Enriching Bézout’s Theorem

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“It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry.”
– Lefschetz, 1924.
How many times do two curves intersect?

**Theorem**

Let $k$ be an algebraically closed field. If $f, g \subset \mathbb{P}_k^2$ are generic algebraic curves of degree $c, d$, respectively, then

$$\sum_{p \in f \cap g} i_p(f, g) = cd.$$ 

What if $k$ is not algebraically closed?
What if $k$ is not algebraically closed?

$$k = \mathbb{R}, \quad f = y - x^3, \quad g = y^2 + x^2 - 1.$$
Algebraic topology: deg valued in $\mathbb{Z}$

$\mathbb{A}^1$-algebraic topology: deg$^{\mathbb{A}^1}$ valued in $GW(k)$

- $GW(k)$ = symmetric, non-degenerate bilinear forms over $k$
- $(x, y) \mapsto axy$ denoted by $\langle a \rangle$

(i) $\langle a^2 \rangle = \langle 1 \rangle$
(ii) $\langle a \rangle \langle b \rangle = \langle ab \rangle$
(iii) If $a + b \neq 0$, then $\langle a \rangle + \langle b \rangle = \langle ab(a + b) \rangle + \langle a + b \rangle$
(iv) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle =: \mathbb{H}$
Can use $\text{deg}^{\mathbb{A}^1}$ to study classical enumerative problems
(Bethea-Kass-Wickelgren, Brazelton, Hoyois, Kass-Wickelgren, Larson-Vogt, Levine,
Pauli, Srinivasan-Wickelgren, Wendt, ...)

$GW(k)$ gives us richer counts than $\mathbb{Z}$:

\[
\begin{align*}
GW(\mathbb{C}) \xrightarrow{\text{rank}} & \mathbb{Z} \\
GW(\mathbb{R}) \xrightarrow{\text{rank} \times \text{sign}} & \mathbb{Z} \times \mathbb{Z} \\
GW(\mathbb{F}_q) \xrightarrow{\text{rank} \times \text{disc}} & \mathbb{Z} \times \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2
\end{align*}
\]

If $k$ is not algebraically closed, we get extra information.

$\mathbb{A}^1$-enumerative geometry: extra information has geometric meaning.
Enriched Bézout’s Theorem

Look at sections $\sigma = (f, g)$ of $O(c) \oplus O(d)$.

**Theorem (M.)**

Let $k$ be a perfect field and $f, g$ curves of degrees $c, d$ with $f \cap g$ isolated. If $c + d$ is odd, then

$$\sum_{p \in f \cap g} \deg_p^{A^1} (f, g) = \frac{cd}{2} \cdot \mathbb{H}.$$ 

$$\deg_p^{A^1} (f, g) = \begin{cases} \text{Tr}_{k(p)/k} \left( \frac{i_p}{2} \cdot \mathbb{H} \right) & \text{if } i_p \text{ even}, \\ \text{Tr}_{k(p)/k} \left( \langle a_p \rangle + \frac{i_p - 1}{2} \cdot \mathbb{H} \right) & \text{if } i_p \text{ odd.} \end{cases}$$

$\deg_p^{A^1} (f, g)$ is determined by geometric information.
Enriched Bézout’s Theorem

\( \deg_{A^1}^p (f, g) \) is determined by geometric information:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \deg_{A^1}^p (f, g) )</th>
<th>( \frac{cd}{2} \cdot H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{C} )</td>
<td>( i_p(f, g) )</td>
<td>( cd )</td>
</tr>
<tr>
<td>( \mathbb{R} )</td>
<td>crossing sign at ( p )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \mathbb{F}_q )</td>
<td>crossing sign at ( p )</td>
<td>( (-1)^{\frac{cd}{2}} )</td>
</tr>
</tbody>
</table>

- Over \( \mathbb{C} \): counts intersection points.
- Over \( \mathbb{R} \): equal number of positive/negative crossings.
- Over \( \mathbb{F}_q \): counts crossing types mod 2.
Example

\[ k = \mathbb{R}, \quad f = y - x^3, \quad g = y^2 + x^2 - 1. \]
Why $c + d$ odd?

Approach uses *motivic Euler class* of $\mathcal{O}(c) \oplus \mathcal{O}(d) \to \mathbb{P}^2$.

- Only well-defined if $c + d$ odd.
- Potential fix (Larson-Vogt): pick a divisor.
- If $c, d$ even and $\{f \cap g\}|_{\{x_0=0\}} = \emptyset$, Enriched Bézout still works.

What’s left to do?

- Explicit calculation of $a_p$ when $i_p > 1$.
- Address $c, d$ odd case.