ON DRUŻKOWSKI'S MORPHISMS OF CUBIC LINEAR TYPE

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Abstract. We use theorems of Müller-Quade and Steinwandt, Scheja and Storch, and van der Waerden to study Drużkowski's morphisms of cubic linear type with invertible Jacobian. In particular, we compare the degree of such morphisms with the dimensions of various related vector spaces. These comparisons result in an inequality that, if true, would show that morphisms of cubic linear type with invertible Jacobian are injective, finite, and induce an equality of function fields.

1. Introduction

We set out to prove that over an algebraically closed field k of characteristic 0, morphisms $f:\mathbb{A}^n_k\to\mathbb{A}^n_k$ of cubic linear type with invertible Jacobian are injective on closed points, finite, and induce an equality $k(\mathbb{A}^n_k)=f^*(k(\mathbb{A}^n_k))$. To do so, we compare the degree of the field extension induced by f to the dimension of the local algebra $\frac{k[x_1,...,x_n]_0}{(f_1,...,f_n)}$ over k. Ultimately, our efforts will come short of our goal, but we do give a proof depending on one conjectural inequality. After stating our main question and discussing this inequality, we will describe the layout of the paper.

Definition 1.1. We say that a morphism $f:\mathbb{A}^n_k\to\mathbb{A}^n_k$ is *of cubic linear type* if it is of the form $(f_1,...,f_n)=(x_1+h_1,...,x_n+h_n)$, where each h_i is either 0 or the cube of a homogeneous linear polynomial. Moreover, we say that a morphism $f:\mathbb{A}^n_k\to\mathbb{A}^n_k$ is a *Drużkowski morphism* if it is of cubic linear type with invertible Jacobian (that is, $\mathrm{Jac}(f)\in k^\times$).

Morphisms of cubic linear type are discussed in [Dru83, Theorem 3]. The main question of the present paper is if Drużkowski morphisms over an algebraically closed field of characteristic 0 are injective on closed points, finite, and induce an equality of function fields $k(\mathbb{A}^n_k)=f^*(k(\mathbb{A}^n_k))$.

Question 1.2. *Let k be an algebraically closed field of characteristic 0. Let $f:\mathbb{A}^n_k\to\mathbb{A}^n_k$ be a Drużkowski morphism. Is f injective on closed points, finite, and is $[k(\mathbb{A}^n_k):f^*(k(\mathbb{A}^n_k))]=1$?*

We give an inequality that would imply an affirmative answer to Question 1.2. To set notation, let $\mathbf{x}=(x_1,...,x_n)$ and $\mathbf{z}=(z_1,...,z_n)$ be indeterminates, fix a Drużkowski morphism $f=(f_1,...,f_n)$, and consider the field $K=k(f_1(\mathbf{z}),...,f_n(\mathbf{z}))$. Let $\mathfrak{p}=$

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$(f_1(\mathbf{x}) - f_1(\mathbf{z}), ..., f_n(\mathbf{x}) - f_n(\mathbf{z}))$ as an ideal of $K[x_1, ..., x_n]$. We wish to compare the vector space dimensions $\dim_K \frac{K[x_1,...,x_n]}{\mathfrak{p}}$ and $\dim_k \frac{k[x_1,...,x_n]}{(f_1,...,f_n)}$.

Theorem 1.3. *Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a Drużkowski morphism. If*

$$\dim_K \frac{K[x_1, ..., x_n]}{\mathfrak{p}} \leq \dim_k \frac{k[x_1, ..., x_n]}{(f_1, ..., f_n)},$$

then f is injective on closed points, is finite, and induces an equality of function fields.

The bulk of the paper's content is done in Section 2. If k is an algebraically closed field of characteristic 0 and f is a Drużkowski morphism, then the work of Scheja and Storch [SS75] allows us to compute $\dim_k \frac{k[x_1,...,x_n]_0}{(f_1,...,f_n)}$. We use this computation to prove that $\dim_k \frac{k[x_1,...,x_n]_0}{(f_1,...,f_n)} = \dim_k \frac{k[x_1,...,x_n]}{(f_1,...,f_n)}$ in Corollary 2.5. In Conjecture 2.7, we ask if the degree of the field extension $k(x_1, ..., x_n)$ over $k(f_1, ..., f_n)$ is bounded above by the dimension of $\frac{k[x_1,...,x_n]_0}{(f_1,...,f_n)}$ as a k-vector space. This conjecture is broken down into one conjecture and two propositions and largely depends on the work of Müller-Quade and Steinwandt [MQS00] and van der Waerden [vdW27]. Next, we use the results of Section 2 to provide a conditional answer to Question 1.2 in Section 3. Finally, we discuss the relation of Question 1.2 to the Jacobian Conjecture in Section 4.

1.1. **Notation.** Throughout this paper, k denotes an algebraically closed field of characteristic 0. The notation $(f_1, ..., f_n)$ will be used to denote both the ideal generated by the polynomials $f_1, ..., f_n$ and the function $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$. The intended meaning will be clear from context. For $(f_1, ..., f_n)$ a function, we denote the Jacobian determinant $\mathrm{Jac}(f) := \det(\frac{\partial f_i}{\partial x_j})$. Given a point $p \in \mathbb{A}^n_k(k)$, we may write $p = (p_1, ..., p_n)$. If $\mathfrak{p}$ is the prime ideal corresponding to p, we will write $k[x_1, ..., x_n]_p$ to denote the localization $k[x_1, ..., x_n]_{\mathfrak{p}}$. When p and q are k-rational points with $f(p) = q$, we define the k-algebra $Q_p := \frac{k[x_1,...,x_n]_p}{(f_1-q_1,...,f_n-q_n)}$. When $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is quasi-finite, we will denote the degree of the field extension $\deg(f) := [k(\mathbb{A}^n_k) : f^*(k(\mathbb{A}^n_k))] = [k(x_1, ..., x_n) : k(f_1, ..., f_n)]$. We note that when f is quasi-finite, $\deg(f)$ is finite by [Sta19, Lemma 01TG].

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2. Comparing degrees and dimensions

Throughout this section, let k be an algebraically closed field of characteristic 0. It is a standard result that endomorphisms of $\mathbb{A}^n_k$ with invertible Jacobian are étale and quasi-finite.

Proposition 2.1. *If $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ has $\mathrm{Jac}(f) \in k^\times$, then f is étale.*

Proof. By [Sta19, Lemma 02GU (1) and (8)], f is étale at a point $p \in \mathbb{A}^n_k$ (with corresponding prime ideal $\mathfrak{p}$) if and only if $\mathrm{Jac}(f) \notin \mathfrak{p}$. Since k is algebraically closed, we have that $\mathfrak{p} = (x_1 - p_1, ..., x_n - p_n)$ for p a closed point or $\mathfrak{p} = (0)$ for p the generic point. But $\mathrm{Jac}(f) \in k^\times$ by assumption, so $\mathrm{Jac}(f) \notin \mathfrak{p}$. □

Proposition 2.2. *If $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ has $\mathrm{Jac}(f) \in k^\times$, then f is quasi-finite.*

Proof. Since $\mathrm{Jac}(f) \in k^\times$, Proposition 2.1 and [Sta19, Lemma 03WS] imply that f is locally quasi-finite. Since $\mathbb{A}^n_k$ is Noetherian, $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is quasi-compact. Thus [Sta19, Lemma 01TJ] implies that f is quasi-finite. □

When f is quasi-finite, each fiber $f^{-1}(q)$ is a discrete set, so the local ring Q_p has finite dimension as a k-vector space. We show that this dimension is minimal when $\mathrm{Jac}(f)$ is invertible.

Lemma 2.3. *Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a morphism with $f(p) = q$ for k-rational points p and q. Moreover, assume $\mathrm{Jac}(f) \in k^\times$. Then $\dim_k Q_p = 1$.*

Proof. By [SS75, (4.7) Korollar], we have that $\mathrm{Jac}(f) = \dim_k Q_p \cdot E_p$, where E_p is a distinguished element that generates the socle of Q_p. Since f is quasi-finite (by Proposition 2.2) with $f(p) = q$, the ring Q_p is a local Artin ring. (Indeed, f is quasi-finite, so f has finite fibers. It follows that Q_p must have Krull dimension zero, which implies that Q_p is Artinian.) Thus the socle is the annihilator of the unique maximal ideal $\mathfrak{m}$ of Q_p. Since $\mathrm{Jac}(f) \in k^\times$, it follows that $E_p \in k^\times$ as well. As a socle element, the unit E_p annihilates $\mathfrak{m}$, which implies that $\mathfrak{m} = 0$, so Q_p is a field. Since Q_p is a finite-dimensional k-algebra, it follows that Q_p is a finite extension of k. As k is algebraically closed, this implies that Q_p is a degree 1 extension of k, so $\dim_k Q_p = 1$. □

For the remainder of the article, we will focus on the case when $p = 0 := (0, 0, ..., 0) \in \mathbb{A}^n_k$. We will show that if f is a Drużkowski morphism, then $\dim_k \frac{k[x_1,...,x_n]_0}{(f_1,...,f_n)} = \dim_k \frac{k[x_1,...,x_n]}{(f_1,...,f_n)}$. First, we need the following proposition.

Proposition 2.4. *If $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is a Drużkowski morphism, then for each i, we either have $h_i = 0$ or $x_i \nmid h_i$.*

Proof. Assume there exists some i such that $x_i \mid h_i$. By permuting columns in the following argument, we may assume that $i = 1$. Then $h_1 = (a_1 x_1)^3$ for some $a_1 \in k$. Thus

$$\begin{aligned}\mathrm{Jac}(f) &= \det \begin{pmatrix} 1 + 3a_1^3 x_1^2 & 0 & \cdots & 0 \\ \frac{\partial h_2}{\partial x_1} & 1 + \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial x_1} & \frac{\partial h_n}{\partial x_2} & \cdots & 1 + \frac{\partial h_n}{\partial x_n} \end{pmatrix} \\ &= (1 + 3a_1^3 x_1^2) \det(A),\end{aligned}$$

where A is the bottom-right $(n-1) \times (n-1)$ minor of the Jacobian matrix of f. By assumption, $\mathrm{Jac}(f) \in k^\times$, so $(1 + 3a_1^3 x_1^2) \det(A) \in k^\times$. This implies that $1 + 3a_1^3 x_1^2$ divides some non-zero element of k. Since $\mathrm{char}(k) = 0$, it follows that $k[x_1, ..., x_n]$ is an integral domain, and hence $1 + 3a_1^3 x_1^2$ must be constant. Thus $a_1 = 0$, as desired. □

Corollary 2.5. *If $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is a Drużkowski morphism, then*

$$\dim_k \frac{k[x_1, ..., x_n]_0}{(f_1, ..., f_n)} = \dim_k \frac{k[x_1, ..., x_n]}{(f_1, ..., f_n)}.$$

Proof. Let $<$ denote graded lexicographic order with $x_n > x_{n-1} > ... > x_1$. Let G be a reduced Gröbner basis (with respect to $<$) of the ideal $(f_1, ..., f_n)$ in $k[x_1, ..., x_n]_0$. Note that $(f_1, ..., f_n) \neq k[x_1, ..., x_n]_0$, so G does not contain any units of $k[x_1, ..., x_n]_0$. In particular, all elements of G are contained in $(x_1, ..., x_n)$. Since $\dim_k \frac{k[x_1,...,x_n]_0}{(f_1,...,f_n)} = 1$ by Lemma 2.3, it follows that for each $1 \leq i \leq n$, there exists $g_i \in G$ such that x_i is the leading monomial of g_i. In fact, requiring G to be a reduced Gröbner basis forces us to have $g_i = x_i$. Indeed, if we had $g_i = x_i +$ (lower order terms), the lower order terms would be canceled out when reducing by G. For each $1 \leq i \leq n$, we will show that x_i is contained in the ideal $(f_1, ..., f_n)$ of $k[x_1, ..., x_n]$. Recall that $f_i = x_i + h_i$, where h_i is either 0 or the cube of a homogeneous linear polynomial. If $h_i = 0$, then $f_i = x_i$ and hence $x_i \in (f_1, ..., f_n)$. We may thus assume $h_i \neq 0$. Since $x_i \in G$, we know that x_i is contained in the ideal $(f_1, ..., f_n) \subseteq k[x_1, ..., x_n]_0$. Thus there exist $a_{ij}, b_{ij} \in k[x_1, ..., x_n]$ with $b_{ij} \notin (x_1, ..., x_n)$ such that

$$x_i = \sum_{j=1}^{n} \frac{a_{ij}}{b_{ij}} f_j. \tag{2.1}$$

Since $b_{ij} \notin (x_1, ..., x_n)$ for all j, we also have $b_i := \prod_{j=1}^{n} b_{ij} \notin (x_1, ..., x_n)$. We may thus write $b_i = \beta_i + \eta_i$, where $\beta_i \in k^\times$ and $\eta_i \in (x_1, ..., x_n)$. Multiplying both sides of Equation 2.1 by $b_i = \beta_i + \eta_i$ and writing $f_j = x_j + h_j$, we obtain

$$\beta_i x_i + x_i \eta_i = \sum_{j=1}^{n} \frac{a_{ij} b_i}{b_{ij}} (x_j + h_j), \tag{2.2}$$

with $\frac{a_{ij} b_i}{b_{ij}} \in k[x_1, ..., x_n]$. If $\eta_i = 0$, then we have $\beta_i x_i \in (f_1, ..., f_n)$. Note that $\beta_i \in k^\times$, and hence $x_i \in (f_1, ..., f_n)$, as desired. Now if $\eta_i \neq 0$, we proceed by comparing the degrees of various terms and drawing a contradiction. Since $\beta_i \in k^\times$, the term $\beta_i x_i$ has degree 1. On the other hand, $\eta_i \in (x_1, ..., x_n)$ is non-zero by assumption, so all terms of $x_i \eta_i$ must have degree at least 2. On the right hand side of Equation 2.2, we can have degree 1 terms only if the polynomial $\frac{a_{ij} b_i}{b_{ij}}$ has a non-zero constant term. Since the only degree 1 term that appears on the left hand side of Equation 2.2 is $\beta_i x_i$, we have that $\frac{a_{ii} b_i}{b_{ii}} = \beta_i + \gamma_i$, where $\gamma_i \in (x_1, ..., x_n)$, and we also have that $\frac{a_{ij} b_i}{b_{ij}} \in (x_1, ..., x_n)$. Altogether, we have

$$\underbrace{\beta_i x_i}_{\deg=1} + \underbrace{x_i \eta_i}_{\deg\geq 2} = \underbrace{\beta_i x_i}_{\deg=1} + \underbrace{x_i \gamma_i}_{\deg\geq 2} + \underbrace{\beta_i h_i}_{\deg=3} + \underbrace{\gamma_i h_i}_{\deg\geq 4} + \underbrace{\sum_{j\neq i} \frac{a_{ij} b_i}{b_{ij}} h_j}_{\deg\geq 4}, \tag{2.3}$$

where the listed degrees refer to the minimal possible degree of a (non-zero) monomial in the given polynomial. (For example, $\gamma_i \in (x_1, ..., x_n)$, so γ_i is either 0 or consists of terms of degree 1 or greater. Thus $x_i\gamma_i$ is either 0 or consists of terms of degree 2 or greater.) We now compare the degree 3 part of each side of Equation 2.3. We introduce the following notation to simplfiy our statement. Given a polynomial $p(x_1,...,x_n)$ and a non-negative integer r, let $p[r]$ be the r^{th}-degree part of p. With this notation, we have $x_i\eta_i[3] = (x_i\gamma_i + \beta_i h_i)[3]$. But $\beta_i h_i$ only consists of degree 3 terms, so $\beta_i h_i = x_i\eta_i[3] - x_i\gamma_i[3]$. Since $\beta_i \in k^\times$, this implies that $h_i = x_i\frac{1}{\beta_i}(\eta_i[2] - \gamma_i[2])$. Thus $x_i \mid h_i$, so $h_i = 0$ by Proposition 2.4. But this contradicts our assumption that $h_i \neq 0$ and $\eta_i \neq 0$, so we either have $h_i = 0$ or $h_i \neq 0$ and $\eta_i = 0$. In either case, we have shown that x_i is contained in the ideal $(f_1,...,f_n)$ of $k[x_1,...,x_n]$. We have thus shown that $(x_1,...,x_n) \subseteq (f_1,...,f_n)$. By definition of morphisms of cubic linear type, we also have $(x_1,...,x_n) \supseteq (f_1,...,f_n)$, and hence $(x_1,...,x_n) = (f_1,...,f_n)$. Thus

$$\dim_k \frac{k[x_1,...,x_n]}{(f_1,...,f_n)} = \dim_k \frac{k[x_1,...,x_n]}{(x_1,...,x_n)} = 1,$$

which is equal to $\dim_k \frac{k[x_1,...,x_n]_0}{(f_1,...,f_n)}$ by Lemma 2.3. □

Paired with Corollary 2.5, Conjecture 2.7 would allow us to compare $\dim_k \frac{k[x_1,...,x_n]_0}{(f_1,...,f_n)}$ and $\deg(f)$. We first show that a certain ideal is zero-dimensional, so that we may later apply the Noether normalization lemma. Recall that an ideal I of a polynomial ring R (finitely generated) over a field F is *zero-dimensional* if $\dim_F(R/I) < \infty$, or equivalently if the Krull dimension of R/I is zero.

Lemma 2.6. *Suppose $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is quasi-finite. Let $\mathbf{x} = (x_1,...,x_n)$ and $\mathbf{z} = (z_1,...,z_n)$ be indeterminates, and let $K = k(f_1(\mathbf{z}),...,f_n(\mathbf{z}))$. Then $\mathfrak{p} = (f_1(\mathbf{x}) - f_1(\mathbf{z}),...,f_n(\mathbf{x}) - f_n(\mathbf{z}))$ is a zero-dimensional ideal of $K[x_1,...,x_n]$.*

Proof. Here, one could note that $\dim_K(K[x_1,...,x_n]/\mathfrak{p})$ is the cardinality of a general fiber of f. Since f is assumed to be quasi-finite, it follows that $\dim_K(K[x_1,...,x_n]/\mathfrak{p})$ is finite, and hence $\mathfrak{p}$ is a zero-dimensional ideal of $K[x_1,...,x_n]$.

For the sake of clarity, we also include a more hands-on proof. Let $\overline{K}$ be an algebraic closure of K. We first show that if $\overline{K}[x_1,...,x_n]/\mathfrak{p}\overline{K}[x_1,...,x_n]$ has Krull dimension zero, then $K[x_1,...,x_n]/\mathfrak{p}$ has Krull dimension zero as well. The inclusion $K \to \overline{K}$ induces an injective map $K[x_1,...,x_n]/\mathfrak{p} \to \overline{K}[x_1,...,x_n]/\mathfrak{p}\overline{K}[x_1,...,x_n]$. Since the ring $\overline{K}[x_1,...,x_n]$ is integral over $K[x_1,...,x_n]$ and $\mathfrak{p}\overline{K}[x_1,...,x_n] \cap K[x_1,...,x_n] = \mathfrak{p}$, it follows that $\overline{K}[x_1,...,x_n]/\mathfrak{p}\overline{K}[x_1,...,x_n]$ is integral over $K[x_1,...,x_n]/\mathfrak{p}$. By [Sta19, Lemma 00OK], the Krull dimension of $K[x_1,...,x_n]/\mathfrak{p}$ is equal to the Krull dimension of $\overline{K}[x_1,...,x_n]/\mathfrak{p}\overline{K}[x_1,...,x_n]$, which is zero by assumption.

Now we show that $\overline{K}[x_1,...,x_n]/\mathfrak{p}\overline{K}[x_1,...,x_n]$ has Krull dimension zero, which will conclude the proof. Let $f_K : \mathbb{A}^n_K \to \mathbb{A}^n_K$ denote the base change of f. Since quasi-finiteness is stable under base change by [Sta19, Remark 02WF (11)], it follows that f_K is quasi-finite. Next, let $g : \mathbb{A}^n_K \to \mathbb{A}^n_K$ be given by $g(x_1,...,x_n) = (x_1 - f_1(\mathbf{z}),...,x_n - f_n(\mathbf{z}))$. Since $\mathbb{A}^n_K$ is Noetherian, g is quasi-compact. Moreover, g is locally of finite type and has finite fibers, so g is quasi-finite by [Sta19, Lemma 02NH]. Since quasi-finiteness is stable under composition by [Sta19, Remark 02WG (11)], it follows that $g \circ f_K$ is quasi-finite. Now let $(g \circ f_K)_{\overline{K}} : \mathbb{A}^n_{\overline{K}} \to \mathbb{A}^n_{\overline{K}}$ be the base change of $g \circ f_K$. This is again quasi-finte by the stability of quasi-finiteness under base change. In particular, $(g \circ f_K)_{\overline{K}}$ has finite fibers. We note that $g \circ f_K = (f_1(\mathbf{x}) - f_1(\mathbf{z}), ..., f_n(\mathbf{x}) - f_n(\mathbf{z}))$, so the fiber $(g \circ f_K)^{-1}_{\overline{K}}(0)$ is equal to the vanishing locus of the ideal $\mathfrak{p}\overline{K}[x_1, ..., x_n]$ of $\overline{K}[x_1, ..., x_n]$. Since $(g \circ f_K)_{\overline{K}}$ is quasi-finite, the fiber $(g \circ f_K)^{-1}_{\overline{K}}(0)$ is finite and hence the Krull dimension of $\overline{K}[x_1, ..., x_n]/\mathfrak{p}\overline{K}[x_1, ..., x_n]$ is zero. □

Conjecture 2.7. *If $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is a Drużkowski morphism, then*

$$\deg(f) \leq \dim_k \frac{k[x_1, ..., x_n]}{(f_1, ..., f_n)}.$$

Remark 2.8. We give a conditional proof in terms of Conjecture 2.9 and Propositions 2.12 and 2.13. Throughout the rest of the section, let $\mathbf{x}, \mathbf{z}$, and K be as in Lemma 2.6. We will consider the ring $K[x_1, ..., x_n]$ with the ideal $\mathfrak{p} = \{p(\mathbf{x}) \in K[x_1, ..., x_n] : p(\mathbf{z}) = 0\}$.

Conjecture 2.9. *If $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is a Drużkowski morphism, then*

$$\dim_K \frac{K[x_1, ..., x_n]}{\mathfrak{p}} \leq \dim_k \frac{k[x_1, ..., x_n]}{(f_1, ..., f_n)}.$$

Remark 2.10. Since $\mathrm{char}(k) = 0$, we have that $K[x_1, ..., x_n]$ is an integral domain and hence $\mathfrak{p}$ is a prime ideal. By [MQS00, Corollary 2], the fact that $f_1, ..., f_n$ are polynomials gives us that $\mathfrak{p} = (f_1(\mathbf{x}) - f_1(\mathbf{z}), ..., f_n(\mathbf{x}) - f_n(\mathbf{z})) + \mathfrak{q}$, where $\mathfrak{q} = \{p(\mathbf{x}) \in k[x_1, ..., x_n] : p(\mathbf{z}) = 0\}$. Since $z_1, ..., z_n$ are indeterminates, it follows that in $k[x_1, ..., x_n]$, we have $p(\mathbf{z}) = 0$ if and only if p is the zero polynomial. As a consequence, we have $\mathfrak{q} = (0)$ and hence $\mathfrak{p} = (f_1(\mathbf{x}) - f_1(\mathbf{z}), ..., f_n(\mathbf{x}) - f_n(\mathbf{z}))$. By Lemma 2.6, we have that $\dim_K \frac{K[x_1, ..., x_n]}{\mathfrak{p}} < \infty$.

Remark 2.11. If f is generically finite-to-one and $(f_1, ..., f_n)$ is a zero-dimensional ideal of $k[x_1, ..., x_n]$, then $\dim_k \frac{k[x_1, ..., x_n]}{(f_1, ..., f_n)} \leq \dim_K \frac{K[x_1, ..., x_n]}{\mathfrak{p}}$. This is because $\dim_k \frac{k[x_1, ..., x_n]}{(f_1, ..., f_n)}$ is the cardinality (counted with multiplicity) of the fiber $f^{-1}(0)$, while $\dim_K \frac{K[x_1, ..., x_n]}{\mathfrak{p}}$ is the cardinality (counted with multiplicity) of a general fiber of f. If f is only assumed to be quasi-finite, then this inequality can even be strict. For example, consider the quasi-finite morphism $f = (x_1x_2 - 1, x_2(x_1x_2 - 1) + x_1)$. In this case, we have $\dim_k \frac{k[x_1, ..., x_n]}{(f_1, ..., f_n)} = 0$ and $\dim_K \frac{K[x_1, ..., x_n]}{\mathfrak{p}} = 2$. This shows that Conjecture 2.9 is false for some quasi-finite morphisms.

Proposition 2.12. *If $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is quasi-finite, then*

$$\dim_K \mathrm{Frac}\left(\frac{K[x_1, ..., x_n]}{\mathfrak{p}}\right) \leq \dim_K \frac{K[x_1, ..., x_n]}{\mathfrak{p}}.$$

Proof. By Lemma 2.6, we have that $\mathfrak{p}$ is a zero-dimensional ideal of $K[x_1, ..., x_n]$, so the quotient $\frac{K[x_1, ..., x_n]}{\mathfrak{p}}$ is a finite-dimensional K-algebra. Moreover, van der Waerden [vdW27, §3, 2.] shows that that the field of fractions of $\frac{K[x_1, ..., x_n]}{\mathfrak{p}}$ is isomorphic to $k(x_1, ..., x_n)$. By the Noether normalization lemma, the Krull dimension of $\frac{K[x_1,...,x_n]}{\mathfrak{p}}$ is equal to the transcendence degree of $k(x_1, ..., x_n)$ over K, which is 0 since $\deg(f) < \infty$ (see Section 1.1). Thus $\frac{K[x_1,...,x_n]}{\mathfrak{p}}$ is an integral domain with Krull dimension 0 (that is, a field), so we have an isomorphism

$$\varphi : \frac{K[x_1, ..., x_n]}{\mathfrak{p}} \xrightarrow{\cong} \operatorname{Frac}\left(\frac{K[x_1, ..., x_n]}{\mathfrak{p}}\right).$$

This isomorphism is given by $\varphi(a) = \frac{a}{1}$, with inverse $\varphi^{-1}(\frac{a}{b}) = ab^{-1}$. Suppose $\{e_1, ..., e_m\}$ is a basis for $\frac{K[x_1,...,x_n]}{\mathfrak{p}}$ over K. If $\frac{a}{b} \in \operatorname{Frac}(\frac{K[x_1,...,x_n]}{\mathfrak{p}})$, then $a, b \in \frac{K[x_1,...,x_n]}{\mathfrak{p}}$. Thus $ab^{-1} = \sum c_i e_i$ for some $c_1, ..., c_m \in K$. By construction, $\frac{a}{b} = \varphi(ab^{-1}) = \sum_{i=1}^{m} c_i \frac{e_i}{1}$. Thus $\operatorname{Frac}(\frac{K[x_1,...,x_n]}{\mathfrak{p}})$ is contained in the K-span of $\{\frac{e_1}{1}, ..., \frac{e_m}{1}\}$, which finishes the proof. □

Proposition 2.13. *If $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is quasi-finite, then*

$$\deg(f) \leq \dim_K \operatorname{Frac}\left(\frac{K[x_1, ..., x_n]}{\mathfrak{p}}\right).$$

Proof. By definition, $\deg(f) = \dim_K k(z_1, ..., z_n)$. Van der Waerden [vdW27, §3, 2.] gives an isomorphism $\psi : \frac{K[x_1,...,x_n]}{\mathfrak{p}} \xrightarrow{\cong} K[z_1, ..., z_n]$ defined by $g(\mathbf{x}) + \mathfrak{p} \mapsto g(\mathbf{z})$, which induces an isomorphism $\Psi : \operatorname{Frac}(\frac{K[x_1,...,x_n]}{\mathfrak{p}}) \xrightarrow{\cong} \operatorname{Frac}(K[z_1, ..., z_n])$ defined by $(a(\mathbf{x}) + \mathfrak{p})/(b(\mathbf{x}) + \mathfrak{p}) \mapsto a(\mathbf{z})/b(\mathbf{z})$. Note that ψ and Ψ are well-defined, as $p(\mathbf{x}) \in \mathfrak{p}$ if and only if $p(\mathbf{z}) = 0$ (see the Remarks 2.8 and 2.10). Moreover, the identity map $\operatorname{Frac}(K[z_1, ..., z_n]) \to K(z_1, ..., z_n) = k(z_1, ..., z_n)$ respects the K-vector space structure of both fields. It thus suffices to show that

$$\dim_K \operatorname{Frac}(K[z_1, ..., z_n]) \leq \dim_K \operatorname{Frac}\left(\frac{K[x_1, ..., x_n]}{\mathfrak{p}}\right). \tag{2.4}$$

To this end, let $\{e_1, ..., e_m\}$ be a K-basis for $\operatorname{Frac}(\frac{K[x_1,...,x_n]}{\mathfrak{p}})$. Given an element $\frac{a(\mathbf{z})}{b(\mathbf{z})}$ of $\operatorname{Frac}(K[z_1, ..., z_n])$, we know that $b(\mathbf{z}) \notin \mathfrak{p}$ and hence $(a(\mathbf{x}) + \mathfrak{p})/(b(\mathbf{x}) + \mathfrak{p}) = \sum c_i e_i$, where $c_1, ..., c_m \in K$. Since $c_i \in K \subseteq k(z_1, ..., z_n)$, we note that $\psi(c_i) = c_i$. Thus $\frac{a(\mathbf{z})}{b(\mathbf{z})} = \Psi((a(\mathbf{x}) + \mathfrak{p})/(b(\mathbf{x}) + \mathfrak{p})) = \sum c_i \Psi(e_i)$. It follows that $\operatorname{Frac}(K[z_1, ..., z_n])$ is contained in the K-span of $\{\Psi(e_1), ..., \Psi(e_m)\}$, which proves Equation 2.4. □

3. Addressing the main question

Let f be a Drużkowski morphism over an algebraically closed field k of characteristic 0. In summary of Section 2, we have shown that if Conjecture 2.9 is true, then $\deg(f) \leq \dim_k Q_p$, where $f(p) = q$ for closed points p, q. In order to conditionally answer Question 1.2, we will prove a related inequality that will bound the possible size of the fibers of f. We assume that k is an algebraically closed field of characteristic 0. Since $\mathbb{A}^n_k$ is locally of finite type and k is algebraically closed, a point of $\mathbb{A}^n_k$ is k-rational if and only if it is closed by [DG67, I, Corollaire 6.5.3].

Lemma 3.1. *Let* $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ *have* $\mathrm{Jac}(f) \in k^\times$. *Let* $q \in \mathbb{A}^n_k(k)$ *be a closed point with* $f^{-1}(q) \subset \mathbb{A}^n_k(k)$ *consisting only of closed points. Then*

$$\deg(f) \geq \sum_{p \in f^{-1}(q)} \dim_k Q_p.$$

Proof. Since f is quasi-finite by Proposition 2.2, Zariski's Main Theorem implies that f can be factored as

$$f = g \circ i : \mathbb{A}^n_k \xhookrightarrow{i} Z \xrightarrow{g} \mathbb{A}^n_k,$$

where i is an open immersion and g is a finite morphism. If $q \in \mathbb{A}^n_k$ is a closed point, then $|g^{-1}(q)| \leq \deg(g)$ by [Sha13, II.6.3, Theorem 2.28]. Since i is injective, it follows that $|f^{-1}(q)| \leq |g^{-1}(q)|$. Since i is an open immersion, we have $\deg(i) = 1$ and hence $\deg(f) = \deg(g)$. It follows that $\deg(f) \geq |f^{-1}(q)|$. The result of Lemma 2.3, namely that $\dim_k Q_p = 1$ for all $p \in f^{-1}(q)$, finishes the proof. □

Proof of Theorem 1.3. Assume that $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is a Drużkowski morphism. By Lemma 2.3, Corollary 2.5, and Conjecture 2.7 (which follows from Conjecture 2.9), we have that $1 \leq \deg(f) \leq \dim_k Q_p = 1$. Thus $\deg(f) = 1$. By Lemma 3.1, we have

$$\sum_{p \in f^{-1}(q)} \dim_k Q_p \leq 1. \tag{3.1}$$

If q is in the image of f, then $|f^{-1}(q)| > 0$ and hence $|f^{-1}(q)| = 1$ by Equation 3.1 and Lemma 2.3. Thus f is injective on closed points. Finally, the Ax-Grothendieck Theorem and the fact that $f : k^n \to k^n$ is injective imply that $f : k^n \to k^n$ is also surjective. Thus the fiber degree $|f^{-1}(q)|$ is constant on closed points, so f is finite by [DG67, IV, Proposition 18.2.8]. □

4. Application to the Jacobian Conjecture

The Jacobian Conjecture was first posed in 1939 by Keller [Kel39].

Conjecture 4.1 (Jacobian Conjecture)**.** *Let* k *be a field of characteristic 0, and let* $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ *be a morphism with* $\mathrm{Jac}(f) \in k^\times$. *Then* f *has a regular inverse.*

The substantial work of Bass, Connell, and Wright [BCW82] and Drużkowski [Dru83] on this conjecture are the main motivation for this paper. In [BCW82, Theorem 2.1], Bass, Connell, and Wright show that if k is a field of characteristic 0 and $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is a morphism with $\mathrm{Jac}(f) \in k^\times$, then the following are equivalent:

(i) f is invertible, and f^{-1} is regular.

(ii) $f : \mathbb{A}^n_k(k) \to \mathbb{A}^n_k(k)$ is injective.

(iii) f is proper.

(iv) $\deg(f) = 1$.

Note that finite morphisms are proper by [Sta19, Lemma 01WN]. By Theorem 1.3, it follows that Conjecture 2.9 implies that Drużkowski morphisms over an algebraically closed field of characteristic 0 satisfy properties (ii), (iii), and (iv). Each of these implies that such morphisms are invertible, so the Jacobian Conjecture is true for morphisms of cubic linear type over an algebraically closed field of characteristic 0, provided that Conjecture 2.9 is true.

Theorem 4.2. *If k is an algebraically closed field of characteristic 0, then Conjecture 2.9 implies the Jacobian Conjecture for morphisms of cubic linear type.*

In [Dru83, Theorem 3], Drużkowski shows that in order to prove the Jacobian Conjecture, it suffices to prove the Jacobian Conjecture for morphisms $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ of cubic linear type for all n. Paired with Theorem 4.2, it follows that Conjecture 2.9 implies the Jacobian Conjecture over algebraically closed fields of characteristic 0. By [BCW82, (1.1) Remark 4], Conjecture 2.9 implies the Jacobian Conjecture over all characteristic 0 fields.

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