Preliminary Exam

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Overview

- Study self-duality of complete intersections
- Scheja-Storch: construct explicit isomorphism Θ for self-duality
- **Goal:** describe Scheja-Storch's isomorphism
- **Context:** A a commutative ring, B a finitely generated projective A-algebra

Notation

- A a commutative ring, B a finitely generated projective A-algebra
- $\chi : B \otimes_A B \to \mathrm{Hom}_A(\mathrm{Hom}_A(B, A), B)$ defined by $\chi(b \otimes c) = (\varphi \mapsto \varphi(b)c)$
- Presentation of B as a complete intersection over A:
    - $A[x]$ denotes either $A[x_1, ..., x_n]$ or $A[[x_1, ..., x_n]]$
    - $\rho : A[x] \to B$ with $\ker \rho = (t_1, ..., t_n)$
- Kähler differentials $dt_j = t_j \otimes 1 - 1 \otimes t_j = \sum_{i=1}^{n} a_{ij}(x_i \otimes 1 - 1 \otimes x_i)$
    - $\mu_{A[x]} : A[x] \otimes_A A[x] \to A[x]$ given by $b \otimes c \mapsto bc$
    - $\ker \mu_{A[x]} = (x_i \otimes 1 - 1 \otimes x_i : 1 \leq i \leq n)$
    - $t_j \otimes 1 - 1 \otimes t_j \in \ker \mu_{A[x]}$
- $\Delta = \rho \otimes \rho(\det(a_{ij})) \in B \otimes_A B$
    - **Remark:** $\mu_B(\Delta) = \rho(\det(\partial t_j / \partial x_i)) := J$

Main Theorem

- Scheja-Storch: construct explicit isomorphism between B and B^*

Theorem

The homomorphism $\Theta := \chi(\Delta) : \mathrm{Hom}_A(B, A) \to B$ is a B-linear isomorphism.

- Proof uses the following lemmas:
  - **Lemma 1:** χ induces a B-linear isomorphism $\mathrm{Ann}_{B \otimes_A B}(\ker \mu_B) \to \mathrm{Hom}_B(\mathrm{Hom}_A(B, A), B)$.
  - **Lemma 2:** $(B \otimes_A B) \cdot \Delta = \mathrm{Ann}_{B \otimes_A B}(\ker \mu_B)$ and $\ker \mu_B = \mathrm{Ann}_{B \otimes_A B}((B \otimes_A B) \cdot \Delta)$.
  - **Lemma 3:** $\mathrm{Hom}_A(B, A)$ is a projective A-module.
  - **Lemma 4:** $\mathrm{Hom}_A(B, A)$ is a free B-module.

Proof of Main Theorem

Theorem: $\Theta := \chi(\Delta) : \mathrm{Hom}_A(B, A) \to B$ is a B-linear isomorphism.

- $(B \otimes_A B) \cdot \Delta = ((B \otimes_A B)/\operatorname{Ann}_{B\otimes_A B} \Delta) \cdot \Delta$
    - **Lemma 2** $\implies$ $\operatorname{Ann}_{B\otimes_A B} \Delta = \ker \mu_B$
    - $(B \otimes_A B)/\operatorname{Ann}_{B\otimes_A B} \Delta = (B \otimes_A B)/\ker \mu_B$
    - **First iso. theorem for modules** $\implies$ $(B \otimes_A B)/\ker \mu_B \cong B$
    - Thus $(B \otimes_A B) \cdot \Delta \cong B \cdot \Delta$
- **Lemma 2** $\implies$ $\operatorname{Ann}_{B\otimes_A B} \ker \mu_B = (B \otimes_A B) \cdot \Delta = B \cdot \Delta$
    - $\operatorname{Ann}_{B\otimes_A B} \ker \mu_B$ is a free B-module with basis Δ
    - **Lemma 1** $\implies$ $\chi(\operatorname{Ann}_{B\otimes_A B} \ker \mu_B)$ is a free B-module with basis $\chi(\Delta)$
    - **Lemma 1** $\implies$ $\mathrm{Hom}_B(\mathrm{Hom}_A(B, A), B)$ is a free B-module with basis $\chi(\Delta)$
- **Lemmas 3,4** $\implies$ $\mathrm{Hom}_A(B, A)$ is a free B-module of rank 1
    - $\chi(\Delta) : \mathrm{Hom}_A(B, A) \to B$ is B-linear homomorphism of rank 1 free B-modules
    - Any $\mathrm{Hom}_A(B, A) \to B$ is a B-multiple of $\chi(\Delta)$, so $\chi(\Delta)$ is an isomorphism

Proof of Lemma 1

Lemma 1: χ induces $\mathrm{Ann}_{B\otimes_A B} \ker \mu_B \cong \mathrm{Hom}_B(\mathrm{Hom}_A(B, A), B)$.

- $\chi : B \otimes_A B \to \mathrm{Hom}_A(\mathrm{Hom}_A(B, A), B)$ is a bijection
  - Look at two B-module structures compatible with χ
- On $B \otimes_A B$, let $a \cdot (b \otimes c) = ab \otimes c$ and $a * (b \otimes c) = b \otimes ac$
  - $\ker \mu_B = (a \otimes 1 - 1 \otimes a : a \in B)$
  - $\mathrm{Ann}_{B\otimes_A B} \ker \mu_B = \{x \in B \otimes_A B : (a \otimes 1 - 1 \otimes a)x = 0 \ \forall a \in B\}$
  - $\mathrm{Ann}_{B\otimes_A B} \ker \mu_B = \{x \in B \otimes_A B : a \cdot x = a * x \ \forall a \in B\}$
  - $\implies$ $\mathrm{Ann}_{B\otimes_A B} \ker \mu_B$ is the largest submodule where $\cdot$ and $*$ agree
- On $\mathrm{Hom}_A(\mathrm{Hom}_A(B, A), B)$, let $a \cdot \varphi = (\psi \mapsto \varphi(a\psi))$ and $a * \varphi = (\psi \mapsto a\varphi(\psi))$
  - $a \cdot \varphi = a * \varphi$ for all $a \in B$ iff φ is B-linear
  - $\implies$ $\mathrm{Hom}_B(\mathrm{Hom}_A(B, A), B)$ is the largest submodule where $\cdot$ and $*$ agree
- $\chi(a \cdot (b \otimes c)) = a \cdot \chi(b \otimes c)$ and $\chi(a * (b \otimes c)) = a * \chi(b \otimes c)$

Proof of Lemma 2

Lemma 2: $(B \otimes_A B) \cdot \Delta = \mathrm{Ann}_{B \otimes_A B} \ker \mu_B$ and $\ker \mu_B = \mathrm{Ann}_{B \otimes_A B}((B \otimes_A B) \cdot \Delta)$

- Reduce to the following lemma:

Lemma 5: Let A be a commutative ring. Assume:

- Ideals $(g_1, ..., g_n) = \mathfrak{b} \subseteq \mathfrak{a} = (f_1, ..., f_n)$
- $g_j = \sum_{i=1}^n a_{ij} f_i$ and $\Delta = \det(a_{ij})$
- $\mathfrak{a}' = \mathfrak{a}/\mathfrak{b}$ in $A' = A/\mathfrak{b}$ and Δ' image of Δ in A'
- $f_1, ..., f_n$ and $g_1, ..., g_n$ are prime sequences in $A_\mathfrak{p}$ for all $\mathfrak{p} \supseteq \mathfrak{a}$

Then:

(a) Δ' is independent of choice of a_{ij}

(b) $\Delta' A' = \mathrm{Fitt}_{A'}(\mathfrak{a}')$

(c) $\Delta' A' = \mathrm{Ann}_{A'}(\mathfrak{a}')$ and $\mathfrak{a}' = \mathrm{Ann}_{A'}(\Delta' A')$

Proof of Lemma 2

Lemma 2: $(B \otimes_A B) \cdot \Delta = \mathrm{Ann}_{B\otimes_A B} \ker \mu_B$ and $\ker \mu_B = \mathrm{Ann}_{B\otimes_A B}((B \otimes_A B) \cdot \Delta)$

$$
\begin{array}{ccc}
A[x] \otimes_A A[x] & \xrightarrow{\mu_{A[x]}} & A[x] \\
\downarrow{\scriptstyle \rho \otimes \mathrm{id}} & & \\
B \otimes_A A[x] & & \downarrow{\scriptstyle \rho} \\
\downarrow{\scriptstyle \mathrm{id} \otimes \rho} & & \\
B \otimes_A B & \xrightarrow{\mu_B} & B
\end{array}
$$

- For $A[x] = A[[x_1, ..., x_n]]$, consider $\mathrm{id} \otimes \rho : B \otimes_A A[x] \to B \otimes_A B$
  - $\ker(\mathrm{id} \otimes \rho) = (-1 \otimes t_1, ..., -1 \otimes t_n)$ plays the role of $\mathfrak{b}$
  - $(\rho(x_1) \otimes 1 - 1 \otimes x_1, ..., \rho(x_n) \otimes 1 - 1 \otimes x_n)$ plays the role of $\mathfrak{a}$
- For $A[x] = A[x_1, ..., x_n]$, reduce to the local case in two steps:
  - Locally in B, i.e. over $B_{\mathfrak{n}} \otimes_A B_{\mathfrak{n}}$
  - Locally in $B \otimes_A B$, i.e. over $(B \otimes_A B)_{\mathfrak{N}}$
  - Clear denominators to go from locally in B to locally in $B \otimes_A B$

Proof of Lemmas 3 and 4

Lemma 3: $\mathrm{Hom}_A(B, A)$ is a projective A-module.

- B finitely generated $\implies \pi : A^m \twoheadrightarrow B$
  - $0 \to \ker\pi \to A^m \to B \to 0$ and B projective $\implies B \oplus \ker\pi \cong A^m$
  - $A^m \cong \mathrm{Hom}_A(A^m, A) \cong \mathrm{Hom}_A(B \oplus \ker\pi, A) \cong \mathrm{Hom}_A(B, A) \oplus \mathrm{Hom}_A(\ker\pi, A)$

Lemma 4: $\mathrm{Hom}_A(B, A)$ is a free B-module.

- If M finitely generated B-module, M projective A-module, and $\mathrm{Hom}_B(M, B)$ projective (free) B-module, then M is a projective (free) B-module.

Proof of Lemma 5

Lemma 5: Let A be a commutative ring. Assume:

- Ideals $(g_1, \ldots, g_n) = \mathfrak{b} \subseteq \mathfrak{a} = (f_1, \ldots, f_n)$
- $g_j = \sum_{i=1}^{n} a_{ij} f_i$ and $\Delta = \det(a_{ij})$
- $\mathfrak{a}' = \mathfrak{a}/\mathfrak{b}$ in $A/\mathfrak{b}$ and Δ' image of Δ in A'
- $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are prime sequences in $A_{\mathfrak{p}}$ for all $\mathfrak{p} \supseteq \mathfrak{a}$

Then:

(a) Δ' is independent of choice of a_{ij}
(b) $\Delta' A' = \mathrm{Fitt}_{A'}(\mathfrak{a}')$
(c) $\Delta' A' = \mathrm{Ann}_{A'}(\mathfrak{a}')$ and $\mathfrak{a}' = \mathrm{Ann}_{A'}(\Delta' A')$

- A *prime sequence* is a regular sequence $f_1, \ldots, f_n$ such that $(f_1, \ldots, f_i)$ is a prime ideal for each $1 \leq i \leq n$
- The *Fitting ideal* $\mathrm{Fitt}_{A'}(\mathfrak{a}')$ is generated by the determinants of all $n \times n$ minors $\det(b_{ij})$ of the collection of syzygies $\sum_{i=1}^{n} b_{ij} f_i = 0$, where $f_1, \ldots, f_n$ generate $\mathfrak{a}'$ over A'

Proof of Lemma 5

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- $\mathfrak{a}' = \mathfrak{a}/\mathfrak{b}$ in $A/\mathfrak{b}$ and Δ' image of Δ in A'
- $f_1, ..., f_n$ and $g_1, ..., g_n$ are prime sequences in $A_\mathfrak{p}$ for all $\mathfrak{p} \supseteq \mathfrak{a}$

Then:

(a) Δ' is independent of choice of a_{ij}
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- (a), (b), (c) are local in A'
- If $\mathfrak{a}' = A'$, then $\mathfrak{a}'$ is one-dimensional over A'
  - Any $n \times n$ minor contains linearly dependent rows, so $\Delta' = 0$
- May assume A' is local, $\{f_i\}$ and $\{g_i\}$ regular sequences contained in unique max ideal

Proof of Lemma 5

(a) Δ' is independent of choice of a_{ij}.

- Suppose $g_j = \sum_{i=1}^{n} a_{ij} f_i = \sum_{i=1}^{n} b_{ij} f_i$
  - Need to show $\det(a_{ij}) - \det(b_{ij}) \in \mathfrak{b}$
  - WLOG, assume first $n-1$ rows of (a_{ij}) and (b_{ij}) agree
  - Set $c_{ij} = a_{ij} = b_{ij}$ for $1 \leq j \leq n-1$ and $c_{in} = a_{in} - b_{in}$
  - $\det(c_{ij}) = \det(a_{ij}) - \det(b_{ij})$
- Cramer's rule $\implies$
  $\det(c_{ij}) \cdot f_i = \det(\text{replace } i^{\text{th}} \text{ column of } (c_{ij}) \text{ with } (g_1, \ldots, g_{n-1}, 0)^T)$
  - $\implies \det(c_{ij}) \cdot \mathfrak{a} \subseteq (g_1, \ldots, g_{n-1})$
  - $g_1, \ldots, g_n$ prime sequence $\implies (g_1, \ldots, g_{n-1})$ a prime ideal
  - $g_1, \ldots, g_n$ a regular sequence in $\mathfrak{a} \implies \mathfrak{a} \not\subseteq (g_1, \ldots, g_{n-1})$
  - $\implies \det(c_{ij}) \in (g_1, \ldots, g_{n-1}) \subseteq \mathfrak{b}$

Proof of Lemma 5

(b) $\Delta' A' = \mathrm{Fitt}_{A'}(\mathfrak{a}')$.

- $\mathrm{Fitt}_{A'}(\mathfrak{a}')$ is the image of $\mathrm{Fitt}_A(\mathfrak{a})$ in A'
- $\mathfrak{a}' \cong A^n/U$
  - U is generated by $(a_{1j}, ..., a_{nj})$ and $(0, ..., f_q, ..., -f_p, ...0)$
  - One of these minors is Δ
  - Using Cramer's rule as in (a), all other minors are in $\mathfrak{b}$
- Fitting ideal does not depend on choice of generators or relations

Proof of Lemma 5

(c) $\Delta' A' = \mathrm{Ann}_{A'}(\mathfrak{a}')$ and $\mathfrak{a}' = \mathrm{Ann}_{A'}(\Delta' A')$.

- Induct on n
- $n = 0$:
  - $\Delta' = \det(\emptyset) = \prod_{\emptyset} \lambda = 1$ and $\mathfrak{a}' = 0$
  - Trivially, $\mathrm{Ann}_{A'}(0) = A'$ and $0 = \mathrm{Ann}_{A'}(A')$
- $n \geq 1$:
  - (b) $\implies$ $\Delta' A'$ is a Fitting ideal
  - $\implies$ $\Delta' A'$ does not depend on choice of $f_1, \ldots, f_n$
  - Pick $f_1, \ldots, f_n$ so that f_1 is prime to $(g_2, \ldots, g_n)$
  - Set $B = A/(g_2, \ldots, g_n)$, $b_{ij} = a_{ij}$ for $j \geq 2$, $b_{11} = 1$ and $b_{i1} = 0$ for $i \geq 2$, $\Delta_b = \det(b_{ij})$
  - Inductive hypothesis on $f_2, \ldots, f_n$ and $g_2, \ldots, g_n$ in $A/f_1 A$ $\implies$:
    (i) $(B/f_1 B) \cdot \Delta_b = \mathrm{Ann}_{B/f_1 B}(\mathfrak{a}B/f_1 B) = (f_1 B : \mathfrak{a}B)/f_1 B$
    (ii) $\mathfrak{a}B/f_1 B = \mathrm{Ann}_{B/f_1 B}((B/f_1 B) \cdot \Delta_b) = (f_1 B : \Delta_b B)/f_1 B$
  - (ii) $\implies$ $\mathfrak{a}B = (f_1 B : \Delta_b B)$
  - Cramer's rule $\implies$ $\Delta f_1, \Delta_b g_1 \in \mathfrak{b}$ $\implies$ $\Delta_b g_1 \equiv \Delta f_1 \mod \mathfrak{b}$

Proof of Lemma 5

(c) $\Delta' A' = \mathrm{Ann}_{A'}(\mathfrak{a}')$ and $\mathfrak{a}' = \mathrm{Ann}_{A'}(\Delta' A')$.

- Inductive hypothesis on $f_2, ..., f_n$ and $g_2, ..., g_n$ in $A/f_1A \implies$:
  (i) $(B/f_1B) \cdot \Delta_b = \mathrm{Ann}_{B/f_1B}(\mathfrak{a}B/f_1B) = (f_1B : \mathfrak{a}B)/f_1B$
  (ii) $\mathfrak{a}B/f_1B = \mathrm{Ann}_{B/f_1B}((B/f_1B) \cdot \Delta_b) = (f_1B : \Delta_b B)/f_1B$
- (ii) $\implies \mathfrak{a}B = (f_1B : \Delta_b B)$
- Cramer's rule $\implies \Delta f_1, \Delta_b g_1 \in \mathfrak{b} \implies \Delta_b g_1 \equiv \Delta f_1 \mod \mathfrak{b}$
  - $\implies (g_1B : \Delta B) = (f_1B : \Delta_b B) \implies (g_1B : \Delta B) = \mathfrak{a}B$
  - $\implies \mathfrak{a}B/g_1B = (g_1B : \Delta B)/g_1B = \mathrm{Ann}_{B/g_1B}((B/g_1B)\Delta)$
- Canonical isomorphism $(f_1B : \mathfrak{a}B)/f_1B \cong (g_1B : \mathfrak{a}B)/g_1B$ sends image of Δ_b in B/f_1B to image of Δ in B/g_1B
  - (i) $\implies (B/g_1B)\Delta = (g_1B : \mathfrak{a}B)/g_1B = \mathrm{Ann}_{B/g_1B}(\mathfrak{a}B/g_1B)$
- Conclude by noting $A' = B/g_1B$, $\mathfrak{a}' = \mathfrak{a}B/g_1B$, $\Delta' A' = \Delta(B/g_1B)$

Corollaries

- Main Theorem $\implies$ $\eta := \Theta^{-1}(1)$ generates $\mathrm{Hom}_A(B, A)$
- $\Theta(\mathrm{Tr}_{B/A}) = J = \mu(\Delta)$ and $\mathrm{Tr}_{B/A} = J \cdot \eta$
- **Corollary:** Let A be reduced with $(x_1, ..., x_n)^m \subseteq (t_1, ..., t_n) \subseteq (x_1, ..., x_n)$ for some m.
  - Write $t_j = \sum_{i=1}^{n} e_{ij} x_i$ and set $E \equiv \det(e_{ij}) \mod (t_1, ..., t_n)$. Then:
  - $J = (\dim_A B) \cdot E$
  - Get bilinear form $\beta_\eta : B \times B \to A$ defined by $\beta_\eta(x, y) = \eta(xy)$
  - Eisenbud-Levine: if $\phi : B \to A$ has $\phi(E) = 1$, then $\beta_\phi \cong \beta_\eta$
  - Kass-Wickelgren: β_η is the local $\mathbb{A}^1$-degree

Questions?