Elliptic Cohomology

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Overview

1. Classical elliptic cohomology
2. E_∞-ring spectra and derived algebraic geometry
3. Oriented group schemes
4. Classifying oriented elliptic curves
5. Applications

Cohomology theories

Definition

A **cohomology theory** is a collection $\{A^n\}_{n\in\mathbb{Z}}$ of contravariant functors from the category of pairs of topological spaces $(Y \subseteq X)$ to abelian groups such that:

- Weak homotopy equivalences induce isomorphisms on cohomology.
- For every triple $Z \subseteq Y \subseteq X$, there is a long exact sequence

$$... \to A^n(X, Y) \to A^n(X, Z) \to A^n(Y, Z) \to A^{n+1}(X, Y) \to ...$$

- If U is open with $\overline{U} \subseteq Y$, then $A^n(X, Y) \cong A^n(X - U, Y - U)$.
- $A^n(\bigsqcup X_\alpha) \cong \prod A^n(X_\alpha)$.

Multiplicative cohomology theories

Definition

A **multiplicative cohomology theory** is a cohomology theory A^* with a graded commutative ring structure.

- $uv = (-1)^{mn} vu$ for $u \in A^m(X), v \in A^n(X)$.

Example

K-theory is a multiplicative cohomology theory.

- $K^{2n+1}(*) = 0$ for all n.
- $K^*(*) \cong \mathbb{Z}[\beta, \beta^{-1}]$, where $\beta \in K^{-2}(*)$ is the **Bott element**.

Even and periodic cohomology theories

Generalizing these properties of K-theory:

Definition

Let A be a multiplicative cohomology theory.

- We say that A is **even** if $A^{2n+1}(*) = 0$ for all n.
- We say that A is **periodic** if there exists $\beta \in A^{-2}(*)$ with $\beta^{-1} \in A^2(*)$.

Example

Periodic cohomology $A^n(X) = \prod_{k\in\mathbb{Z}} H^{n+2k}(X; R)$ is even and periodic.

Chern classes

If A is even periodic, then $A(\mathbb{C}P^\infty) \cong A(*)[[t]]$.

- t corresponds to $c_1(\mathcal{O}(1))$.
- For any line bundle $\mathcal{L} \to X$, we have $\mathcal{L} \cong \phi^*\mathcal{O}(1)$ for $\phi : X \to \mathbb{C}P^\infty$.
- May thus define $c_1(\mathcal{L}) = \phi^* t \in A(X)$.

Question

What does $c_1(\mathcal{L} \otimes \mathcal{L}')$ look like?

- In periodic cohomology, $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$.
- In K-theory, $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}') + c_1(\mathcal{L})c_1(\mathcal{L}')$.

Chern classes

Question

What does $c_1(\mathcal{L} \otimes \mathcal{L}')$ look like?

$A(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong A(*)[[t_1, t_2]]$

- $t_i = c_1(\pi_i^* \mathcal{O}(1))$, where $\pi_i : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is projection.
- $c_1(\pi_1^* \mathcal{O}(1) \otimes \pi_2^* \mathcal{O}(1)) = f(t_1, t_2) \in A(*)[t_1, t_2]$.
- If X admits a finite cell decomposition, then $c_1(\mathcal{L})$ is nilpotent and $c_1(\mathcal{L} \otimes \mathcal{L}') = f(c_1(\mathcal{L}), c_1(\mathcal{L}'))$ makes sense.

$f(u, v)$ should satisfy the following:

- $f(t, 0) = f(0, t) = t$ (since $c_1(\text{trivial bundle}) = 0$).
- $f(u, v) = f(v, u)$ (since $\mathcal{L} \otimes \mathcal{L}' \cong \mathcal{L}' \otimes \mathcal{L}$).
- $f(u, f(v, w)) = f(f(u, v), w)$ (since $\otimes$ of line bundles is associative up to isomorphism).

Formal group laws

$f(u, v)$ should satisfy the following:

- $f(t, 0) = f(0, t) = t$ (since $c_1(\text{trivial bundle}) = 0$).
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- $f(u, f(v, w)) = f(f(u, v), w)$ (since $\otimes$ of line bundles is associative up to isomorphism).

Definition

Such a power series is called a **commutative, one-dimensional formal group law** over $A(*)$.

- Such formal group laws give a group structure to the formal scheme $\operatorname{Spf} A(*)[[t]] = \operatorname{Spf} A(\mathbb{C}P^\infty)$.
- The formal group law f depends on t, but the formal group $(\operatorname{Spf} A(\mathbb{C}P^\infty), \times)$ does not.

Cohomology theories from formal group laws

Example

- $f(u, v) = u + v$ defines the *formal additive group* $\hat{\mathbb{G}}_a$.
- $f(u, v) = u + v + uv$ defines the *formal multiplicative group* $\hat{\mathbb{G}}_m$.

Question

Can we classify all cohomology theories coming from a commutative, one-dimensional formal group law?

- We will use **periodic complex cobordism** MP.
- $MP = \bigvee_{n \in \mathbb{Z}} \Sigma^{2n} MU$.
- Landweber's exact functor theorem will be a key tool.

Cohomology theories from formal group laws

Question

Can we classify all cohomology theories coming from a commutative, one-dimensional formal group law?

MP is "universal" for first Chern classes and commutative, one-dimensional formal group laws:

- There is a **canonical** isomorphism $MP(\mathbb{C}P^\infty) \cong MP(*)[[t]]$.
- Quillen: $MP(*)$ is the Lazard ring, which is the universal coefficient ring for commutative, one-dimensional formal group laws.
- Bijection from $\mathrm{Hom}(MP(*), R)$ to set of commutative, one-dimensional formal group laws.

Cohomology theories from formal group laws

Question

Can we classify all cohomology theories coming from a commutative, one-dimensional formal group law?

- R a commutative ring.
- $f(u, v) \in R[[u, v]]$ a formal group law.
- $\mathbb{G} = \operatorname{Spf} A(\mathbb{C}P^\infty)$ a formal group (over R) determined by f.

$(\mathbb{G}, R, f)$ is uniquely determined by $MP(*) \to R$. For finite complexes X, our candidate for A is

$$A^n_{\mathbb{G}}(X) = MP^n(X) \otimes_{MP(*)} R.$$

When is $A_{\mathbb{G}}$ a cohomology theory?

Landweber exactness and algebraic stacks

$$A^n_{\mathbb{G}}(X) = MP^n(X) \otimes_{MP(*)} R.$$

- Landweber exact functor theorem: $A_{\mathbb{G}}$ is a cohomology theory if $(\mathbb{G}, R, f)$ satisfies certain properties.
- $A_{\mathbb{G}}$ satisfies Landweber if R is flat over $MP(*)$.
- More general situation can be phrased in terms of **algebraic stacks**.
  - $\mathcal{M}_{fgl} = \operatorname{Spec} MP(*)$ the moduli stack of formal group laws.
  - $\operatorname{Hom}(\operatorname{Spec} R, \mathcal{M}_{fgl})$ is the set of commutative, one-dimensional formal group laws in $R[[u, v]]$.
  - $G = (\{\sum_{i=1}^{\infty} a_i x^i \mid a_i \in R,\ a_1 \in R^\times\}, \circ)$.
  - G acts on $\mathcal{M}_{fgl}$ by $f^g(u, v) = g^{-1} f(g(u), g(v))$.
  - $\mathcal{M}_{fg} = \mathcal{M}_{fgl}/G$.

Landweber exactness and algebraic stacks

For X finite, $MP^n(X)$ are quasi-coherent sheaves with G-action on $\mathcal{M}_{fgl}$.

- Interpret $MP^n(X)$ as quasi-coherent sheaves $\mathcal{MP}^n(X)$ on $\mathcal{M}_{fg}$.
- $(\mathbb{G}, R, f)$ is classified by $\phi : \operatorname{Spec} R \to \mathcal{M}_{fg}$.
- $A^n(X) = \phi^* \mathcal{MP}^n(X)$ is a cohomology theory when ϕ is flat.
- $A^n(X)$ coincides with $A^n_{\mathbb{G}}(X) = MP^n(X) \otimes_{MP(*)} R$.

Elliptic cohomology

$\hat{\mathbb{G}}_a, \hat{\mathbb{G}}_m$ are the formal completions of the algebraic groups $\mathbb{G}_a, \mathbb{G}_m$.

- Interested in commutative, one-dimensional formal group laws.
- Every commutative, one-dimensional algebraic group is isomorphic to $\mathbb{G}_a$, $\mathbb{G}_m$, or an elliptic curve.
    - $\mathbb{G}_a$ gives rise to (periodic) ordinary cohomology.
    - $\mathbb{G}_m$ gives rise to K-theory.
    - Elliptic curves give rise to **elliptic cohomology**.

Definition

An **elliptic cohomology theory** consists of:

(a) A commutative ring R.

(b) An elliptic curve E over R.

(c) An even, **weakly** periodic multiplicative cohomology theory A.

- $A^2(*)$ a free module over $A(*)$.

(d) Isomorphisms $A(*) \cong R$ and $\hat{E} \cong \operatorname{Spf} A(\mathbb{C}P^\infty)$.

Elliptic cohomology and $\mathcal{M}_{1,1}$

If $\hat{E}$ is Landweber-exact (as a formal group), then (c) and (d) are uniquely determined.

- **Upshot:** In the Landweber-exact case, an elliptic curve gives an elliptic cohomology theory.
- **Problem:** There is no universal elliptic curve over a ring.
    - Moduli stack $\mathcal{M}_{1,1}$ of elliptic curves is not a scheme.
    - $\mathrm{Hom}(\mathrm{Spec}\, R, \mathcal{M}_{1,1}) = \{E \to \mathrm{Spec}\, R\}$.
- **Solution:** E_∞-rings.
    - $\mathcal{M}_{1,1}$ is Deligne-Mumford $\Rightarrow$ have étale maps $\phi : \mathrm{Spec}\, R \to \mathcal{M}_{1,1}$.
    - Étale $\phi \Rightarrow$ elliptic curve $E_\phi \Rightarrow$ Landweber-exact $\hat{E}_\phi \Rightarrow A_\phi$.
    - $\mathcal{O} : [\phi : \mathrm{Spec}\, R \to \mathcal{M}_{1,1}] \mapsto A_\phi$ is a presheaf on $\mathcal{M}_{1,1}$.
    - $\mathcal{O}$ valued in cohomology theories, so $\Gamma(\mathcal{M}_{1,1}, \mathcal{O})$ doesn't make sense.
    - Try representing $\{A_\phi\}$ with spectra.

Elliptic cohomology and E_∞-rings.

$$\begin{array}{ccc} & & \{\text{Spectra}\} \\ & \nearrow & \downarrow \\ \{\phi : \operatorname{Spec} Rto\mathcal{M}_{1,1}\} & \longrightarrow & \{\text{Cohomology theories}\} \end{array}$$

- Try supplying dotted line by obstruction theory – this is hard.
- Elliptic cohomology is multiplicative; multiplicative cohomology theories are represented by E_∞-ring spectra.
- Brown representability gives a spectrum for A_ϕ, but not necessarily an E_∞-structure for multiplicative structure.
- E_∞-rings are more rigid, so obstruction theoretic approach has a better chance of working.

tmf

Theorem (Goerss-Hopkins-Miller)

There exists a commutative diagram

$$\begin{array}{ccc} & \overset{\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}}{\nearrow} & \{E_\infty\text{-rings}\} \\ & & \downarrow \\ \{\phi : \operatorname{Spec} R \to \mathcal{M}_{1,1}\} & \longrightarrow & \{\text{Multiplicative cohomology theories}\}, \end{array}$$

and $\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}$ is uniquely determined up to homotopy.

- Global sections: $\mathrm{tmf}[\Delta^{-1}]$ the homotopy limit of $\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}$.
- tmf comes from working over $\overline{\mathcal{M}}_{1,1}$.
- Since $\mathcal{M}_{1,1}$ not affine, passing from $\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}$ to $\mathrm{tmf}[\Delta^{-1}]$ loses information.

Derived algebraic geometry

Want to extend cohomology theory A to G-equivariant cohomology.

Example

G-equivariant K-theory should be the Grothendieck group $K_G(X)$ of G-equivariant vector bundles on X.

- $A_G^{\mathrm{Bor}}(X) = A((X \times EG)/G)$
  - Only contains the same information as A.
  - Does not give the correct G-equivariant K-theory.
- Finding equivariant versions of A boils down to realizing $\hat{\mathbb{G}} = \operatorname{Spf} A(\mathbb{C}P^\infty)$ as the formal completion of some $\mathbb{G}$.
  - Given $\mathbb{G}$ with $\hat{\mathbb{G}} = \operatorname{Spf} A(\mathbb{C}P^\infty)$, define $A_{S^1}(*) = \Gamma(\mathbb{G}, \mathcal{O}_{\mathbb{G}})$.
  - **Issue:** Knowing $A_G(*)$ is not enough.
  - Need to get cohomology **theories** from $\mathbb{G}$, not just cohomology rings.
  - **Idea:** $\mathcal{O}_{\mathbb{G}}$ valued in E_∞-rings.

E_∞-rings

- E_∞: homotopy **everything** (i.e. associative and commutative).
- E_∞-rings are to commutative rings as commutative rings are to reduced rings.
- $\pi_0 A$ is a commutative ring, and $\pi_n A$ is a $\pi_0 A$-module.

Definition

An A-module M is **flat** if

(a) $\pi_0 M$ is flat over $\pi_0 A$;

(b) $\pi_n M \otimes_{\pi_0 A} \pi_0 M \to \pi_n M$ is an isomorphism.

$A \to B$ is flat if B is flat over A.

Derived schemes

Definition

A **scheme** is a pair $(X, \mathcal{O}_X)$ that is locally isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

- For derived schemes, want $\mathcal{O}_X$ to be valued in E_∞ rings.
- What should $\operatorname{Spec} A$ be?
  - $\operatorname{Spec} A = \operatorname{Spec} \pi_0 A$.
  - Basic opens $U_f = \{\mathfrak{p} : f \notin \mathfrak{p}\}$.
- $\mathcal{O}_{\operatorname{Spec} A}(U_f) = A[f^{-1}]$.
  - $A \to A[f^{-1}]$ a map of E_∞-rings.
  - $\operatorname{Hom}(A[f^{-1}], B) \to \operatorname{Hom}(A, B)$ should be a homotopy equivalence of $\operatorname{Hom}(A[f^{-1}], B)$ to maps $A \to B$ taking f to an invertible element of $\pi_0 B$.
- **Derived scheme:** A topological space X and sheaf of E_∞-rings $\mathcal{O}_X$ that is locally isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for an E_∞-ring A.

Sheaves of E_∞-rings

- $\mathcal{C}$ a model category for E_∞-rings.
- $\{\mathcal{C}$-valued presheaves on $X\}$:
    - $\mathcal{F} \to \mathcal{G}$ a **cofibration** if $\mathcal{F}(U) \to \mathcal{G}(U)$ is a cofibration for every open $U \subseteq X$.
    - $\mathcal{F} \to \mathcal{G}$ a **weak equivalence** if $\mathcal{F}_x \to \mathcal{G}_x$ is a weak equivalence for every $x \in X$.
    - A **sheaf** on $\mathcal{C}$ is a fibrant and cofibrant object on $\{\mathcal{C}$-valued presheaves$\}$.
- $\mathrm{Hom}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y))$ is a **space** of morphisms.
- $p : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is **flat** if $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is flat for all open affines $U \subseteq X, V \subseteq Y$ such that $p(U) \subseteq V$.
    - Fibers of flat morphisms are ordinary schemes.

Deligne-Mumford stacks

Here are a few ways to think about Deligne-Mumford stacks:

1. **Like an orbifold:** a quotient stack of a scheme over the étale site.
   - Site: a category with a Grothendieck topology.
   - Finite automorphism groups.
2. **Representability:** S a scheme. An S-stack is a category fibered in groupoids over $(\mathrm{Aff}/S)_{\mathrm{et}}$, satisfying descent.
   - **Deligne-Mumford** when $\Delta : X \to X \times_S X$ is representable, separable, and quasi-finite.
   - Also need a surjective, representable, étale covering $p : A \to X$ by an algebraic space A.
3. **As a topos:** a DM stack is a topos X with a sheaf $\mathcal{O}_X$ of commutative rings that is locally isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.
   - Topos $\approx$ sheaves over a site (looks like a space).
   - $\operatorname{Spec} A$ is the étale topos of A.

Derived Deligne-Mumford stacks

As with schemes, we can consider **derived stacks** by letting $\mathcal{O}_X$ be a sheaf of E_∞-rings.

Example

$\mathcal{M}_{1,1} = (\mathcal{M}, \mathcal{O}_{\mathcal{M}_{1,1}})$: moduli stack of elliptic curves with étale topos $\mathcal{M}$.

- The hard work of Goerss-Hopkins-Miller extends the canonical sheaf $\mathcal{O}$ of $\mathcal{M}$ to a sheaf $\mathcal{O}_{\mathcal{M}^{\mathrm{Der}}}$ of E_∞-rings.
- $\mathcal{M}^{\mathrm{Der}} = (\mathcal{M}, \mathcal{O}_{\mathcal{M}^{\mathrm{Der}}})$ is a derived DM stack.
- $\pi_0 \mathcal{O}_{\mathcal{M}^{\mathrm{Der}}} \simeq \mathcal{O}_{\mathcal{M}_{1,1}}$.
- **Payoff:** get equivariant elliptic cohomology.

Derived group schemes

Want equivariant versions of a cohomology theory A.

- Want derived A-scheme $\mathbb{G} \to \operatorname{Spec} A$ instead of ordinary $A(*)$-scheme $\mathbb{G} \to \operatorname{Spec} A(*)$.
- What should group structure look like in derived setting?
  - Commutative X-group for derived scheme X:
  - Topological functor $\mathbb{G} : \{X\text{-schemes}\} \to \{\text{topological abelian groups}\}$.
  - Want composite $\{X\text{-schemes}\} \to \{\text{topological abelian groups}\} \to \{\text{topological spaces}\}$ to be representable by a flat, derived X-scheme.
  - Flatness makes fiber products work when going between ordinary and derived schemes.

Orientations of derived group schemes

Definition

X a derived scheme, $\mathbb{G}$ a commutative X-group. A **preorientation** of $\mathbb{G}$ is a map of topological groups $\mathbb{C}P^\infty \to \mathbb{G}$.

- May equivalently consider maps $\mathbb{C}P^1 \to \mathbb{G}(X)$ that take the base point of $\mathbb{C}P^1$ to the identity of $\mathbb{G}(X)$.
- Preorientations of $\mathbb{G}$ are classified by elements of $\pi_2\mathbb{G}(X)$.
- To get an **orientation**, need to rule out the 0 map.

Definition

A an E_∞-ring, $\mathbb{G}$ a commutative A-group, $\sigma : S^2 \to \mathbb{G}(A)$ a preorientation. We say that σ is an **orientation** if σ satisfies

(1) a (specific) smoothness/flatness condition.

(2) a (specific) weak periodicity condition.

Oriented elliptic curves

Definition

A an E_∞-ring. An **elliptic curve over** A is a commutative A-group $E \to \operatorname{Spec} A$ such that the underlying map $E_0 \to \operatorname{Spec} \pi_0 A$ is an elliptic curve.

Theorem

Let $E(A)$ be the classifying space for the topological category of **oriented** *elliptic curves over A. Then there is a natural homotopy equivalence*

$$\operatorname{Hom}(\operatorname{Spec} A, \mathcal{M}^{\mathrm{Der}}) \simeq E(A).$$

- **Upshot:** $\mathcal{M}^{\mathrm{Der}}$ is the moduli stack of **oriented** elliptic curves.
- **Compare:** $\mathcal{M}_{1,1}$ is the moduli stack of **ordinary** elliptic curves.

Proof sketch

Proposition

A functor from E_∞-rings to spaces is represented by a derived Deligne-Mumford stack if it satisfies ordinary representability, étale descent, and has a well-behaved deformation theory.

- Reduce to the local case.
- Do some intense algebraic geometry.
- **Applications:** physics, equivariant elliptic cohomology, higher equivariance.

Thanks!