Equations for lines on cubic surfaces

Stephen McKean (Duke)

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Overview

- Enumerative problems can be described by incidence varieties
- Monodromy of covers assigns Galois groups to enumerative problems
- Solvable Galois groups imply existence of explicit equations
- **Question:** can you write down the equations?

Lines on cubic surfaces

- Smooth cubic surfaces over $\mathbb{C}$ have 27 lines
- Harris: $\Phi_{27} \to \Phi$ has Galois group $W(E_6)$
- Harris: $\Phi_{27} \to \Phi_{3\text{ skew}}$ has solvable Galois group
- **Upshot:** formulas for all 27 lines in terms of any 3 skew lines
- Farb: what are the equations?
- Answer in joint work with Dan Minahan and Tianyi Zhang

Harris's outline

(i) Start with 3 skew lines: E_1, E_2, E_3

(ii) E_1, E_2, E_3 lie in a ruling of a quadric Q

(iii) $S \cap Q$ contains 3 other lines: C_4, C_5, C_6

- Visualization by Steve Trettel

(iv) Plane containing E_i, C_j meets S in L_{ij}: get 9 more lines

(v) Solve quadratic equation from Plücker relation: C_3 and $L_{1,2}$

- Easier: use quadric to solve for lines through $E_1, E_2, L_{3,4}, L_{3,5}$

(vi) Final 10 lines are residually determined as in step (iv)

- Easier: use a quadric to solve for $E_4, E_5, E_6, L_{4,5}, L_{4,6}, L_{5,6}$
- Visualization by Steve Trettel

Question: when does Galois theory happen?

Galois theory and real lines

- If S is a real cubic surface, only extend field of definition for:
  - C_4, C_5, C_6
  - $C_3, L_{1,2}$
- To find C_4, C_5, C_6, solve:

$$g(t) = \alpha_{2,0,1,0}t^3 + (\alpha_{2,0,0,1} + \alpha_{1,1,1,0})t^2 + (\alpha_{0,2,1,0} + \alpha_{1,1,0,1})t + \alpha_{0,2,0,1} = 0$$

- To find $C_3, L_{1,2}$, solve:

$$h(t) = v_2 t^2 + (u_1 v_2 - u_2 + v_3)t + (u_1 v_3 - u_3) = 0$$

- **Question:** do $g(t)$ and $h(t)$ determine all real lines on S?

Real cubic surfaces

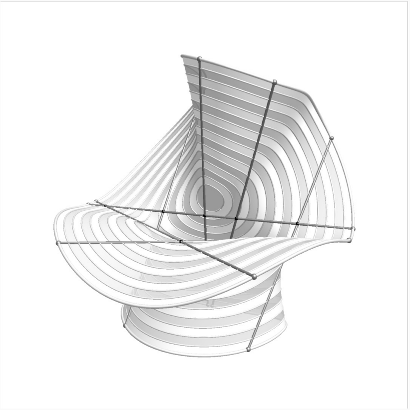
Question: do $g(t)$ and $h(t)$ determine all real lines on S?

- **Theorem** (Segre, 1942): every real smooth cubic surface has 3, 7, 15, or 27 lines.
- **Prop:** S contains 3 skew lines iff S contains at least 7 lines.
- $g(t)$ always has a real root
  - If $g(t)$ has only one real root, only get 7 lines
  - If $g(t)$ has three real roots, get 15 lines
- Both roots of $h(t)$ are real or not real
  - If $h(t)$ has non-real roots, stay at 15 lines
  - If $h(t)$ has real roots, get 27 lines
- **Theorem** (M-Minahan-Zhang): if S has 3 skew lines, then $g(t)$ and $h(t)$ determine how many lines S contains
- More images by Steve Trettel:

Real cubic surfaces

$$f = x_0^2 x_2 - x_0 x_2^2 + 2x_0^2 x_3 - 2x_0 x_1 x_2 + x_1^2 x_2 - x_0 x_1 x_3 + x_1^2 x_3 - x_1 x_3^2$$

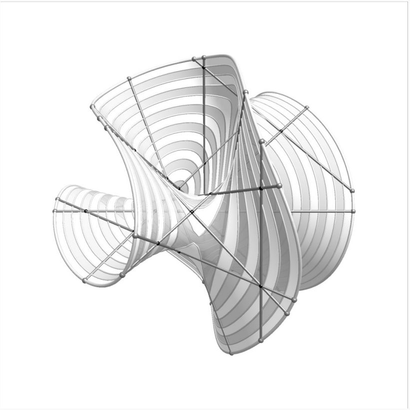
$$g(t) = t^3 + 1$$



Real cubic surfaces

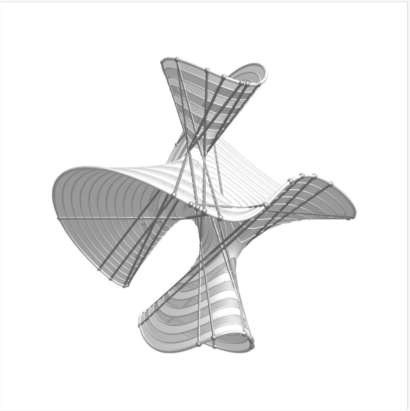
$$f = x_0^2x_2 - x_0x_2^2 + x_0^2x_3 - x_0x_1x_2 + x_1^2x_2 - 2x_0x_1x_3 + x_1x_2^2 - x_0x_2x_3 - x_0x_3^2 + 2x_1x_2x_3$$

$$g(t) = t^3 - t, \qquad h(t) = t^2 - t + 2$$



Real cubic surfaces

$$f = x_0^2x_2 - x_0x_2^2 + x_0^2x_3 - x_0x_1x_2 + \tfrac{17}{39}x_1x_2^2 - \tfrac{17}{39}x_0x_2x_3$$
$$+ 2x_1^2x_2 - 3x_0x_1x_3 + \tfrac{12}{13}x_0x_3^2 + \tfrac{1}{13}x_1x_2x_3$$
$$g(t) = t^3 - t, \qquad h(t) = \tfrac{1}{36}(14t^2 + 367t - 255)$$



Thank you!