Yoga of Motives

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Overview

1. Motives of topological spaces
   - Want intuition: what **are** motives?
2. Motives for algebraic varieties
   - Want intuition: **how** do motives behave?
3. Things we won't do today:
   - Anthology of motives: Ayoub, Beilinson, Chow, Morel, Nori, Spitzweck, Tate, Voevodsky
   - Constructing categories of motives, t-structures
   - Motivic cohomology and algebraic cycles

What are motives?

In order to express this kinship of these different cohomological theories, I formulated the notion of 'motive' associated to an algebraic variety. By this term I want to suggest that it is the 'common motive' (or 'common reason') behind this multitude of cohomological invariants attached to an algebraic variety, or indeed, behind all cohomological invariants that are a priori possible.

Grothendieck, *Recoltes et Semailles*.

- Cohomology theories give functors $H : \mathrm{Sm}_k \to$ "Ab"
- Cohomology theories behave in similar ways:
  - Expect H to factor through some intermediate category M_k
  - Also expect a functor $M : \mathrm{Sm}_k \to \mathrm{M}_k$
  - $M(X)$ is the **motive** of X
  - M sometimes called the **universal** cohomology theory
- To understand motives, want a category between Sm_k and "Ab"
  - Easier: a category between Top and "Ab"

Topological motives

- To understand motives, want a category between Sm_k and "Ab"
  - Easier: a category between Top and "Ab"
  - $\mathrm{Ab}(\mathrm{Top})$ = abelian group objects in Top, i.e. topological abelian groups
- Examples of objects in $\mathrm{Ab}(\mathrm{Top})$:
  - Discrete abelian groups
  - Tori $S^1 \times \cdots \times S^1$ and extensions of tori by discrete abelian groups
  - All CW complexes in $\mathrm{Ab}(\mathrm{Top})$ are of this form
    - n-dimensional CW complex has points with $\mathbb{R}^n$ neighborhood
    - In topological group, all points locally look the same $\implies$ manifold
    - Extra work $\implies$ compact abelian Lie group
  - Some infinite dimensional objects of $\mathrm{Ab}(\mathrm{Top})$
    - $\mathrm{Ab}(X)$ = free abelian group on set underlying X
    - $\mathrm{Ab}(X)$ inherits topology from X
    - Existence: Freyd's adjoint functor theorem gives left adjoint to forgetful functor $\mathrm{Ab}(\mathrm{Top}) \to \mathrm{Top}$ **(need X completely regular)**
    - Uniqueness: universal property (see MO: 19829)

Topological motives

- $\mathrm{Ab} : \mathrm{Top} \to \mathrm{Ab}(\mathrm{Top})$
- (X, x) pointed topological space $\implies \widetilde{\mathrm{Ab}}(X) = \mathrm{Ab}(X)/\langle x \rangle$
  - $\widetilde{\mathrm{Ab}}(S^1) \cong S^1$
  - $\widetilde{\mathrm{Ab}}(\mathbb{R}P^2) \cong \mathbb{R}P^\infty$
  - $\widetilde{\mathrm{Ab}}(S^2) \cong \mathbb{C}P^\infty$
- **Dold-Thom Theorem:** canonical isomorphisms $\pi_i(\mathrm{Ab}(X), 0) \cong H_i(X; \mathbb{Z})$ and $\pi_i \widetilde{\mathrm{Ab}}(X) \cong \widetilde{H}_i(X; \mathbb{Z})$
  - $\widetilde{\mathrm{Ab}}(S^n)$ is a $K(\mathbb{Z}, n)$
  - If $M(n, q)$ is cofiber of multiplication by q $S^n \to S^n$, then $\widetilde{\mathrm{Ab}}(M(n, q))$ is a $K(\mathbb{Z}/q, n)$
- Model structure on $\mathrm{Ab}(\mathrm{Top})$:
  - Weak equivalences/fibration iff weak equivalence/fibration in Top
  - $\mathrm{Ab} : \mathrm{Top} \to \mathrm{Ab}(\mathrm{Top})$ is a left Quillen functor
  - If $X \in \mathrm{Top}$ cofibrant and $Z \in \mathrm{Ab}(\mathrm{Top})$, get $\mathrm{Ho}(\mathrm{Top})(X, Z) \cong \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(\mathrm{Ab}(X), Z)$ and $\mathrm{Ho}(\mathrm{Top}_*)(X, Z) \cong \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(\widetilde{\mathrm{Ab}}(X), Z)$

Topological motives

- $\mathrm{Ho}(\mathrm{Top})(X, Z) \cong \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(\mathrm{Ab}(X), Z)$ and $\mathrm{Ho}(\mathrm{Top}_*)(X, Z) \cong \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(\widetilde{\mathrm{Ab}}(X), Z)$
- $\implies H^n(X; \mathbb{Z}) \cong [X, K(\mathbb{Z}, n)] \cong \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(\mathrm{Ab}(X), \widetilde{\mathrm{Ab}}(S^n))$
- $\implies H_n(X; \mathbb{Z}) \cong \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(\widetilde{\mathrm{Ab}}(S^n), \mathrm{Ab}(X))$
- **Upshot:** singular (co)homology are representable in $\mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))$
    - Also representable in $\mathrm{Ho}(\mathrm{Top})$, but not all info in $\mathrm{Ho}(\mathrm{Top})$ is seen by (co)homology
    - All info in $\mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))$ is seen by (co)homology
- Top is Quillen equivalent to sSet
- $\mathrm{Ab}(\mathrm{Top}) \simeq \mathrm{Ab}(\mathrm{sSet}) \simeq \mathrm{Ch}_{\geq 0}(\mathbb{Z})$
    - $\mathrm{Ab}(X) \simeq$ singular chain complex of X
- Complex in $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ is quasi-isomorphic to homology, considered as chain complex with zero maps
    - $\implies$ every topological abelian group decomposes as product of Eilenberg-MacLane spaces
    - $\implies \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(\mathrm{Ab}(X), \mathrm{Ab}(Y)) = \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(\mathrm{Ab}(X), \prod K(G, n)) = \prod H^n(X; G)$

Definitions and notation (for topological motives)

- $\mathrm{Ab}(X)$ = **unstable topological motive** of X
- $\widetilde{\mathrm{Ab}}(X)$ = **unstable reduced topological motive** of X
- $\mathbb{Z}[n] = \widetilde{\mathrm{Ab}}(S^n)$ = chain complex with $\mathbb{Z}$ in dimension n
- $\mathrm{Ab}(\mathrm{Top})$ = **category of unstable topological motives**
- $\mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))$ = **derived category of unstable topological motives**
- Tensor product: tensor underlying abelian groups, inherit natural topology from original topological abelian groups
  - $X \otimes \mathbb{Z}[1]$ = suspension in $\mathrm{Ab}(\mathrm{Top})$
  - $X[k] := X \otimes \mathbb{Z}[k]$
  - $H^n(X;\mathbb{Z}) \cong \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(\mathrm{Ab}(X), \mathbb{Z}[n])$ and $H_n(X;\mathbb{Z}) \cong \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(\mathbb{Z}[n], \mathrm{Ab}(X))$
- Stabilization of $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ is $\mathrm{Ch}(\mathbb{Z})$
  - Stabilization of $\mathrm{Ab}(\mathrm{Top})$ Quillen equivalent to $\mathrm{Ch}(\mathbb{Z})$
  - $\mathrm{Ab}(\mathrm{Top})$ semi-stable since $\mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(X, Y) \cong \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(X \otimes \mathbb{Z}[1], Y \otimes \mathbb{Z}[1])$

Stabilization

- Stabilization of $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ is $\mathrm{Ch}(\mathbb{Z})$
    - Stabilization of $\mathrm{Ab}(\mathrm{Top})$ Quillen equivalent to $\mathrm{Ch}(\mathbb{Z})$
    - $\mathrm{Ab}(\mathrm{Top})$ semi-stable since $\mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(X, Y) \cong \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(X[1], Y[1])$
    - **Easier to see:** $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ is semi-stable
    - To stabilize $\mathrm{Ab}(\mathrm{Top})$, just need to add formal desuspensions $X[-n]$ and colimits
    - $\mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(X, Y) \cong \mathrm{Ho}(\mathrm{Spectra}(\mathrm{Ab}(\mathrm{Top})))(X[\infty], Y[\infty])$

Lessons learned from topological motives

- Abelian cohomology theories: singular cohomology with various coefficients and shifts
  - Spectrum E over $\mathrm{Ab}(\mathrm{Top})$
  - $\mathrm{Map}(\mathrm{Ab}(X), E)$ can be modeled by chain complex
  - E-cohomology groups of X are cohomology groups of chain complexes
  - Brown representability + some work $\implies$ every abelian cohomology theory comes from spectrum over $\mathrm{Ab}(\mathrm{Top})$
- Non-abelian cohomology theories: K-theory, cobordism, etc.
- **Topological motives:** geometric objects encoding all abelian cohomology theories, but nothing more
  - Dugger: $\mathrm{Ab}(X)$ a "shadow" of X
  - $\mathrm{Spectra}(\mathrm{Ab}(\mathrm{Top}))$ captures all abelian cohomology theories
- This picture isn't surprising for topological spaces
  - Singular chain complexes $\iff$ singular cohomology
  - Quillen: $\mathrm{Ab}(\mathrm{Top}) \simeq \mathrm{Ch}$ the **universal setting** for singular cohomology

Motives for algebraic varieties

- Abelian cohomology theories: singular, ℓ-adic, de Rham, crystalline, etc. (Not algebraic K-theory, etc.)
- Want a category M_k between Sm_k and "Ab"
- Want a functor $M : \mathrm{Sm}_k \to \mathrm{M}_k$ with $M(X)$ the **motive** (Dugger: "shadow") of X
- Idea: $\mathrm{M}_k = \mathrm{Ab}(\mathrm{Sm}_k)$
    - These are just abelian varieties
    - For X a curve, $M = \mathrm{Ab}$
    - For higher dimensional varieties, motives are **not** just products of abelian varieties
- **Yoga of motives:** understand motives by how they behave (even before knowing/understanding what they are)

Yoga: first properties

- M_k should be stable, additive category
- Expect an embedding $j : \mathrm{Ab}(\mathrm{Sm}_k) \to \mathrm{M}_k$
  - Every abelian variety A **has** a motive $M(A)$ and **is** a motive $j(X)$
  - $M(A)$ and $j(A)$ are almost always different
  - $M(A)$ only depends on underlying variety
  - $j(A)$ depends on underlying variety **and** underlying group structure
- $X \in \mathrm{M}_k$ has suspension $\Sigma X = X[1]$
- $\mathbb{Z}(0) := M(\mathrm{Spec}\, k)$
  - Often, $\mathbb{Z} := \mathbb{Z}(0)$ (this motive is not just the abelian group $\mathbb{Z}$)
  - $\mathbb{Z}[i] = \mathbb{Z}(0)[i]$
- Reduced motive $\widetilde{M}(X)$: homotopy fiber of $M(X) \to M(\mathrm{Spec}\, k)$
  - Rational point $\mathrm{Spec}\, k \to X$ induces splitting
  - If X has rational point, $M(X) \simeq \widetilde{M}(X) \oplus \mathbb{Z}(0)$

Yoga: stability and the Tate object

- M_k should have tensor product that descends to homotopy category
  - Tensor unit is $\mathbb{Z}(0)$
- **Tate object:** $\mathbb{Z}(1)$
  - In topology, circle $\mathbb{A}^1 - \{0\}$ has reduction $\mathbb{Z}[1] = \Sigma\{pt\}$, also a circle
  - In algebraic geometry, reduction $\widetilde{M}(\mathbb{A}^1 - \{0\}) \cong \mathbb{Z}(1)[1]$ is suspension of the Tate object
  - $\mathbb{Z}(1) = \widetilde{M}(\mathbb{A}^1 - \{0\})[-1]$ carries info about cohomology in degree 1
- $\mathbb{Z}(q) = \mathbb{Z}(1)^{\otimes q}$ for $q \geq 0$
- Inverting the Tate object: special object $\mathbb{Z}(q)$ for $q < 0$
- $\mathbb{Z}(i) \otimes \mathbb{Z}(j) \cong \mathbb{Z}(i+j)$
- For $A \in \mathrm{M}_k$, denote $A(q) = A \otimes \mathbb{Z}(q)$
- $\mathbb{Z}(q)[n] =$ motivic Eilenberg-MacLane space
- M_k stable $\implies \mathrm{Ho}(\mathrm{M}_k)(X, Y) \cong \mathrm{Ho}(\mathrm{M}_k)(X[1], Y[1])$
- $- \otimes Z$ additive functor gives $\mathrm{Ho}(\mathrm{M}_k)(X, Y) \to \mathrm{Ho}(\mathrm{M}_k)(X \otimes Z, Y \otimes Z)$
  - This is an isomorphism if Z is tensor-invertible
  - $\mathrm{Ho}(\mathrm{M}_k)(X, Y) \cong \mathrm{Ho}(\mathrm{M}_k)(X(q), Y(q))$ for $q \geq 0$

Yoga: motivic cohomology

- $\mathrm{Ho}(\mathrm{M}_k)(X, Y) \cong \mathrm{Ho}(\mathrm{M}_k)(X[n], Y[n])$
- $\mathrm{Ho}(\mathrm{M}_k)(X, Y) \cong \mathrm{Ho}(\mathrm{M}_k)(X(q), Y(q))$
- M_k is stable in two directions
- For any $A \in \mathrm{M}_k$, get bigraded sequence of functors $X \mapsto A^{p,q}(X) = \mathrm{Ho}(\mathrm{M}_k)(M(X), A(q)[p])$
- $A^{p,q}$ = bigraded A-cohomology
- **Motivic cohomology:** $H^{p,q}(X;\mathbb{Z}) = H^p(X;\mathbb{Z}(q)) = \mathrm{Ho}(\mathrm{M}_k)(M(X), \mathbb{Z}(q)[p])$
- **Motivic homology:** $H_{p,q}(X;\mathbb{Z}) = H_p(X;\mathbb{Z}(q)) = \mathrm{Ho}(\mathrm{M}_k)(\mathbb{Z}(q)[p], M(X))$
- $\pi_{p,q}(X) = \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(\mathbb{Z}(q)[p], X)$

Yoga: homotopy

- **Motivic cohomology:**
  $H^{p,q}(X;\mathbb{Z}) = H^p(X;\mathbb{Z}(q)) = \mathrm{Ho}(\mathrm{M}_k)(M(X), \mathbb{Z}(q)[p])$
- **Motivic homology:**
  $H_{p,q}(X;\mathbb{Z}) = H_p(X;\mathbb{Z}(q)) = \mathrm{Ho}(\mathrm{M}_k)(\mathbb{Z}(q)[p], M(X))$
- $\pi_{p,q}(X) = \mathrm{Ho}(\mathrm{Ab}(\mathrm{Top}))(\mathbb{Z}(q)[p], X)$
- Projection $X \times \mathbb{A}^1 \to X$ should induce weak equivalence $M(X \times \mathbb{A}^1) \simeq M(X)$
- Zariski open cover $U \cup V \supseteq X$ should induce homotopy cofiber sequence $M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1]$
  - Applying $\mathrm{Ho}(\mathrm{M}_k)(-, A(q)[*])$ will induce long exact Mayer-Vietoris sequence for $A^{*,q}(-)$
  - $\implies$ Mayer-Vietoris for motivic cohomology

Yoga: splittings

- Rank n algebraic vector bundle $E \to X$ gives $M(\mathbb{P}(E)) \simeq \bigoplus_{i=0}^{n}(M \otimes \mathbb{Z}(i)[2i])$
- If X smooth projective curve, $M(X) \simeq \mathbb{Z}(0) \oplus j(\mathrm{Jac}(X)) \oplus \mathbb{Z}(1)[2]$
  - Topologically, if X compact Riemann surface of genus g, $\mathrm{Ab}(X) \simeq$ product of Eilenberg-MacLane spaces
  - Dold-Thom: $\mathrm{Ab}(X) \simeq \mathbb{Z} \times (\prod_{i=1}^{2g} K(\mathbb{Z},1)) \times K(\mathbb{Z},2)$
  - $\mathrm{Jac}(X) = 2g$-dimensional torus, i.e. $\prod_{i=1}^{2g} K(\mathbb{Z},1)$
- If X smooth projective of dimension d, $M(X) \simeq \mathbb{Z}(0) \oplus e(X) \oplus \mathbb{Z}(d)[2d]$
- Grothendieck: expect splitting $M(X) \simeq h_0(X) \oplus h_1(X) \oplus \cdots \oplus h_d(X)$
  - $h_i(X) =$ "motive of X related to i-dimensional cohomology"
  - Get such splittings in $\mathrm{Ab}(\mathrm{Top})$
  - If Künneth components of diagonal $\Delta \in H^*_{\text{ét}}(X \times X; \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell$ are algebraic, get this splitting

What don't we know?

- Topological motives split as products of Eilenberg-MacLane spaces
  - For maps of motives, suffices to look at maps of singular cohomology
- Algebro-geometric motives don't seem to have such simple decompositions
  - Motivic (co)homology is not enough to understand maps of motives
  - Example: Friedlander-Voevodsky's **bivariant cycle cohomology** $\mathrm{Ho}(\mathrm{M}_k)(M(X), M(Y)(q)[p])$
- Grothendieck wanted M_k as an abelian category
  - Beilinson: try derived category of motives (Voevodsky)
  - Construction of abelian category M_k related to conjectures on algebraic cycles
- Can think about homological or cohomological motives
  - Dugger's discussion uses homological motives (clearer for $\mathrm{Ab}(\mathrm{Top})$)
  - Algebraic geometers use cohomological motives
- Everything done rationally
  - Hom sets are rational vector spaces
  - Rationally $\implies$ can use Künneth isomorphism to approach ad hoc constructions of motives

Questions?