defined over $\mathbb{Q}$. Thus $S(p_1, \ldots, p_6)$ contains 5 rational lines. Moreover, $\{p_1, \ldots, p_6\}$ can contain zero, one, or two complex conjugate pairs, so $S(p_1, \ldots, p_6)$ contains 7, 15, or 27 real lines.

(6) If p_1 is defined over $\mathbb{Q}$, p_2, p_3, p_4 are Galois conjugates, and p_5, p_6 are Galois conjugates, then precisely the lines E_1, C_1, and L_{56} are defined over $\mathbb{Q}$. Thus $S(p_1, \ldots, p_6)$ contains 3 rational lines, and these lines are skew. Moreover, $\{p_1, \ldots, p_6\}$ can contain zero, one, or two complex conjugate pairs, so $S(p_1, \ldots, p_6)$ contains 7, 15, or 27 real lines.

(7) If p_i, p_j is a Galois conjugate pair for $(i,j) = (1,2)$, $(3,4)$, and $(5,6)$, then precisely the lines L_{12}, L_{34}, and L_{56} are defined over $\mathbb{Q}$. Thus $S(p_1,\ldots,p_6)$ contains 3 rational lines, and these lines are not skew. Moreover, $\{p_1,\ldots,p_6\}$ can contain zero, one, two, or three complex conjugate pairs, so $S(p_1,\ldots,p_6)$ contains 3, 7, 15, or 27 real lines.

(8) If p_1 is defined over $\mathbb{Q}$ and $p_2,\ldots,p_6$ are Galois conjugates, then precisely the lines E_1 and C_1 are defined over $\mathbb{Q}$. Thus $S(p_1,\ldots,p_6)$ contains 2 rational lines, and these lines are skew. Moreover, $\{p_1,\ldots,p_6\}$ can contain zero, one, or two complex conjugate pairs, so $S(p_1,\ldots,p_6)$ contains 7, 15, or 27 real lines.

(9) If $p_1,\ldots,p_4$ are Galois conjugates and p_5, p_6 are Galois conjugates, then precisely the line L_{56} is defined over $\mathbb{Q}$. Thus $S(p_1,\ldots,p_6)$ contains 1 rational line. Moreover, $\{p_1,\ldots,p_6\}$ can contain zero, one, two, or three complex conjugate pairs, so $S(p_1,\ldots,p_6)$ contains 3, 7, 15, or 27 real lines.

(10) If p_1, p_2, p_3 are Galois conjugates and p_4, p_5, p_6 are Galois conjugates, then $S(p_1,\ldots,p_6)$ contains no rational lines. Moreover, $\{p_1,\ldots,p_6\}$ can contain zero, one, or two complex conjugate pairs, so $S(p_1,\ldots,p_6)$ contains 7, 15, or 27 real lines.

(11) If $p_1,\ldots,p_6$ are Galois conjugates, then $S(p_1,\ldots,p_6)$ contains no rational lines. Moreover, $\{p_1,\ldots,p_6\}$ can contain zero, one, two, or three complex conjugate pairs, so $S(p_1,\ldots,p_6)$ contains 3, 7, 15, or 27 real lines.

In summary, we have proved the following.

Lemma 3.2. *Let n be 0, 1, 2, 3, 5, 7, 9, 15, or 27. Then there is a smooth cubic surface over $\mathbb{Q}$ that contains exactly n rational lines.*

Proof. It suffices to show that every scenario in Example 3.1 occurs for some set of six points in $\mathbb{P}^2_{\overline{\mathbb{Q}}}$, where the six points do not lie a conic and no three of the points lie on a line. We describe an approach (suggested to us by Jean-Pierre Serre) that allows one to explicitly construct the desired set of six points.

Let $C = \{[1:t:t^3] : t \in \overline{\mathbb{Q}}\}$ be a parameterized cubic in $\mathbb{P}^2_{\overline{\mathbb{Q}}}$. Three distinct points $[1:t_i:t_i^3]$ lie on the line $\mathbb{V}(ax+by+cz)$ if and only if each t_i is a root of $F(t) = a + bt + ct^3$. The sum of these roots is a scalar multiple of the coefficient of the degree 2 term of $F(t)$, so $[1:t_i:t_i^3]$ lie on a line for $i = 1,2,3$ if and only if $t_1 + t_2 + t_3 = 0$. Likewise, six distinct points $[1:t_i:t_i^3]$ lie on the conic $\mathbb{V}(ax^2+bxy+cy^2+dxz+eyz+fz^2)$ if and only if each t_i is a root of $G(t) = a + bt + ct^2 + dt^3 + et^4 + ft^6$. The sum of these roots is a scalar multiple of the coefficient of the degree 5 term of $G(t)$, so $[1:t_i:t_i^3]$ lie on a conic for $i = 1,\ldots,6$ if and only if $t_1 + \ldots + t_6 = 0$.

Next, note that $[1:s:s^3]$ and $[1:t:t^3]$ are $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugate if and only if s and t are $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugate. In order to construct a set of six general points $\{p_1,\ldots,p_6\}$ with the desired Galois conjugacies, it thus suffices to write down a degree six polynomial $G(t) \in \mathbb{Q}[t]$ satisfying the following properties.

(i) The degree 5 term of $G(t)$ is non-zero.

(ii) No three roots of $G(t)$ sum to zero.

(iii) If $\{p_1, \ldots, p_6\}$ has a minimal Galois-invariant subset of order n, then $G(t)$ has a degree n factor that is irreducible over $\mathbb{Q}$.

This can be done for each of the scenarios in Example 3.1. As mentioned in Section 1.1, this method can also be used to construct examples of lines on smooth cubic surfaces over other fields. □

4. Rational cubic surfaces with no skew triples

In general, a smooth cubic surface over $\mathbb{Q}$ need not contain a skew triple. For example, such a cubic surface might contain no rational lines at all.

Example 4.1. Consider the Fermat cubic $S^a = \mathbb{V}(x_0^3 + ax_1^3 + ax_2^3 + ax_3^3)$, where $a \in \mathbb{Q}$ is not a cube. All 27 lines on $S^a_{\mathbb{C}}$ are of the form $\mathbb{V}(x_0 + cx_i, x_j + \zeta x_k)$, where c is a cube root of a, ζ is a third root of unity, and $i, j, k \in \{1, 2, 3\}$ are pairwise distinct. Since a is not a cube in $\mathbb{Q}$, it follows that $c \notin \mathbb{Q}$ and hence these lines are not defined over $\mathbb{Q}$. We remark that 3 of these lines are defined over $\mathbb{R}$. Compare to Example 3.1 (11).

If a smooth cubic surface S over $\mathbb{Q}$ does not contain a skew triple but does contain a rational line, then we can apply Bézout's Theorem to constrain the number of lines that S contains. To do so, we first define a graph associated to a smooth cubic surface and its lines.

Definition 4.2. Let S be a smooth cubic surface over a field k. The *intersection graph* $G(S)$ of S is the graph whose vertices are given by the lines contained in S and whose edges correspond to intersections of lines on S.

Next, we recall a proposition from [MMZ20].

Proposition 4.3. *[MMZ20, Proposition 9.2] Let G be a graph with at least seven vertices such that for any triple of vertices $v_1, v_2, , v_3$, at least two of v_1, v_2, v_3 are connected by an edge. Then G contains two distinct 3-cycles that share an edge.*

Remark 4.4. If a smooth cubic surface S does not contain a skew triple, then the graph $G(S)$ must satisfy the property that for any triple of vertices v_1, v_2, v_3, at least two of v_1, v_2, v_3 are connected by an edge. We say that a graph with this property *has no disjoint triples.*

Corollary 4.5. *If a smooth cubic surface S does not contain a skew triple, then S contains at most six lines.*

Proof. By Bézout's Theorem, any plane contains at most three lines contained in S, so the line graph $G(S)$ may not contain any distinct 3-cycles that share an edge. By the contrapositive of Proposition 4.3, it follows that $G(S)$ contains at most six vertices and hence S contains at most six lines. □

We can consider all possible configurations of lines on a rational cubic surface S with no skew triples by studying the intersection graph $G(S)$. In particular, if S contains exactly n lines, then $G(S)$ must be a graph of order n with no disjoint triples. Moreover, $G(S)$ may not contain any disjoint 3-cycles that share an edge. By Corollary 2.6, any edge in $G(S)$ must be contained in a 3-cycle. We call any order n graph satisfying these conditions a *permissible* graph. By considering all finite simple graphs of order $1 \leq n \leq 6$, we can classify all permissible graphs of order at most six. It turns out that for each $1 \leq n \leq 6$, there is exactly one permissible graph of order n. For $n = 1$ and $n = 2$, the permissible graph of order n consists of n vertices and no edges. The permissible graphs of order $3 \leq n \leq 6$ are illustrated in Figure 1.

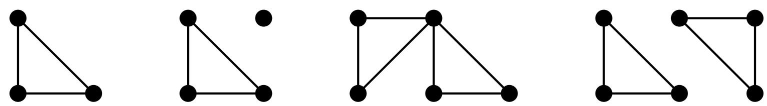

FIGURE 1. Permissible graphs of order $3 \leq n \leq 6$.

Given a rational cubic surface S, we will sometimes compare its intersection graph $G(S)$ with the intersection graph $G(S_{\mathbb{R}})$ of the corresponding real cubic surface. Using the topological classification of real smooth cubic surfaces (due to Schläfli, Klein, and Zeuthen [Sch58, Kle73, Zeu74] and summarized by Segre [Seg42]) we can completely classify the intersection graphs of real cubic surfaces.

Proposition 4.6. *If two smooth cubic surfaces are projectively equivalent, then they have the same intersection graph.*

Proof. Let k be a field, and suppose $S = \mathbb{V}(f), S' = \mathbb{V}(f')$ are smooth cubic surfaces over k that are projectively equivalent. Then there exists $\varphi \in \mathrm{PGL}_3(k)$ such that $S' = \varphi S$, or in other words $f' = f \circ \varphi^{-1}$. Note that a line L is contained in S if and only if $f|_L \equiv 0$. Moreover, φL is also a line, and $f \circ \varphi^{-1}|_{\varphi L} \equiv 0$, so S' contains the line φL. The same line of reasoning shows that if L' is a line in S', then $\varphi^{-1} L'$ is a line in S. We thus see that φ induces a bijection on the lines in S and S', which induces a bijection on the vertices of $G(S)$ and $G(S')$.

It remains to show that two distinct lines $L_1, L_2 \subset S$ intersect if and only if $\varphi L_1, \varphi L_2 \subset S'$ intersect. If $L_1 \cap L_2 = \{p\}$, then both φL_1 and φL_2 contain the point $\{\varphi(p)\}$ and therefore must intersect. Likewise, if $\varphi L_1 \cap \varphi L_2 = \{q\}$, then both L_1 and L_2 contain the point $\{\varphi^{-1}(q)\}$. □

Proposition 4.7. *If two real smooth cubic surfaces in $\mathbb{P}^3_{\mathbb{R}}$ are topologically equivalent, then they are projectively equivalent.*

Proof. By the work of Klein [Kle73], the following equivalence relations for real smooth cubic surfaces in $\mathbb{P}^3_{\mathbb{R}}$ all coincide.

(a) Topological equivalence. Two real smooth cubic surfaces are said to be *topologically equivalent* if there is a homeomorphism between their real points.

(b) Isotopy. Here, the ambient space is $\mathbb{P}^3_{\mathbb{R}}$.

(c) Rigid isotopy. Two real smooth cubic surfaces are said to be *rigidly isotopic* if there exists an isotopy from one to the other that preserves singularity types.

(d) Rough projective equivalence. Two real smooth cubic surfaces are said to be *rough projectively equivalent* if they are projectively equivalent and rigidly isotopic.

For a discussion of this result in modern language, see [DK00, Section 3.5.1]. □

Corollary 4.8. *Any two real smooth cubic surfaces with the same number of lines have the same intersection graph.*

Proof. Segre shows that for $n = 7, 15$, or 27, there is one topological equivalence class of real smooth cubic surfaces containing n lines [Seg42]. These equivalence classes are also projective equivalence classes by Proposition 4.7, so Proposition 4.6 implies that each equivalence class of real smooth cubic surfaces has a fixed intersection graph. There are two topological equivalence classes of real smooth cubic surfaces containing three lines, but [Seg42, III, §31, (iv) and (v)] implies that any cubic surface in either of these equivalence classes have intersection graph K_3, the complete graph of order 3. □

Alternate proof. Corollary 4.8 also follows from [MMZ20, Theorem 9.4] and [Seg42, III, §31, (iv) and (v)]. The latter reference shows that all real smooth cubic surfaces with exactly three lines have the same intersection graph. The former reference shows that for real smooth cubic surfaces containing at least seven lines, the number of lines is determined by the (non)-reality of the roots of a cubic polynomial and a quadratic polynomial. Moreover, these roots determine which lines the real cubic surface contains, and the intersection properties of these lines depend only on the number of lines within the cubic surface. □

Yet another proof. [Seg42, III, §31, (iv) and (v)] shows that all real smooth cubic surfaces with exactly three lines have the same intersection graph. All other real smooth cubic surfaces are the blow-up of $\mathbb{P}^2$ at a set of six general points $\{p_1, \ldots, p_6\} \subset \mathbb{P}^2$ that is invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action. The lines on the resulting cubic surface are as follows.

(i) E_i is the exceptional divisor corresponding to the point p_i. This line is defined over $\mathbb{R}$ if and only if p_i is a real point.

(ii) C_i is the strict transform of the conic through the points $\{p_1, \ldots, p_6\} - \{p_i\}$. This line is defined over $\mathbb{R}$ if and only if $\{p_1, \ldots, p_6\} - \{p_i\}$ is invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action.

(iii) L_{ij} is the strict transform of the line through p_i and p_j. This line is defined over $\mathbb{R}$ if and only if $\{p_i, p_j\}$ is invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action.

If $\{p_1, \ldots, p_6\}$ consists of six real points, four real points and a pair of complex conjugate points, or two real points and two pairs of complex conjugate points, then the resulting real smooth cubic surface has 27, 15, or 7 real lines, respectively. The intersection graph does not depend on the choice of labeling of $\{p_1, \ldots, p_6\}$, which proves the desired result. □

As previously mentioned, the intersection graph of a real smooth cubic surface with 3 lines is the complete graph K_3. The intersection graph of a smooth cubic surface with 27 lines is the complement of the Schläfli graph and has been the subject of extensive study. By considering Examples 5.13 and 5.14, Corollary 4.8 allows us to describe the intersection graphs of real smooth cubic surfaces containing 7 or 15 lines.

Proposition 4.9. *Any real smooth cubic surface S with seven lines has intersection graph F_3, the friendship graph of order 7 (see Figure 2).*

Proof. It in fact suffices to notice that S contains a line that meets the other six lines. Thus $G(S)$ contains S_6, the star graph of order 7. By generalizing the proof of Corollary 2.6 for real smooth cubic surfaces, we note that every edge in the intersection graph must be contained in a 3-cycle. Thus $G(S)$ contains F_3. As previously remarked, any 3-cycle in the intersection graph of a smooth cubic surface corresponds to three coplanar lines. By Bézout's Theorem, any hyperplane section of S may contain at most three lines, so the intersection graph $G(S)$ may not contain any distinct 3-cycles that share an edge. There are hence no additional edges in $G(S)$, so $G(S)$ must be equal to F_3. □

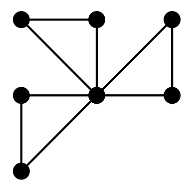

FIGURE 2. Intersection graph of a real smooth cubic surface with 7 lines.

4.1. **Rational cubic surfaces with one line.** Assume that S is a rational cubic surface containing exactly one line L. Suppose $S_{\mathbb{R}}$ contains exactly 3 lines. Then $S_{\mathbb{R}}$ would contain L (considered as a line over $\mathbb{R}$) as well as two real lines L_1, L_2 that are not defined over $\mathbb{Q}$. By modifying the coefficients of the cubic surface in Example 4.1, we can describe some cubic surfaces that match this description.

Example 4.10. Consider the Fermat cubic $S^a = \mathbb{V}(a^4x_0^3 + ax_1^3 + x_2^3 + x_3^3)$, where $a \in \mathbb{Q}$ is not a cube. The smooth cubic surface $S^a_{\mathbb{C}}$ contains the 9 lines $\mathbb{V}(bx_0 + x_1, x_2 + \zeta x_3)$, where b is a cube root of a^3 and ζ is a third root of unity. Precisely one of these 9 lines is defined over $\mathbb{Q}$, namely when $b = a$ and $\zeta = 1$. The other 18 lines on $S^a_{\mathbb{C}}$ are of the form $\mathbb{V}(bx_0 + x_i, cx_1 + x_j)$, where b is a cube root of a^4, c is a cube root of a, and $i \neq j$. Since a is not a cube in $\mathbb{Q}$, it follows that $c \notin \mathbb{Q}$ and hence these lines are not defined over $\mathbb{Q}$. Thus S^a contains exactly one rational line. Compare to Example 3.1 (9).

Remark 4.11. In [MMZ20], the authors note that $S_{\mathbb{R}}$ contains three skew lines if and only if $S_{\mathbb{R}}$ contains 7, 15, or 27 real lines. By [Seg42, III, §31, (iv) and (v)], $S_{\mathbb{R}}$ does not contain three skew lines if and only if $S_{\mathbb{R}}$ contains exactly 3 lines, all of which are pairwise-intersecting.

4.2. **Rational cubic surfaces with two lines.** Assume that S is a rational cubic surface containing exactly two lines L_1, L_2. By Corollary 2.6, we know that L_1 and L_2 do not intersect. If $S_{\mathbb{R}}$ were to contain only three lines, then the rational lines L_1, L_2 in S would necessarily intersect by Remark 4.11. We can thus conclude that $S_{\mathbb{R}}$ contains at least seven lines. By Example 3.1 (8), there is a smooth cubic surface over $\mathbb{Q}$ that contains two rational lines.

Example 4.12. [Pan09, Section 4.3.3] Pannekoek shows that the smooth cubic surface $\mathbb{V}(f)$ contains two rational lines, where

$$\begin{aligned} f &= 175959x_0^2x_2 + 518643x_0x_1x_2 - 131841x_0x_2^2 + 19x_0x_2x_3 \\ &\quad + 27x_0^2x_3 + 400653x_0x_1x_3 + 121068x_0x_3^2 + 52326x_1^2x_2 \\ &\quad + 11799x_1x_2^2 + 383211x_1x_2x_3 + 235467x_1^2x_3 + 108243x_1x_3^2. \end{aligned}$$

4.3. **Rational cubic surfaces with three lines.** Let $S_{\mathbb{R}}$ be a real cubic surface that contains exactly three lines. As previously remarked, Segre shows that the lines in $S_{\mathbb{R}}$ are all pairwise-intersecting. To construct a rational cubic surface with exactly three lines, we simply need to construct a real cubic surface containing exactly three lines, all of which are rational.

Example 4.13. Consider the Fermat cubic $S = \mathbb{V}(x_0^3+x_1^3+x_2^3+x_3^3)$. As with the Fermat cubics in Examples 4.1 and 4.10, $S_{\mathbb{R}}$ contains three real lines. Over $\mathbb{R}$, the only lines contained in this surface are $\mathbb{V}(x_0+x_1, x_2+x_3)$, $\mathbb{V}(x_0+x_2, x_1+x_3)$, and $\mathbb{V}(x_0+x_3, x_1+x_2)$. Since all of these lines are defined over $\mathbb{Q}$, it follows that S contains exactly three lines over $\mathbb{Q}$. Compare to Example 3.1 (7).

4.4. **Rational cubic surfaces with four lines.** We will show that there are no rational cubic surfaces containing exactly four lines with no skew triple.

Proposition 4.14. *Let S be a rational smooth cubic surface containing three pairwise-intersecting lines. Every other rational line on S must intersect one of these three lines.*

Proof. Without loss of generality, we may assume that the three pairwise-intersecting lines are E_1, C_6, and L_{16}. Over $\mathbb{C}$, the lines C_i and L_{1i} for $2 \leq i \leq 6$ intersect E_1, the lines E_j and L_{j6} for $1 \leq j \leq 5$ intersect C_6, and the lines E_6, C_1, and L_{mn} for $m > 1$ and $n < 6$ intersect L_{16}. This accounts for all 27 lines on $S_{\mathbb{C}}$. If S contains E_1, C_6, L_{16}, and another line L, then $L_{\mathbb{C}}$ intersects $L'_{\mathbb{C}}$, where L' is one of E_1, C_6, or L_{16}. Since L and L' are defined over $\mathbb{Q}$, they must also intersect over $\mathbb{Q}$. □

Corollary 4.15. *A rational smooth cubic surface with no skew triple cannot contain exactly four lines.*

Proof. The permissible graph of order 4 (see Figure 1) implies that any rational cubic surface with no skew triple and exactly four lines must contain three pairwise-intersecting lines. Proposition 4.14 implies that the intersection graph of such a cubic surface must be connected. Since the permissible graph of order 4 is not connected, there is no rational smooth cubic surface with no skew triple that contains exactly four lines. □

4.5. **Rational cubic surfaces with five lines.** Assume that S is a rational cubic surface containing exactly five lines with no skew triple. It follows that $S_{\mathbb{R}}$ contains at least seven lines, so $S_{\mathbb{R}}$ contains a skew triple. By Example 3.1 (5), there is a smooth cubic surface over $\mathbb{Q}$ that contains five rational lines.

4.6. **Rational cubic surfaces with six lines.** As in the case of rational cubic surfaces with no skew triples and four lines, Proposition 4.14 implies that there are no rational smooth cubic surface with no skew triples and exactly six lines.

Corollary 4.16. *A rational smooth cubic surface with no skew triple cannot contain exactly six lines.*

Proof. The permissible graph of order 6 (see Figure 1) implies that any rational cubic surface with no skew triple and exactly six lines must contain three pairwise-intersecting lines. Proposition 4.14 implies that the intersection graph of such a cubic surface must be connected. Since the permissible graph of order 6 is not connected, there is no rational smooth cubic surface with no skew triple that contains exactly six lines. □

In summary of this section, we have proved the following lemma.

Lemma 4.17. *A smooth cubic surface over $\mathbb{Q}$ with no skew triple must contain 0, 1, 2, 3, or 5 rational lines.*

5. Rational cubic surfaces with a skew triple

If S is a rational smooth cubic surface that contains a skew triple, then we may assume that S contains the lines $E_1 = \mathbb{V}(x_0, x_1)$, $E_2 = \mathbb{V}(x_2, x_3)$, and $E_3 = \mathbb{V}(x_0 - x_2, x_1 - x_3)$ by a projective change of coordinates. In this context, we may apply the results of [MMZ20]. We recall the relevant details here.

Proposition 5.1. *[MMZ20, Proposition 3.1] There is a cubic polynomial $g(t) \in \mathbb{Q}[t]$ with distinct roots t_4, t_5, t_6 such that the lines C_i contained in $S_{\mathbb{C}}$ are defined over $\mathbb{Q}(t_i)$ for $4 \leq i \leq 6$.*

Proof. Since S is defined over $\mathbb{Q}$, we have $S = \mathbb{V}(f)$ for $f \in \mathbb{Q}[x_0, x_1, x_2, x_3]$. This implies that $g(t) \in \mathbb{Q}[t]$. Explicitly, we have the line $C_i = \mathbb{V}(x_0 - t_i x_1, x_2 - t_i x_3)$, so this line is defined over $\mathbb{Q}(t_i)$. □

This allows us study the rationality of C_4, C_5, C_6. There are three cases to consider.

(1) If $g(t)$ is irreducible over $\mathbb{Q}$, then t_4, t_5, t_6 are algebraic numbers not contained in $\mathbb{Q}$. In this case, C_4, C_5, C_6 are not rational.

(2) If $g(t)$ has only one rational root, then we may assume without loss of generality that $t_4 \in \mathbb{Q}$ and t_5, t_6 are algebraic numbers not contained in $\mathbb{Q}$. In this case, C_4 is rational and C_5, C_6 are not rational.

(3) If $g(t)$ has at least two rational roots, then all three roots of $g(t)$ are rational and hence C_4, C_5, C_6 are rational.

For $1 \leq i \leq 3$ and $4 \leq j \leq 6$, the lines E_i and C_j intersect. By [MMZ20, Proposition 4.1], $S_{\mathbb{C}}$ also contains the line L_{ij}, and the lines E_i, C_j, L_{ij} are coplanar.

Proposition 5.2. *If C_j is not rational, then L_{ij} is not rational.*

Proof. Since E_i is rational, if L_{ij} were rational, then Proposition 2.5 would imply that C_j is also rational. □

The previous proposition allows us to determine the (non)-rationality of the nine lines L_{ij}. As before, we have the following three cases.

(1) If C_4, C_5, C_6 are not rational, then L_{ij} is not rational for $1 \leq i \leq 3$ and $4 \leq j \leq 6$.

(2) If C_4 is rational and C_5, C_6 are not rational, then L_{i4} is rational and L_{i5}, L_{i6} are not rational for $1 \leq i \leq 3$.

(3) If C_4, C_5, C_6 are rational, then L_{ij} is rational for $1 \leq i \leq 3$ and $4 \leq j \leq 6$.

Proposition 5.3. *[MMZ20, Proposition 5.5] There is a quadratic polynomial $h(s) \in \mathbb{Q}(t_4, t_5)[s]$ with distinct roots s_1, s_2 such that the lines C_3 and L_{12} contained in $S_{\mathbb{C}}$ are defined over $\mathbb{Q}(s_1, t_4)$ and $\mathbb{Q}(s_2, t_4)$, respectively. In particular, we have*

$$C_3 = \mathbb{V}(x_0 + (-s_1c_1 - t_4)x_1, (1 + s_1c_2)x_2 + (s_1c_3 - t_4)x_3),$$
$$L_{12} = \mathbb{V}(x_0 + (-s_2c_1 - t_4)x_1, (1 + s_2c_2)x_2 + (s_2c_3 - t_4)x_3),$$

where $c_1x_1 + c_2x_2 + c_3x_3$ is one of the linear forms defining L_{34}.

This proposition allows us to analyze the rationality of C_3 and L_{12} in terms of the rationality of s_1, s_2, and t_4.

Corollary 5.4. *If t_4 and s_1 (respectively s_2) are rational, then C_3 (respectively L_{12}) is defined over $\mathbb{Q}$.*

Proof. If $t_4, s_i \in \mathbb{Q}$, then $\mathbb{Q}(s_i, t_4) = \mathbb{Q}$. □

Corollary 5.5. *If t_4 is rational and s_1 (respectively s_2) is not rational, then C_3 (respectively L_{12}) is not defined over $\mathbb{Q}$.*

Proof. By a slight modification of [MMZ20, Proposition 5.2], we have that $c_1 \neq 0$. Thus if t_4 is rational and s_i is not rational for $i = 1$ (respectively $i = 2$), then $s_ic_1 + t_4 \in \mathbb{Q}(s_i) \backslash \mathbb{Q}$, so C_3 (respectively L_{12}) is not defined over $\mathbb{Q}$. □

Corollary 5.6. *If t_4 is not rational and s_1 (respectively s_2) is rational, then C_3 (respectively L_{12}) is not defined over $\mathbb{Q}$.*

Proof. If t_4 is not rational, then the cubic polynomial $g(t)$ from Proposition 5.1 is a $\mathbb{Q}$-scalar multiple of the minimal polynomial for t_4 over $\mathbb{Q}$. By [MMZ20, Propositions 3.1 and 4.1], we have $c_1 = at_4^2 + xt_4 + y$ for some $a, x, y \in \mathbb{Q}$ with $a \neq 0$. Thus c_1 is not rational. Moreover, if s_i is rational for $i = 1$ (respectively $i = 2$), then $s_i c_1 + t_4 = (s_i a)t_4^2 + (s_i x + 1)t_4 + s_i y$ is not rational, so C_3 (respectively L_{12}) is not defined over $\mathbb{Q}$. □

It turns out that if one of C_3 and L_{12} is not defined over $\mathbb{Q}$, then the other line is also not defined over $\mathbb{Q}$.

Proposition 5.7. *If C_3 is not defined over $\mathbb{Q}$, then L_{12} is not defined over $\mathbb{Q}$. Likewise, if L_{12} is not defined over $\mathbb{Q}$, then C_3 is not defined over $\mathbb{Q}$.*

Proof. It suffices to prove the statement under the assumption that C_3 is not defined over $\mathbb{Q}$. The other case follows by relabeling s_1 and s_2. Since E_2 is rational and C_3 is not rational, Proposition 2.5 implies that L_{23} is not rational. Since E_3 is rational and L_{23} is not rational, C_2 is not rational. Finally, since E_1 is rational and C_2 is not rational, L_{12} is not rational, as desired. □

When t_4, t_5, t_6 are not rational, our cubic surface has either 3 rational lines or at least 15 real lines.

Lemma 5.8. *If $t_4, t_5, t_6 \notin \mathbb{Q}$, then $S_{\mathbb{C}}$ has either exactly 3 rational lines or at least 15 real lines.*

Proof. By assumption, the lines E_1, E_2, E_3 are rational, and the lines C_4, C_5, C_6, and L_{ij} for $1 \leq i \leq 3$ and $4 \leq j \leq 6$ are not rational. If C_3 and L_{12} are rational, then Proposition 2.5 implies that C_1, C_2, L_{13}, and L_{23} are rational, and the remaining lines on $S_{\mathbb{C}}$ are not rational. Thus S contains 9 rational lines. Since all rational lines are also real lines, this implies that $S_{\mathbb{R}}$ contains at least 15 lines.

If C_3 and L_{12} are not rational, then E_1, E_2, E_3 are rational and C_i, L_{jk} are not rational for $1 \leq i \leq 6$, $1 \leq j \leq 3$ and $4 \leq k \leq 6$, and $(j,k) = (1,2)$, $(1,3)$, or $(2,3)$. The remaining lines are $E_4, E_5, E_6, L_{45}, L_{46}$, and L_{56}. If all of these remaining lines are not rational, then S contains exactly three rational lines. Moreover, these rational lines are skew, so $S_{\mathbb{R}}$ contains at least 7 lines by Corollary 4.8.

If one of the remaining lines $E_4, E_5, E_6, L_{45}, L_{46}$, and L_{56} is rational, then S contains at least 4 rational lines. Moreover, E_1, E_2, E_3, and any one of the remaining lines are all pairwise skew, so the intersection graph $G(S_{\mathbb{Q}})$ contains 4 non-adjacent vertices. Since there is no such collection of vertices in the complete graph K_3 or the friendship graph F_3, and since $G(S_{\mathbb{Q}})$ is a subgraph of $G(S_{\mathbb{R}})$, it follows that $S_{\mathbb{R}}$ must contain at least 15 lines. □

We can now consider the following cases.

(a) Both s_1, s_2 are not rational.

(b) One of s_1, s_2 is not rational. Without loss of generality, we may assume in this case that s_1 is not rational and s_2 is rational.

(c) Both s_1, s_2 are rational.

As mentioned by Harris [Har79, p. 719], the remaining ten lines in $S_{\mathbb{C}}$ are residually determined. In particular, L_{ij}, E_i, C_j are coplanar for $i \neq j$, and L_{ij}, L_{mn}, L_{pq} are coplanar for i, j, m, n, p, q all distinct. This allows us to solve for the lines C_1, C_2, E_4, E_5, E_6, L_{13}, L_{23}, L_{45}, L_{46}, and L_{56}.

Remark 5.9. Combining with cases (1), (2), and (3) and applying Proposition 2.5 to the relevant triples of coplanar lines, we can now list all possible line counts for rational cubic surfaces containing a skew triple.

(1) If t_4, t_5, t_6 are not rational, then C_4, C_5, C_6, and L_{ij} are not rational for $1 \leq i \leq 3$ and $4 \leq j \leq 6$.

  (a) If s_2 is not rational, then L_{12} may or may not be rational. If L_{12} is rational, then C_3 is also rational by Proposition 5.7. It follows that C_1, C_2, L_{13}, L_{23} are rational, and $E_4, E_5, E_6, L_{45}, L_{46}, L_{56}$ are not rational. Thus S contains nine lines. While this line count depends on the assumption that L_{12} is rational, Example 3.1 (3) proves that there is in fact a rational cubic surface that contains exactly nine lines.

  (b) If L_{12} is not rational (for example, if s_2 is rational), then C_3 is not rational by Proposition 5.7. Thus C_1, C_2, L_{13}, L_{23} are not rational. The remaining lines to consider are $E_4, E_5, E_6, L_{45}, L_{46}$, and L_{56}. If at least one of these lines is rational, then S contains four skew rational lines. We will show that this forces C_3 or L_{12} to be rational, which is a contradiction.

  Let L be a rational line from the set of $E_4, E_5, E_6, L_{45}, L_{46}$, and L_{56}. Recall that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action preserves the incidence properties of the lines on $S_{\mathbb{C}}$. In particular, the set of lines meeting E_1 and E_2 but not E_3 and L is invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, since E_1, E_2, E_3, L are fixed under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action as rational lines. If L is E_4, E_5, or E_6, then C_3 is the unique line meeting E_1 and E_2 but not E_3 or L, so C_3 is fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. In particular, C_3 is rational, which contradicts the fact that s_2 is rational. On the other hand, if L is L_{45}, L_{46}, or L_{56}, then L_{12} is the unique line meeting E_1 and E_2 but not E_3 or L. In this case, we conclude that L_{12} is rational, again obtaining a contradiction. We thus conclude that if s_2 is rational, then S contains exactly three rational lines, and these lines are skew. Compare to Example 3.1 (6).

(2) If t_4 is rational and t_5, t_6 are not rational, then C_4 and L_{i4} are rational and C_5, C_6, L_{i5}, L_{i6} are not rational for $1 \leq i \leq 3$.

  (a) If s_1, s_2 are not rational, then C_3, L_{12} are not rational. Thus the remaining lines are not rational. In this case, $S_{\mathbb{C}}$ contains seven rational lines, so the rational cubic surface S contains seven lines. Compare to Example 3.1 (4).

  (b) If s_1 is not rational and s_2 is rational, then C_3 is not rational and L_{12} is rational. By Proposition 5.7, this scenario cannot occur.

  (c) If s_1, s_2 are rational, then C_3, L_{12} are rational. Thus $C_1, C_2, E_4, L_{13}, L_{23}, L_{56}$ are rational and E_5, E_6, L_{45}, L_{46} are not rational. In this case, $S_{\mathbb{C}}$ contains fifteen rational lines, so the rational cubic surface S contains fifteen lines. Compare to Example 3.1 (2).

(3) If t_4, t_5, t_6 are rational, then C_j and L_{ij} are rational for $1 \leq i \leq 3$ and $4 \leq j \leq 6$.

   (a) If s_1, s_2 are not rational, then C_3, L_{12} are not rational. Thus the remaining lines are not rational. In this case, $S_{\mathbb{C}}$ contains fifteen rational lines, so the rational cubic surface S contains fifteen lines. Compare to Example 3.1 (2).

   (b) If s_1 is not rational and s_2 is rational, then C_3 is not rational and L_{12} is rational. By Proposition 5.7, this scenario cannot occur.

   (c) If s_1, s_2 are rational, then C_3, L_{12} are rational. Thus the remaining lines are rational, so all twenty-seven lines on $S_{\mathbb{C}}$ are rational. In this case, the rational cubic surface S contains twenty-seven lines. Compare to Example 3.1 (1).

In summary, we have proved the following lemma.

Lemma 5.10. *A smooth cubic surface over $\mathbb{Q}$ that contains a skew triple must contain 3, 7, 9, 15, or 27 lines.*

We have also proved our main theorem.

Proof of Theorem 1.1. By Lemma 4.17 and Lemma 5.10, a smooth cubic surface over $\mathbb{Q}$ can only have 0, 1, 2, 3, 5, 7, 9, 15, or 27 rational lines. By Lemma 3.2, each of these line counts is realized by a smooth cubic surface over $\mathbb{Q}$. ☐

Remark 5.11. There are ten distinct intersection graphs for rational cubic surfaces that occur as blow-ups of $\mathbb{P}^2$ (see Example 3.1). Our work in Sections 4 and 5 shows that no other intersection graphs occur. For example, any rational cubic surface that is not a blow-up of $\mathbb{P}^2$ has the intersection graph of a rational cubic surface that is a blow-up of $\mathbb{P}^2$.

5.1. **Examples.** We conclude this section by giving examples of smooth cubic surfaces over $\mathbb{Q}$ that satisfy the various situations considered in Remark 5.9. We remark that the formulas for $h(s)$ depend on our choice of labels t_4, t_5, t_6 of the roots of $g(t)$. However, the overall count of lines on our cubic surface with a prescribed field of definition does not depend on this choice.

Example 5.12. Let $f = x_0^2x_2 - x_0x_2^2 + 2x_0^2x_3 - 2x_0x_1x_2 + x_1^2x_2 - x_0x_1x_3 + 2x_1^2x_3 - 2x_1x_3^2$. The rational cubic surface $S = \mathbb{V}(f)$ has $g(t) = t^3+2$, so $t_4, t_5, t_6 \notin \mathbb{Q}$. Setting $t_4 = -\sqrt[3]{2}$, $t_5 = \frac{\sqrt[3]{2}(1-i\sqrt{3})}{2}$, and $t_6 = \frac{\sqrt[3]{2}(1+i\sqrt{3})}{2}$, the roots of $h(s)$ are

$$\frac{8i\sqrt{3}(4\sqrt[3]{2}+1) \pm \sqrt{1152i\sqrt{3} - 1152} - 96\sqrt[3]{2} - 24}{32i\sqrt{3}(\sqrt[3]{4}+2\sqrt[3]{2}+1) - 96\sqrt[3]{4} - 192\sqrt[3]{2} - 96}.$$

We have $c_1 = \sqrt[3]{4} + 2\sqrt[3]{2} + 1$, and hence $s_ic_1 + t_4 \notin \mathbb{R}$, where s_i is a root of $h(s)$. By [MMZ20, Proposition 5.5], the lines C_3 and L_{12} of $S_{\mathbb{C}}$ are not real. Since L_{12} is not rational, Remark 5.9 (1b) implies that S contains exactly three rational lines. These are $E_1 = \mathbb{V}(x_0, x_1)$, $E_2 = \mathbb{V}(x_2, x_3)$, and $E_3 = \mathbb{V}(x_0 - x_2, x_1 - x_3)$.

We note that $s_2 \notin \mathbb{Q}$, which was the initial assumption in Remark 5.9 (1a). This shows that, in contrast to [MMZ20, Theorem 9.4] for real cubic surfaces, the (non)-rationality of the roots of $g(t)$ and $h(s)$ do not always determine the number of lines on a rational cubic surface with a skew triple.

Example 5.13. Let $f = x_0^2x_2 - x_0x_2^2 + 2x_0^2x_3 - 2x_0x_1x_2 + x_1^2x_2 - x_0x_1x_3 + x_1^2x_3 - x_1x_3^2$. The rational cubic surface $S = \mathbb{V}(f)$ has $g(t) = t^3 + 1$. Setting $t_4 = -1, t_5 = \frac{1+i\sqrt{3}}{2}, t_6 = \frac{1-i\sqrt{3}}{2}$, we have $h(s) = -\frac{i\sqrt{3}+3}{2}(16s^2 - 11s + 2)$. Thus S contains seven lines by Remark 5.9 (2a).

Example 5.14. Let $f = x_0^2x_2 - x_0z_2^2 + x_0^2x_3 - x_0x_1x_2 + x_1^2x_2 - 2x_0x_1x_3 + x_1x_2^2 - x_0x_2x_3 - x_0x_3^2 + 2x_1x_2x_3$. The rational cubic surface $S = \mathbb{V}(f)$ has $g(t) = t^3 - t$. Setting $t_4 = 0, t_5 = 1, t_6 = -1$, we have $h(s) = s^2 - 2s + 3$. Thus S contains fifteen lines by Remark 5.9 (3a).

Example 5.15. Let $f = x_0^2x_2 - x_0x_2^2 + x_0^2x_3 - x_0x_1x_2 + \frac{17}{39}x_1x_2^2 - \frac{17}{39}x_0x_2x_3 + 2x_1^2x_2 - 3x_0x_1x_3 + \frac{12}{13}x_0x_3^2 + \frac{1}{13}x_1x_2x_3$. The rational cubic surface $S = \mathbb{V}(f)$ has $g(t) = t^3 - t$. Setting $t_4 = 0, t_5 = 1, t_6 = -1$, we have $h(s) = \frac{1}{72}(28s^2 + 108s - 81)$. Thus S contains twenty-seven lines by Remark 5.9 (3c).

Appendix A. Sage code

In this appendix, we include some Sage code that we use to study smooth cubic surfaces over $\mathbb{Q}$ with a skew triple. In [MMZ20], we assume that our cubic surface contains the skew triple $E_1 = \mathbb{V}(x_0, x_1)$, $E_2 = \mathbb{V}(x_2, x_3)$, and $E_3 = \mathbb{V}(x_0 - x_2, x_1 - x_3)$ to simplify calculations. We make this assumption here as well. Cubic surfaces containing this particular skew triple are of the form $\mathbb{V}\left(\sum_{i+j+k+\ell=3} \alpha_{i,j,k,\ell} x_0^i x_1^j x_2^k x_3^\ell\right)$ with

$$\begin{aligned}
\alpha_{3,0,0,0} = \alpha_{0,3,0,0} = \alpha_{0,0,3,0} = \alpha_{0,0,0,3} = 0,\\
\alpha_{2,1,0,0} = \alpha_{1,2,0,0} = \alpha_{0,0,2,1} = \alpha_{0,0,1,2} = 0,\\
\alpha_{0,2,0,1} + \alpha_{0,1,0,2} = \alpha_{2,0,1,0} + \alpha_{1,0,2,0} = 0,\\
\alpha_{0,2,1,0} + \alpha_{1,0,0,2} + \alpha_{1,1,0,1} + \alpha_{0,1,1,1} = 0,\\
\alpha_{0,1,2,0} + \alpha_{2,0,0,1} + \alpha_{1,0,1,1} + \alpha_{1,1,1,0} = 0.
\end{aligned}$$

Given $\alpha_{i,j,k,\ell}$ that satisfy the above relations, we first check if the resulting cubic surface is smooth.

```
P3 = ProjectiveSpace(3, QQ, 'x');
P3.inject_variables();
var('a0201 a0102 a2010 a1020 a0210 a1002
     a1101 a0111 a0120 a2001 a1011 a1110');
f = a0201*x1^2*x3+a0102*x1*x3^2+a2010*x0^2*x2
   +a1020*x0*x2^2+a0210*x1^2*x2+a1002*x0*x3^2
   +a1101*x0*x1*x3+a0111*x1*x2*x3+a0120*x1*x2^2
```

```
    +a2001*x0^2*x3+a1011*x0*x2*x3+a1110*x0*x1*x2;
f = f.substitute(
    {a0201:$\alpha_{0,2,0,1}$, a0102:$\alpha_{0,1,0,2}$, a2010:$\alpha_{2,0,1,0}$, a1020:$\alpha_{1,0,2,0}$,
     a0210:$\alpha_{0,2,1,0}$, a1002:$\alpha_{1,0,0,2}$, a1101:$\alpha_{1,1,0,1}$, a0111:$\alpha_{0,1,1,1}$,
     a0120:$\alpha_{0,1,2,0}$, a2001:$\alpha_{2,0,0,1}$, a1011:$\alpha_{1,0,1,1}$, a1110:$\alpha_{1,1,1,0}$});
P3.subscheme([f]).is_smooth()
```

If the resulting cubic surface is smooth, we can compute t_4, t_5, t_6.

```
var('a0201 a0102 a2010 a1020 a0210 a1002
     a1101 a0111 a0120 a2001 a1011 a1110');
R.<t> = QQbar[];
g = a2010*t^3+(a2001+a1110)*t^2+(a0210+a1101)*t+a0201;
g = g.substitute(
    {a0201:$\alpha_{0,2,0,1}$, a0102:$\alpha_{0,1,0,2}$, a2010:$\alpha_{2,0,1,0}$, a1020:$\alpha_{1,0,2,0}$,
     a0210:$\alpha_{0,2,1,0}$, a1002:$\alpha_{1,0,0,2}$, a1101:$\alpha_{1,1,0,1}$, a0111:$\alpha_{0,1,1,1}$,
     a0120:$\alpha_{0,1,2,0}$, a2001:$\alpha_{2,0,0,1}$, a1011:$\alpha_{1,0,1,1}$, a1110:$\alpha_{1,1,1,0}$});
g.roots()
```

We label the roots of $g(t)$ as t_4, t_5, t_6. Permuting the labels for these roots only permutes the names of the remaining 24 lines on our smooth cubic surface. In this paper, our convention has been to pick t_4 to be rational if $g(t)$ has a rational root. Finally, we can compute s_1, s_2.

```
var('a0201 a0102 a2010 a1020 a0210 a1002
     a1101 a0111 a0120 a2001 a1011 a1110');
var('t4 t5 t6');
R.<t> = QQbar[];
l3 = vector([a2010*t^2+a1110*t+a0210,
             a2010*t+a0120+a1110,
             a2001*t-a1002]);
l34 = l3.substitute(t=t4);
l35 = l3.substitute(t=t5);
c1 = l34[0]; c2 = l34[1]; c3 = l34[2];
d1 = l35[0]; d2 = l35[1]; d3 = l35[2];
u1 = (t4-t5)/c1;
u2 = -(c3+c2*t5)/(c3+c2*t4);
u3 = (t4-t5)/(c3+c2*t4);
v2 = (c1/d1)*(d2*c3-d3*c2)/(c3+c2*t4);
v3 = -(c1/d1)*(d2*t4+d3)/(c3+c2*t4);
h = v2*t^2+(u1*v2-u2+v3)*t+(u1*v3-u3);
h = h.substitute(
    {a0201:$\alpha_{0,2,0,1}$, a0102:$\alpha_{0,1,0,2}$, a2010:$\alpha_{2,0,1,0}$, a1020:$\alpha_{1,0,2,0}$,
     a0210:$\alpha_{0,2,1,0}$, a1002:$\alpha_{1,0,0,2}$, a1101:$\alpha_{1,1,0,1}$, a0111:$\alpha_{0,1,1,1}$,
```

```
        a0120:α_{0,1,2,0}, a2001:α_{2,0,0,1}, a1011:α_{1,0,1,1}, a1110:α_{1,1,1,0},
        t4:t_4, t5:t_5, t6:t_6});
    h.roots()
```

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