Enriching Bézout's Theorem

Stephen McKean (Georgia Tech)

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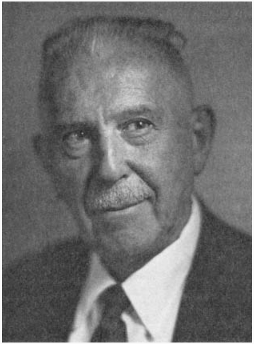
AMS Fall Central Sectional Meeting

Enriching Bézout's Theorem



"It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry."
– Lefschetz, 1924.

Bézout's Theorem

How many times do two curves intersect?

Theorem

Let k be an algebraically closed field. If $f, g \subset \mathbb{P}^2_k$ are generic algebraic curves of degree c, d, respectively, then

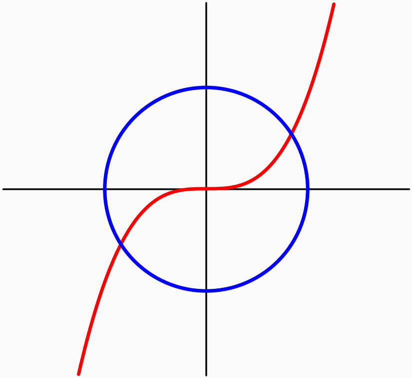
$$\sum_{p \in f \cap g} i_p(f, g) = cd.$$

What if k is not algebraically closed?

Bézout's Theorem

What if k is not algebraically closed?

$$k = \mathbb{R}, \quad f = y - x^3, \quad g = y^2 + x^2 - 1.$$



Algebraic topology: deg valued in $\mathbb{Z}$
$\mathbb{A}^1$-algebraic topology: $\deg^{\mathbb{A}^1}$ valued in $\mathrm{GW}(k)$

- $\mathrm{GW}(k) =$ symmetric, non-degenerate bilinear forms over k
- $(x, y) \mapsto axy$ denoted by $\langle a \rangle$

(i) $\langle a^2 \rangle = \langle 1 \rangle$

(ii) $\langle a \rangle \langle b \rangle = \langle ab \rangle$

(iii) If $a + b \neq 0$, then $\langle a \rangle + \langle b \rangle = \langle ab(a + b) \rangle + \langle a + b \rangle$

(iv) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle =: \mathbb{H}$

$\mathbb{A}^1$-Enumerative Geometry

Can use $\deg^{\mathbb{A}^1}$ to study classical enumerative problems

(Bethea-Kass-Wickelgren, Brazelton, Hoyois, Kass-Wickelgren, Larson-Vogt, Levine, Pauli, Srinivasan-Wickelgren, Wendt, ...)

$\mathrm{GW}(k)$ gives us richer counts than $\mathbb{Z}$:

$$\mathrm{GW}(\mathbb{C}) \xrightarrow{\text{rank}} \mathbb{Z}$$

$$\mathrm{GW}(\mathbb{R}) \xrightarrow{\text{rank}\times\text{sign}} \mathbb{Z} \times \mathbb{Z}$$

$$\mathrm{GW}(\mathbb{F}_q) \xrightarrow{\text{rank}\times\text{disc}} \mathbb{Z} \times \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$$

If k is not algebraically closed, we get extra information.

$\mathbb{A}^1$-enumerative geometry: extra information has geometric meaning.

Enriched Bézout's Theorem

Look at sections $\sigma = (f, g)$ of $\mathcal{O}(c) \oplus \mathcal{O}(d)$.

Theorem (M.)
Let k be a perfect field and f, g curves of degrees c, d with $f \cap g$ isolated. If $c + d$ is odd, then

$$\sum_{p \in f \cap g} \deg_p^{\mathbb{A}^1}(f, g) = \frac{cd}{2} \cdot \mathbb{H}.$$

$$\deg_p^{\mathbb{A}^1}(f, g) = \begin{cases} \mathrm{Tr}_{k(p)/k}\left(\frac{i_p}{2} \cdot \mathbb{H}\right) & i_p \text{ even,} \\ \mathrm{Tr}_{k(p)/k}\left(\langle a_p \rangle + \frac{i_p - 1}{2} \cdot \mathbb{H}\right) & i_p \text{ odd.} \end{cases}$$

$\deg_p^{\mathbb{A}^1}(f, g)$ is determined by geometric information.

Enriched Bézout's Theorem

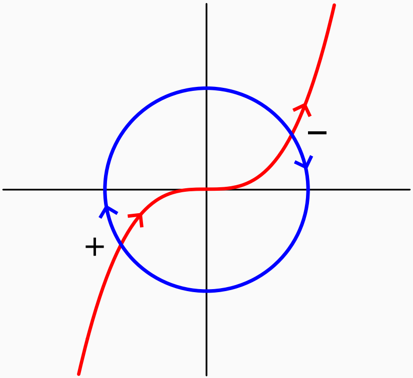
$\deg_p^{\mathbb{A}^1}(f, g)$ is determined by geometric information:

k	$\deg_p^{\mathbb{A}^1}(f, g)$	$\frac{cd}{2} \cdot \mathbb{H}$
$\mathbb{C}$	$i_p(f, g)$	cd
$\mathbb{R}$	crossing sign at p	0
$\mathbb{F}_q$	crossing sign at p	$(-1)^{\frac{cd}{2}}$

- Over $\mathbb{C}$: counts intersection points.
- Over $\mathbb{R}$: equal number of positive/negative crossings.
- Over $\mathbb{F}_q$: counts crossing types mod 2.

Example

$$k = \mathbb{R}, \quad f = y - x^3, \quad g = y^2 + x^2 - 1.$$



Relative Orientability

Why $c + d$ odd?

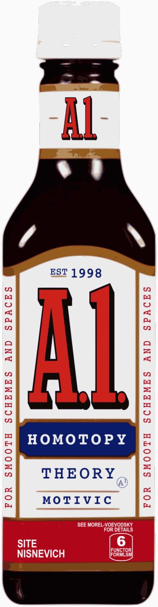
Approach uses *motivic Euler class* of $\mathcal{O}(c) \oplus \mathcal{O}(d) \to \mathbb{P}^2$.

- Only well-defined if $c + d$ odd.
- Potential fix (Larson-Vogt): pick a divisor.
- If c, d even and $\{f \cap g\}|_{\{x_0=0\}} = \emptyset$, Enriched Bézout still works.

What's left to do?

- Explicit calculation of a_p when $i_p > 1$.
- Address c, d odd case.

Thanks!



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