RANDOM SUBSHIFTS OF FINITE TYPE

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Let $X$ be an irreducible shift of finite type (SFT) of positive entropy, and let $B_n(X)$ be its set of words of length $n$. Define a random subset $\omega$ of $B_n(X)$ by independently choosing each word from $B_n(X)$ with some probability $\alpha$. Let $X_\omega$ be the (random) SFT built from the set $\omega$. For each $0 \leq \alpha \leq 1$ and $n$ tending to infinity, we compute the limit of the likelihood that $X_\omega$ is empty, as well as the limiting distribution of entropy for $X_\omega$. For $\alpha$ near 1 and $n$ tending to infinity, we show that the likelihood that $X_\omega$ contains a unique irreducible component of positive entropy converges exponentially to 1. These results are obtained by studying certain sequences of random directed graphs. This version of “random SFT” differs significantly from a previous notion by the same name, which has appeared in the context of random dynamical systems and bundled dynamical systems.

1. Introduction. A shift of finite type (SFT) is a dynamical system defined by finitely many local transition rules. These systems have been studied for their own sake [36, 40], and they have also served as important tools for understanding other dynamical systems [30, 9, 21].

Each SFT can be described as the set of bi-infinite sequences on a finite alphabet that avoid a finite list of words over the alphabet. Thus there are only countably many SFTs up to the naming of letters in an alphabet.

For the sake of simplicity, we state our results in terms of SFTs in the introduction, even though we prove more general results in terms of sequences of directed graphs in the subsequent sections. Let $X$ be a non-empty SFT (for definitions, see Section 2.1). Let $B_n(X)$ be the set of words of length $n$ that appear in $X$. For $\alpha$ in $[0,1]$, let $P_\alpha$ be the probability measure on the power set of $B_n(X)$ given by choosing each word in $B_n(X)$ independently with probability $\alpha$. The case $\alpha = 1/2$
puts uniform measure on the subsets of $B_n(X)$. For notation, let $\Omega_n$ be the power set of $B_n(X)$. To each subset $\omega$ of $B_n(X)$, we associate the SFT $X_\omega$ consisting of all points $x$ in $X$ such that each word of length $n$ in $x$ is contained in $\omega$. With this association, we view $P_\alpha$ as a probability measure on the SFTs $X_\omega$ that can be built out of the subsets of $B_n(X)$. Briefly, if $X$ has entropy $h(X) = \log \lambda > 0$ and $n$ is large, then a typical random SFT $X_\omega$ is built from about $\alpha \lambda^n$ words, an $\alpha$ fraction of all the words in $B_n(X)$, but not all of these words will occur in any point in $X_\omega$.

Our main results can be stated as follows. Let $\zeta_X(t)$ denote the Artin-Mazur zeta function of $X$ (see Definition 2.11). The first theorem deals with the likelihood that a randomly chosen SFT is empty.

**Theorem 1.1.** Let $X$ be a non-empty SFT with entropy $h(X) = \log \lambda$. Let $E_n \subset \Omega_n$ be the event that $X_\omega$ is empty. Then for $\alpha$ in $[0, 1]$,

$$
\lim_{n \to \infty} P_\alpha(E_n) = \begin{cases} 
(\zeta_X(\alpha))^{-1}, & \text{if } \alpha \in [0, 1/\lambda) \\
0, & \text{if } \alpha \in [1/\lambda, 1], 
\end{cases}
$$

Thus when $\alpha$ is in $[0, 1/\lambda)$, there is an asymptotically positive probability of emptiness. The next theorem gives more information about what happens when $\alpha$ lies in $[0, 1/\lambda)$.

**Theorem 1.2.** Let $X$ be a non-empty SFT with entropy $h(X) = \log \lambda$. Let $Z_n \subset \Omega_n$ be the event that $X_\omega$ has zero entropy, and let $I_n$ be the random variable on $\Omega_n$ which is the number of irreducible components of $X_\omega$. Then for $0 \leq \alpha < 1/\lambda$,

1. $\lim_{n \to \infty} P_\alpha(Z_n) = 1$;
2. the sequence $(I_n)$ converges in distribution to the random variable $I_\infty$ such that $P(I_\infty = 0) = (\zeta_X(\alpha))^{-1}$ and for $k \geq 1$,

$$
P(I_\infty = k) = (\zeta_X(\alpha))^{-1} \sum_{S \subseteq \mathbb{N}} \prod_{|S| = k} \frac{\alpha^{|\gamma_s|}}{1 - \alpha^{|\gamma_s|}},
$$

where $\{\gamma_i\}_{i=1}^\infty$ is an enumeration of the periodic orbits in $X$;
3. the random variable $I_\infty$ has exponentially decreasing tail and therefore finite moments of all orders.

Our next result describes the entropy of the typical random SFT when $\alpha$ lies in $(1/\lambda, 1]$. 


Theorem 1.3. Let $X$ be an SFT with positive entropy $h(X) = \log \lambda$. Then for $1/\lambda < \alpha \leq 1$ and $\epsilon > 0$, 
\[ \lim_{n \to \infty} P_{\alpha}(|h(X_\omega) - \log(\alpha \lambda)| \geq \epsilon) = 0, \]
and the convergence to this limit is exponential in $n$.

Finally, we have a result concerning the likelihood that a random SFT will have a unique irreducible component of positive entropy when $\alpha$ is near 1.

Theorem 1.4. Let $X$ be an irreducible SFT with positive entropy $h(X) = \log \lambda$. Let $W_n \subset \Omega_n$ be the event that $X_\omega$ has a unique irreducible component $C$ of positive entropy and $C$ has the same period as $X$. Then there exists $c > 0$ such that for $1 - c < \alpha \leq 1$,
\[ \lim_{n \to \infty} P_{\alpha}(W_n) = 1; \]
Furthermore, the convergence to this limit is exponential in $n$.

There have been studies of other objects called random subshifts of finite type in the literature [8, 7, 25, 31–35], but the objects studied here are rather different in nature. The present work is more closely related to perturbations of SFTs, which have already appeared in works by Lind [38] in dimension 1 and by Pavlov [47] in higher dimensions. In those works, the main results establish good uniform bounds for the entropy of an SFT obtained by removing any single word of length $n$ from a sufficiently mixing SFT as $n$ tends to infinity. Random SFTs may also be interpreted as dynamical systems with holes [12, 11, 13–15, 18, 17, 19, 20, 41, 42], in which case the words of length $n$ in $X$ that are forbidden in the random SFT $X_\omega$ are viewed as (random) holes in the original system $X$. The question of whether an SFT defined by a set of forbidden words is empty has been studied in formal language theory and automata theory, and in that context it amounts to asking whether the set of forbidden words is unavoidable [4, 10, 29]. Also, the random SFTs considered here can be viewed as specific instances of random matrices (see [3, 43]) or random graphs (see [2, 5, 22–24, 28, 27, 44]), and the concept of directed percolation on finite graphs has appeared in the physics literature in the context of directed networks [46, 49]. To the best of our knowledge, the specific considerations that arise for our random SFTs seem not to have appeared in any of this wider literature.
The paper is organized as follows. Section 2 contains the necessary background and notation, as well as some preliminary lemmas. The reader familiar with SFTs and directed graphs may prefer to skip Sections 2.1 and 2.2, referring back as necessary. In Section 3 we discuss the likelihood that a random SFT is empty, and in particular we prove Theorem 1.1. The remainder of the main results are split into two sections according to two cases: $\alpha \in [0, 1/\lambda)$ and $\alpha \in (1/\lambda, 1]$. The case $\alpha \in [0, 1/\lambda)$ is treated in Section 4, and the case $\alpha \in (1/\lambda, 1]$ is addressed in Section 5. Section 6 discusses some corollaries of the main results.

2. Preliminaries.

2.1. Shifts of finite type and their presentations. For a detailed treatment of SFTs and their presentations, see [40]. In this section we describe three ways to present an SFT: with a finite list of forbidden words over a finite alphabet, with a finite, directed graph, or with a square, non-negative integer matrix.

Let $A$ be a finite set, which we will call the alphabet. An element $b \in A^n$ is called a word of length $n$. Let $\Sigma = A^Z$, endowed with the product topology induced by the discrete topology on $A$. Then $\Sigma$ is a compact metrizable space, which is called the full shift on $A$. Let $\sigma : \Sigma \to \Sigma$ be the left shift, i.e. for $x = (x_i)$ in $\Sigma$, let $(\sigma(x))_i = x_{i+1}$. With this definition $\sigma$ is a homeomorphism of $\Sigma$.

A subset $X$ of $\Sigma$ is called shift-invariant if $\sigma(X) = X$. A closed, shift-invariant subset of $\Sigma$ is called a subshift. For any subshift $X$, the language $B(X)$ of $X$ is the collection of all finite words (blocks) that appear in some sequence $x$ in $X$. Note that $B(X) = \bigcup B_n(X)$, where $B_n(X)$ is the set of all words of length $n$ that appear in some sequence $x$ in $X$. (By convention we set $B_0(X) = \{\epsilon\}$, where $\epsilon$ denotes the empty word). Given a set $F$ of words on $A$, we may define a subshift $X(F)$ as the set of sequences $x$ in $\Sigma$ such that no word in $F$ appears in $x$. One may check that this procedure indeed defines a subshift. If $X$ is a subshift and there exists a finite set of words $F = \{F_1, \ldots, F_k\}$ such that $X = X(F)$, then $X$ is called a subshift of finite type (SFT).

The natural notion of isomorphism for SFTs is called conjugacy. Two SFTs $X$ and $Y$ are conjugate, written $X \cong Y$, if there exists a homeomorphism $\phi : X \to Y$ such that $\phi \circ \sigma = \sigma \circ \phi$. An SFT $X$ is irreducible if for every two non-empty open sets $U$ and $V$ and every $N$ in $\mathbb{N}$, there exists $n \geq N$ such that $\sigma^n(U) \cap V \neq \emptyset$. An SFT $X$ is mixing if for every two non-empty open sets
U and V in X, there exists $n_0$ in $\mathbb{N}$ such that for all $n \geq n_0$, we have $\sigma^n(U) \cap V \neq \emptyset$. Mixing and irreducibility are conjugacy-invariant. We now define the higher block presentations of an SFT.

**Definition 2.1.** Let $X$ be an SFT. The $n$-block presentation of $X$, denoted $X^{[n]}$ is defined as follows. The alphabet for $X^{[n]}$ is $B_n(X)$. We define the code $\phi_n : X \to B_n(X)^\mathbb{Z}$ by the equation

$$\phi_n(x)_i = x[i, i + n - 1],$$

for all $x$ in $X$. Then $X^{[n]} = \phi_n(X)$. For all $n \geq 1$, we have that $X^{[n]} \cong X$, where the conjugacy is given by $\phi_n$.

**Definition 2.2.** The entropy of an SFT $X$ is defined as $h(X) = \lim_{n \to \infty} \frac{1}{n} \log |B_n(X)|$.

Alternatively, one may define SFTs in terms of finite directed graphs. A directed graph $G = (V, E)$ consists of a set of vertices $V$ and a set of edges $E$ such that for each edge $e \in E$, there is a unique initial vertex, $i(e) \in V$, and a unique terminal vertex, $t(e) \in V$. We view the edge $e$ as going from $i(e)$ to $t(e)$. We allow self-loops, but for the sake of convenience we assume (without loss of generality for our considerations) that there are no multiple edges. In this paper, we make the standing convention that “graph” means directed graph. We will collect our standing assumptions in Standing Assumptions 2.21.

**Definition 2.3.** Given a directed graph $G$, we define the edge shift $X_G$ to be the set of all bi-infinite (oriented) walks on $G$, i.e. $X_G = \{x \in E^\mathbb{Z} : t(x_j) = i(x_{j+1}) \text{ for all } j \in \mathbb{Z}\}$.

Any edge shift is an SFT (trivially). Let us show that any SFT is conjugate to an edge shift. If $X = X(\mathcal{F})$ is an SFT and $\mathcal{F}$ is a finite set of forbidden words, then $X \cong X_G$, where $G = (V, E)$ is defined as follows. Let $n_0 = \max\{|F| : F \in \mathcal{F}\}$. Then let $V = B_{n_0 - 1}(X)$ and $E = B_{n_0}(X)$. Further, for any edge $e \in B_{n_0}(X)$, we let $i(e) = e[1, n_0 - 1]$ and $t(e) = e[2, n_0]$. The same construction works with $n$ in place of $n_0$ for any $n \geq n_0$.

If $G$ is a graph such that $X \cong X_G$, we say that $X_G$ is an edge presentation of $X$, or sometimes just a presentation of $X$. The adjacency matrix $A$ of a directed graph $G$ may be defined as follows. Fix an enumeration of the vertices in $G$. Then let $A_{ie}$ be the number of distinct edges $e$ in $G$ such that $i(e) = v_k$ and $t(e) = v_{k'}$. A square, non-negative integral matrix $A$ is irreducible if for each pair $i, j$ and each $N$, there exists $n > N$ such that $(A^n)_{ij} > 0$. A matrix $A$ is non-degenerate if it
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has no zero row and no zero column. If $A$ is non-degenerate, then the edge shift $X_G$ is irreducible if and only if $A$ is irreducible. Also, if $A$ is non-degenerate, then the edge shift $X_G$ is mixing if and only if there exists $n_0$ such that for all $n \geq n_0$ and all pairs $i, j$, it holds that $(A^n)_{ij} > 0$. A matrix is \textbf{primitive} if it satisfies the latter property. A path in $G$ is a finite sequence $\{e_j\}_{j=1}^n$ of edges such that $t(e_j) = i(e_{j+1})$ for $j = 1, \ldots, n - 1$. If $b = b_1 \ldots b_n$ is a path in $G$, we say that $b$ goes from vertex $i(b_1)$ to vertex $t(b_n)$. We denote by $B_k(G)$ the set of paths of length $k$ in $G$. By convention, we set $B_0(G) = V$.

\textbf{Definition 2.4.} For a path $b$ in $G$, let $V(b)$ and $E(b)$ be the set of vertices and the set of edges traversed by $b$, respectively.

\textbf{Definition 2.5.} Let $X$ be an SFT. An \textbf{irreducible component} $Y$ of $X$ is a non-empty, maximal SFT contained in $X$ such that $Y$ is irreducible. Let $G$ be a graph. An \textbf{irreducible component} $C$ of $G$ is a non-empty, maximal subgraph of $G$ such that the adjacency matrix of $C$ is irreducible. The reader should be advised that in some papers the definition of irreducible component includes trivial components (a single vertex with no edges adjacent to it), but the definition given here does not include trivial components.

\textbf{Definition 2.6.} Let $G$ be a finite, directed graph. For $n \geq 1$, define $G^{[n]} = (V^{[n]}, E^{[n]})$, the $n$-\textbf{block graph} of $G$, as follows. Let $V^{[n]} = B_{n-1}(G)$ and $E^{[n]} = B_n(G)$, such that if $e \in E^{[n]}$, then $i(e) = e[1, n - 1]$ and $t(e) = e[2, n]$. Note that $G^{[1]} = G$.

If $X = X_G$ for some graph $G$, then it follows immediately from the definitions that $X^{[n]} = X_G^{[n]}$.

\textbf{Definition 2.7.} Let $G = (V, E)$ be a graph. For $p$ in $\mathbb{N}$, we define the $p$-th power graph, $G^p = (V^p, E^p)$ as follows. Let $V^p = V$ and $E^p = B_p(G)$. If $b = b_1 \ldots b_p$ is an edge in $G^p$, then we let $i(b) = i(b_1)$ and $t(b) = t(b_1)$.

\textbf{Definition 2.8.} Let $G = (V, E)$ be a graph. Define the transpose graph, $G^T = (V^T, E^T)$, as follows. Let $V^T = V$ and $E^T = E$, where an edge $e$ in $G^T$ goes from $t(e)$ to $i(e)$. In other words, the transpose graph is just the graph formed by reversing the direction of all the edges in $G$.

Given a square, non-negative, integral matrix $A$, one may also define an SFT $X_A$ as follows. Let
$G$ be a directed graph whose adjacency matrix is exactly $A$ (such a graph always exists). Then let $X_A$ be the edge shift defined by $G$.

Recall the following basic facts (which may be found in [40]). For an SFT $X$, we have $h(X) = \inf_n \frac{1}{n} \log |B_n(X)|$. If $X$ is a non-empty SFT and $X = X_A$ for a square, non-negative integral matrix $A$, then $h(X) = \log \lambda$, where $\lambda$ is the spectral radius of $A$. By the Perron-Frobenius Theorem, if $A$ is non-negative and irreducible, then there exists a strictly positive (column) vector $v$ such that $Av = \lambda v$, and there exists a strictly positive (row) vector $w$ such that $wA = \lambda w$. Furthermore, $v$ and $w$ are each unique up to a positive scalar.

**Definition 2.9.** For any non-negative integer matrix $A$, let $\lambda_A$ be the spectral radius of $A$, and let $\chi_A$ be the characteristic polynomial of $A$. Then let $\text{Sp}_x(A)$ be the non-zero spectrum of the matrix $A$, which is defined as the multiset of non-zero roots of $\chi_A$ listed according to their multiplicity. If $A$ is the adjacency matrix of the graph $G$, we define $\lambda_G = \lambda_A$ and $\text{Sp}_x(G) = \text{Sp}_x(A)$.

If $X_A \cong X_B$ for two non-negative integral matrices $A$ and $B$, then $\text{Sp}_x(A) = \text{Sp}_x(B)$. Also, if $A$ is primitive, then $\max \{|\beta| : \beta \in \text{Sp}_x(A) \setminus \{\lambda_A\}\} < \lambda_A$. Finally, if $A$ is irreducible, then there exists a unique $\sigma$-invariant Borel probability measure $\mu$ on $X_A$ of maximal entropy. Let us describe some basic properties of $\mu$. We associate a word $b = b_1 \ldots b_k$ in $X$ to the cylinder set $C_b = \{x \in X : x[1,k] = b\}$. In this way we interpret the measure of words in $\mathcal{B}(X)$ as the measure of the corresponding cylinder set. Let $v$ be a positive right eigenvector of $A$ and $w$ a positive left eigenvector of $A$, and suppose they are normalized so that $w \cdot v = 1$. Our standing assumption that there are no multiple edges means that that $A_{ij} \leq 1$ for all $i, j$. Then for a vertex $u$ in $V$, we have $\mu(u) = w_u v_u$, and for $b \in B_n(X_A)$, we have that

(2.2) $\mu(b) = w_{i(b_1)} \lambda_A^{-n} v_{t(b_n)}$.

Now we define two objects, the period and the zeta function, which contain combinatorial information about the cycles in a graph $G$ (alternatively, one may refer to the periodic points in an SFT $X$).

**Definition 2.10.** For an SFT $X$, let $\text{per}(X)$ be the greatest common divisor of the sizes of all periodic orbits in $X$. For a graph $G$, let $\text{per}(G)$ be the greatest common divisor of the lengths of all cycles in $G$. 

Definition 2.11. Let $X$ be an SFT and $N_p = |\{x \in X : \sigma^p(x) = x\}|$. Then the Artin-Mazur zeta function of $X$ (see [40]) is, by definition,

$$
\zeta_X(t) = \exp \left( \sum_{p=1}^{\infty} \frac{N_p}{p} t^p \right).
$$

For a graph $G$, let $\zeta_G = \zeta_{X_G}$.

For a graph $G$, note that $|\{x \in X_G : \sigma^p(x) = x\}|$ is the number of cycles of (not necessarily least) period $p$ in $G$, and

$$
\zeta_G(t) = \frac{1}{\det(I-tA)} = \prod_{\lambda \in \Sp(G)} \frac{1}{1-\lambda t}.
$$

Also, $\zeta_G$ has radius of convergence $1/\lambda_G$ and $\lim_{t \to 1/\lambda_G^-} \zeta_G(t) = +\infty$.

2.2. Sequences of graphs under consideration. In this work we consider sequences of graphs $(G_n)$ that grow in some way. A particular example of such a sequence is the sequence of $n$-block graphs of an SFT $X$. Indeed, by taking $(G_n)$ to be such a sequence in Theorems 3.1, 4.2, 5.13, and 5.15, we obtain the theorems stated in the introduction. Generalizing to the graph setting also allows one to consider sequences of graphs presenting SFTs which are conjugate to a fixed SFT $X$, where the sequences need not be the $n$-block sequence for $X$. To indicate the generality of the arguments further, though, we formulate and prove the results for sequences of graphs that do not necessarily present conjugate SFTs. Before we move on to these results, we need to define several notions regarding the manner of growth of the sequence $(G_n)$.

Let $G$ be a finite, directed graph with adjacency matrix $A$. We will have use for the following notations.

Definition 2.12. Let

$$
\text{Per}_p(G) = \{ b \in B_p(G) : i(b_1) = t(b_p) \}, \quad \text{and} \quad \text{Per}(G) = \cup_{p=1}^{\infty} \text{Per}_p(G).
$$

For $b$ in $\text{Per}_p(G)$, let $\theta(b)$ be the set of all paths $c$ in $\text{Per}_p(G)$ such that there exists a natural number $\ell$ such that $c = b_{\tau(1)} \ldots b_{\tau(p)}$, where $\tau$ is the permutation of $\{1, \ldots, k\}$ defined in cycle notation by $(1 \ldots k)$.

Definition 2.13. For each vertex $u$ in $G$, let $d_{\text{out}}(u) = |\{e \in E : i(e) = u\}|$ and $d_{\text{in}}(u) = |\{e \in E : t(e) = u\}|$. Then let

$$
d_{\text{max}}(G) = \max \{ \max(d_{\text{out}}(u), d_{\text{in}}(u)) : u \in V \}.
$$
In order to measure the separation of periodic orbits in $G$, we make the following definition.

**Definition 2.14.** Let

$$z(G) = \max \{ n \geq 0 : \forall b,c \in \bigcup_{p=1}^{n} \text{Per}_{p}(G) \text{ with } c \notin \theta(b), V(b) \cap V(c) = \emptyset \},$$

where $V(b)$ is the set of vertices traversed by the path $b$.

As a measure of the size of $G$, we consider the following quantity.

**Definition 2.15.** If $A$ has spectral radius $\lambda > 1$, then let

$$m(G) = \lceil \log_{\lambda} |V| \rceil.$$

To measure a range for uniqueness of paths in $G$, we make the following definitions.

**Definition 2.16.** Let

$$U_{1}(G) = \sup \{ n : \forall i,j \text{ it holds that } (A^{n})_{ij} \leq 1 \}$$
$$U_{2}(G) = \sup \{ n : \forall u \in V \text{ and } 1 \leq s < t \leq n, \ |\{ b \in B_{t}(X) : i(b_{1}) = u, b_{s} = b_{t} \}| \leq 1 \}$$
$$U(G) = \min(U_{1}(G), U_{2}(G)).$$

We use the transition length as a type of diameter of $G$.

**Definition 2.17.** Let

$$R(G) = \inf \{ n : \forall i,j, \exists k \leq n, (A^{k})_{ij} > 0 \}.$$

Here we briefly recall the notion of the weighted Cheeger constant of an irreducible, directed graph $G$. The weighted Cheeger constant was defined and studied in [16]. Let $\mu$ be the measure of maximal entropy of $X_{G}$, and let $F : E \to [0,1]$ be given by $F(e) = \mu(e)$. For any vertex $v$ in $V$, let $F(v) = \sum_{i(e) = v} F(e) = \sum_{t(e) = v} F(e)$. Then for any subset of vertices $S \subseteq V$, let $F(S) = \sum_{v \in S} F(v)$, and for any two subsets $S,T \subseteq V$, let

$$F(S,T) = \sum_{i(e) \in S \atop t(e) \in T} F(e).$$

In general $F(S,T)$ is not symmetric in $S$ and $T$ since $G$ is directed. Let $E(S,T)$ be the set of edges $e$ in $G$ such that $i(e) \in S$ and $t(e) \in T$. Let $\overline{S} = V \setminus S$. 

Definition 2.18. The weighted Cheeger constant of \( G \) is defined as

\[
c_w(G) = \inf_{\emptyset \subseteq S \subseteq V} \frac{F(S, \overline{S})}{\min(F(S), F(\overline{S}))},
\]

and the unweighted Cheeger constant of \( G \) is defined as

\[
c(G) = \inf_{0 < |S| \leq |V|/2} \frac{|E(S, \overline{S})|}{|S|}.
\]

Definition 2.19. We say that \( G \) is a directed \( b \)-expander graph if \( c(G) \geq b \). Also, a sequence of directed graphs \( (G_n) \) is a uniform expander sequence, if there exists a \( b > 0 \) such that \( G_n \) is a directed \( b \)-expander graph for each \( n \).

We will also have use for the following quantity related to the spectral gap of \( G \).

Definition 2.20. Let \( g(G) = \min \{ 1 - \frac{\| \lambda_i \|}{\lambda_i} : \lambda_i \in \text{Sp}_x(G) \setminus \{ \lambda \} \} \).

We make the following standing assumptions, even though some of the statements we make may hold when these restrictions are relaxed. In particular, Theorems 3.1 and 4.2 do not require that \( A_n \) is irreducible, nor do they require that \( \lambda > 1 \) (see Remark 6.1).

Standing Assumptions 2.21. Recall that “graph” means directed graph. Let \( (G_n) \) be a sequence of graphs with associated sequence of adjacency matrices \( (A_n) \). Unless otherwise stated, we will make the following assumptions:

- for each \( n \), each entry of \( A_n \) is contained in \( \{0, 1\} \);
- each \( A_n \) is irreducible;
- for each \( n \), \( \text{Sp}_x(A_n) = \text{Sp}_x(A_1) \);
- \( \lambda := \lambda_{A_1} > 1 \);
- \( \lim_n m(G_n) = \infty \).

Remark 2.22. Note that \( |\text{Per}_p(G_n)| = \text{tr}(A_n^p) \), which depends only on \( \text{Sp}_x(A_n) \) and \( p \). Therefore the standing assumptions imply that \( |\text{Per}_p(G_n)| \) does not depend on \( n \), and therefore \( \text{per}(G_n) \) and \( \zeta_{G_n} \) do not depend on \( n \).

Additional conditions that we place on sequences of graphs will come from the following list. (Different theorems will require different assumptions, but the sequence of \( n \)-block graphs of an
irreducible graph with spectral radius greater than 1 will satisfy conditions (C1)-(C8) below by Proposition 2.29.)

**Definition 2.23.** We define the following conditions on a sequence of graphs \((G_n)\) with sequence of adjacency matrices \((A_n)\):

(C1) there exists \(\Delta > 0\) such that \(d_{\text{max}}(G_n) \leq \Delta\) for all \(n\) (bounded degree).

(C2) \(z(G_n)\) tends to infinity as \(n\) tends to infinity (separation of periodic points);

(C3) there exists \(C > 0\) such that \(z(G_n) \geq Cm(G_n)\) for all \(n\) (fast separation of periodic points);

(C4) there exists \(C > 0\) such that \(U(G_n) \geq m(G_n) - C\) for all \(n\) (local uniqueness of paths);

(C5) there exists \(C > 0\) such that \(R(G_n) \leq m(G_n) + C\) for all \(n\) (small diameter);

(C6) there exists \(K > 0\) such that \(\max_{u \in V_n} \mu(u) \leq K \min_{u \in V_n} \mu(u)\) for all \(n\) (bounded distortion of vertices) and \(\max_{e \in E_n} \mu(e) \leq K \min_{e \in E_n} \mu(e)\) for all \(n\) (bounded distortion of edges);

(C7) there exists \(K > 0\) such that \(\max_i w^n_i \leq K \min_i w^n_i\) and \(\max_i v^n_i \leq K \min_i v^n_i\) for all \(n\), where \(w^n\) is a positive left eigenvector of \(A_n\) and \(v^n\) is a positive right eigenvector of \(A_n\) (bounded distortion of weights);

(C8) \((G_n)\) is a uniform expander sequence, and \((G^n_T)\) is a uniform expander sequence (forward/backward expansion).

Now we establish some lemmas, which will be used in the subsequent sections.

**Lemma 2.24.** Let \((G_n)\) be a sequence of graphs satisfying the Standing Assumptions 2.21. Then (C7) implies (C1) and (C6) for both \((G_n)\) and \((G^n_T)\).

**Proof.** First note that if (C7) holds for \((G_n)\), then it also holds for \((G^n_T)\) since a positive left eigenvector for \(A^n_T\) is given by \((v^n)^T\) and a positive right eigenvector for \(A^n_T\) is given by \((w^n)^T\). Therefore we only need to show that (C7) for \((G_n)\) implies (C1) and (C6) for \((G_n)\) (since the same argument will apply to \((G^n_T)\)).

Let \(w^n\) and \(v^n\) be positive left and right eigenvectors for \(A_n\), respectively, and assume that \(w^n \cdot v^n = 1\). Recall with this normalization, if \(u\) is a vertex in \(V_n\), then \(\mu(u) = w^n_u v^n_u\). Then
condition (C7) implies that there exists $K > 0$ such that for all $n$,

$$\max_u \mu(u) \leq \max_u w^u_n \max_u v^u_n \leq K^2 \min_u w^u_n \min_u v^u_n \leq K^2 \min_u w^u_n v^u_n = K^2 \min_u \mu(u).$$

Similarly, (C7) implies that there exists $K' > 0$ such that for all $n$, we have that $\max_{e \in E_n} \mu(e) \leq K' \min_{e \in E_n} \mu(e)$ (recall that $\mu(e) = w^n_{i(e)} \lambda^{-1} v^n_{t(e)}$). Thus (C7) implies (C6).

Note that for $e$ in $E_n$, we have that $\mu(e|i(e)) = w^n_{i(e)} \lambda^{-1} v^n_{t(e)} = v^n_{t(e)} \lambda v^n_{i(e)}$. Then condition (C7) implies that there exists a uniform constant $K > 0$ such that $\mu(e|i(e)) \geq K^{-1}$ for all $n$ and all $e$ in $E_n$. We also have that

$$\mu(u) = \sum_{e: i(e) = u} \mu(e) \geq \sum_{e: i(e) = u} K^{-1} \mu(u) = |\{e : i(e) = u\}| K^{-1} \mu(u).$$

Since $G_n$ is irreducible (by Standing Assumptions 2.21), we know that $\mu(u) > 0$, and therefore we have that for any $n$, and any $u$ in $V_n$,

$$|\{e \in E_n : i(e) = u\}| \leq K,$$

which implies that $\max_u d_{out}(u)$ is uniformly bounded in $n$. A similar argument shows that $\max_u d_{in}(u)$ is uniformly bounded in $n$, which shows that $d_{\max}(G_n)$ is uniformly bounded in $n$ and gives (C1).

Recall that for a graph $G$, the quantities $g(G)$ and $c_w(G)$ were defined in Definitions 2.20 and 2.18, respectively.

**Lemma 2.25.** Let $G$ be a graph with primitive adjacency matrix $A$. Then it holds that $c_w(G) \geq \frac{1}{2} g$.

**Proof.** This lemma is a consequence of [16, Theorems 4.3 and 5.1], as we now explain. Since $A$ is primitive, there exists a strictly positive vector $v$ and $\lambda \geq 1$ such that $Av = \lambda v$. Let $P$ be the stochastic matrix defined by $P_{ij} = \frac{A_{ij} v_j}{\lambda v_i}$. Then $P$ is the transition probability matrix corresponding to the random walk defined by the measure of maximal entropy $\mu$ on $X_G$. We have that $\text{Sp}_x(P) = \frac{1}{\lambda} \text{Sp} \times (A)$. Given such a transition probability matrix, Chung defines a Laplacian
and proves [16, Theorem 4.3] that the smallest non-zero eigenvalue of \( L \), denoted \( \lambda_1 \), satisfies the following inequality:

\[
(2.3) \quad \min \left\{ 1 - |\rho| : \rho \in \text{Sp}_x(P) \setminus \{1\} \right\} \leq \lambda_1.
\]

We remark that the left-hand side of the inequality in [16, Theorem 4.3] is equal to the left-hand side of Equation (2.3) since \( A \) is primitive (not just irreducible). Note that the left-hand side of Equation (2.3) equals \( g(G) \), as defined in Definition 2.20. After defining the weighted Cheeger constant (as in Definition 2.18), Chung proves [16, Theorem 5.1] that

\[
(2.4) \quad c_w(G) \geq \frac{1}{2} \lambda_1.
\]

Combining the inequalities in Equations (2.3) and (2.4), we obtain the desired inequality. \( \square \)

Recall that the \( p \)-th power graph was defined in Definition 2.7.

**Lemma 2.26.** Let \( G \) be a graph with irreducible adjacency matrix. Let \( p = \text{per}(G) \). Let \( G^{p,0} \) be an irreducible component of \( G^p \), the \( p \)-th power graph of \( G \). Let \( g = g(G^{p,0}) \) (which does not depend on the choice of irreducible component in \( G^p \)). Then there exists \( b > 0 \), depending only on \( g \) and \( p \), such that \( c_w(G) \geq b \).

**Proof.** Let \( G \), \( p \), and \( g \) be as in the statement of the lemma. If \( p = 1 \), then Lemma 2.25 immediately gives the result. Now we assume \( p \geq 2 \). The fact that \( G \) is irreducible and \( \text{per}(G) = p \) implies that there is a partition of the vertices into \( p \) non-empty subsets, \( V = \cup_{j=0}^{p-1} V^j \), such that for each edge \( e \) with \( i(e) \in V^j \), it holds that \( t(e) \in V^{j+1} \), where the superscripts are taken modulo \( p \). Let \( X = X_G \) (Definition 2.3), and for each \( j = 0, \ldots, p-1 \), let \( X_j = \{ x \in X : i(x) \in V^j \} \). For any set \( S \subset V \) with \( 0 < |S| < |V| \) and \( j = 0, \ldots, p-1 \), define

\[
C_S = \{ x \in X : i(x) \in S \}, \quad \overline{C}_S = X_G \setminus C_S,
\]

\[
C_S^j = X_j \cap C_S, \quad \text{and} \quad \overline{C}_S^j = X_j \cap \overline{C}_S.
\]

Recall that we denote by \( \mu \) the measure of maximal entropy on \( X \), and we may write \( c_w(G) \) as follows:

\[
c_w(G) = \inf_{\emptyset \subset S \subset V} \frac{\mu(C_S \cap \sigma^{-1} \overline{C}_S)}{\min(\mu(C_S), \mu(\overline{C}_S))}
= \inf_{\emptyset \subset S \subset V} \max \left( \frac{\mu(C_S \cap \sigma^{-1} \overline{C}_S)}{\mu(C_S)}, \frac{\mu(C_S \cap \sigma^{-1} \overline{C}_S)}{\mu(\overline{C}_S)} \right).
\]
We also use the following notation:

\[(2.5) \quad r_i = \frac{\mu(C_S^i)}{\mu(C_S)}, \quad \tau_i = \frac{\mu(C_S)}{\mu(C_S^i)}.\]

Let us establish a useful inequality. For \(i = 0, \ldots, p-1\) and \(1 \leq \ell \leq p\), note that each point \(x\) in \(C_S^i \cap \sigma^{-\ell}C_S^{i+\ell}\) also lies in \(C_S^j \cap \sigma^{-1}C_S^{j+1}\) for \(j = \min\{k > 0 : \sigma^k x \not\in C_S\}\). Thus

\[(2.6) \quad \mu\left(C_S^i \cap \sigma^{-\ell}C_S^{i+\ell}\right) \leq \sum_{j=0}^{p-1} \mu\left(C_S^j \cap \sigma^{-1}C_S^{j+1}\right) = \mu\left(C_S \cap \sigma^{-1}C_S\right).\]

To complete the proof, we will find \(b > 0\) in terms of \(g\) and \(p\) so that for \(S \subseteq V\) with \(0 < |S| < |V|\), we have that

\[(2.7) \quad b \leq \max\left(\frac{\mu(C_S \cap \sigma^{-1}C_S)}{\mu(C_S)}, \frac{\mu(C_S \cap \sigma^{-1}C_S)}{\mu(C_S)}\right).\]

The bound \(b\) will be the minimum of four bounds, each coming from a particular type of set \(S \subseteq V\).

Consider the following conditions on the set \(S\), which we will use to break our proof into cases:

(I) there exists \(i \in \{0, \ldots, p-1\}\) such that \(\mu(C_S^i) \in \{0, 1\}\);

(II) \(\mu(C_S^i) \leq 1/2p\) for each \(i\), or \(\mu(C_S^i) \geq 1/2p\) for each \(i\);

(III) \(1/4p \leq \mu(C_S^i) \leq 3/4p\) for each \(i\).

Now we consider cases.

**Case:** (I) holds, *i.e.* there exists \(i \in \{0, \ldots, p-1\}\) such that \(\mu(C_S^i) \in \{0, 1\}\). Assume first that \(\mu(C_S^i) = 0\), which implies that \(\mu(C_S^i) = \mu(X_i)\). Choose \(j\) such that \(\mu(C_S^j) = \max_k \mu(C_S^k)\), and finally choose \(1 \leq \ell \leq p\) such that \(j + \ell = i \pmod{p}\). Then by inequality (2.6) and the shift-invariance of \(\mu\), we have that

\[\frac{\mu(C_S \cap \sigma^{-1}C_S)}{\mu(C_S)} \geq \frac{\mu(C_S^j \cap \sigma^{-\ell}C_S^{j+\ell})}{\mu(C_S)} \geq \frac{\mu(C_S \cap \sigma^{-\ell}X_{j+\ell})}{p \max_k \mu(C_S^k)} = \frac{\mu(C_S^j)}{p \mu(C_S^j)} = \frac{1}{p}.\]

Now assume \(\mu(C_S^i) = 1\). Choose \(j\) such that \(\mu(C_S^j) = \max_k \mu(C_S^k)\), and finally choose \(1 \leq \ell \leq p\) such that \(i + \ell = j \pmod{p}\). Then by (2.6) and the shift-invariance of \(\mu\),

\[\frac{\mu(C_S \cap \sigma^{-1}C_S)}{\mu(C_S)} \geq \frac{\mu(C_S \cap \sigma^{-\ell}C_S^{i+\ell})}{\mu(C_S)} \geq \frac{\mu(X_i \cap \sigma^{-\ell}C_S^j)}{p \max_k \mu(C_S^k)} = \frac{\mu(C_S^j)}{p \mu(C_S^j)} = \frac{1}{p}.\]

Let \(b_1 = 1/p\), and note that if condition (I) holds, then the inequality in (2.7) holds with \(b_1\) in place of \(b\).
Case: (I) does not hold, but (II) holds, i.e. $0 < \mu(C^i_S) \leq 1/2p$ for all $i$, or $1 > \mu(C^i_S) \geq 1/2p$ for all $i$. Assume first that $0 < \mu(C^i_S) \leq 1/2p$ for all $i$. Since $\sum_i r_i = 1$ and $r_i \geq 0$ for all $i$, there exists $j$ such that $r_j \geq 1/p$. Then by (2.6) and the definition of $r_i$ in (2.5),
\[
\frac{\mu(C_S \cap \sigma^{-1}C_S)}{\mu(C_S)} \geq \frac{\mu(C^j_S \cap \sigma^{-p}C^j_S)}{\mu(C^j_S)} = r_j \frac{\mu(C^j_S \cap \sigma^{-p}C^j_S)}{\mu(C^j_S)} \geq \frac{1}{p} \frac{\mu(C^j_S \cap \sigma^{-p}C^j_S)}{\mu(C^j_S)}.
\]

Let $G^{p-j}$ be the irreducible component of $G^p$ with vertex set $V^j$. Then $G^{p-j}$ has primitive adjacency matrix, and $g = g(G^{p-j}) > 0$. Lemma 2.25 gives that $c_w(G^{p-j}) \geq \frac{1}{2}g$. Since $\mu(C^j_S) \leq 1/2p$ and $\mu(C^j_S) \mu(C^j_S) = \mu(X_j) = 1/p$, we have that $\min(\mu(C^j_S), \mu(C^j_S)) = \mu(C^j_S)$, and thus
\[
\frac{\mu(C^j_S \cap \sigma^{-p}C_S)}{\mu(C^j_S)} \geq c_w(G^{p-j}) \geq \frac{1}{2}g.
\]

Let $b_2 = g/2p$. We have shown that for $S$ such that $\mu(C^i_S) \leq 1/2p$ for each $i$, the inequality in (2.7) holds with $b_2$ in place of $b$. For $S$ such that $1 > \mu(C^i_S) \geq 1/2p$ for each $i$, choose $j$ such that $\tau_j \geq 1/p$. Then an analogous argument gives that the inequality in (2.7) holds with $b_2$ in place of $b$.

Case: (III) holds, i.e. $1/4p \leq \mu(C^i_S) \leq 3/4p$ for all $i$. A simple calculation yields that $r_i \geq 1/3p$ and $\tau_i \geq 1/3p$ for each $i$. Using (2.6), we see that for each $j$,
\[
\frac{\mu(C_S \cap \sigma^{-1}C_S)}{\mu(C_S)} \geq \frac{\mu(C^j_S \cap \sigma^{-p}C^j_S)}{\mu(C^j_S)} = r_j \frac{\mu(C^j_S \cap \sigma^{-p}C^j_S)}{\mu(C^j_S)} \geq \frac{1}{3p} \frac{\mu(C^j_S \cap \sigma^{-p}C^j_S)}{\mu(C^j_S)},
\]
and
\[
\frac{\mu(C_S \cap \sigma^{-1}C_S)}{\mu(C_S)} \geq \frac{\mu(C^j_S \cap \sigma^{-p}C^j_S)}{\mu(C^j_S)} = r_j \frac{\mu(C^j_S \cap \sigma^{-p}C^j_S)}{\mu(C^j_S)} \geq \frac{1}{3p} \frac{\mu(C^j_S \cap \sigma^{-p}C^j_S)}{\mu(C^j_S)}.
\]

Then since $G^{p-j}$ has primitive adjacency matrix, Lemma 2.25 and inequalities (2.8) and (2.9) give that the inequality in (2.7) holds with $b_3 := g/6p$ in place of $b$.

Case: each of (I), (II), and (III) does not hold, i.e. we assume that $S$ is such that $0 < \mu(C^i_S) < 1$ for each $i$, there exists $i_1$ and $i_2$ such that $\mu(C^{i_1}_S) > 1/2p$ and $\mu(C^{i_2}_S) < 1/2p$, and there exists $i_3$ such that either $\mu(C^{i_3}_S) < 1/4p$ or $\mu(C^{i_3}_S) > 3/4p$. Suppose first that $\mu(C^{i_3}_S) < 1/4p$. Choose $j$ such that $\mu(C^j_S) = \max_k \mu(C^k_S)$, and choose $1 \leq \ell \leq p$ such that $j + \ell = i_3 \pmod{p}$. Calculation gives that $\mu(C^{i_3}_S) < \frac{1}{2} \mu(C^j_S)$. Then by (2.6) and the shift-invariance of $\mu$,
\[
\frac{\mu(C_S \cap \sigma^{-1}C_S)}{\mu(C_S)} \geq \frac{\mu(C^j_S \cap \sigma^{-\ell}C^j_S)}{p \mu(C^j_S)} \geq \frac{\mu(C^j_S) - \mu(C^{i_3}_S)}{p \mu(C^j_S)} \geq \frac{\mu(C^j_S) - \frac{1}{2} \mu(C^j_S)}{p \mu(C^j_S)} = \frac{1}{2p}.
\]
Now assume $\mu(C^i_S) > 3/4p$. Choose $j$ such that $\mu(C^i_S) = \max_k \mu(C^k_S)$, and choose $1 \leq \ell \leq p$ such that $j + \ell = i_2 \pmod{p}$. Calculation reveals that $\mu(C^j_S) < \frac{2}{3} \mu(C^i_S)$. Then by (2.6) and the shift-invariance of $\mu$,

$$\frac{\mu(C_S \cap \sigma^{-1}C_S)}{\mu(C_S)} \geq \frac{\mu(C_S \cap \sigma^{-\ell}C_S)}{p\mu(C_S)} \geq \frac{\mu(C^j_S) - \mu(C^{i_2})}{p\mu(C^i_S)} \geq \frac{\mu(C^j_S) - \frac{2}{3} \mu(C^i_S)}{p\mu(C^i_S)} = \frac{1}{3p}.$$ 

Let $b_4 = 1/3p$. We have shown that for $S$ in this case, the inequality in (2.7) holds with $b_4$ in place of $b$.

Now let $b = \min(b_1, b_2, b_3, b_4) = \min(1/p, g/2p, g/6p, 1/3p) = g/6p$, which depends only on $g$ and $p$. We have shown that $c_w(G) \geq b$. \hfill $\square$

Recall that the transpose graph $G^T$ of a graph $G$ was defined in Definition 2.8.

**Lemma 2.27.** Let $(G_n)$ be a sequence of graphs satisfying the Standing Assumptions 2.21 and such that both $(G_n)$ and $(G^T_n)$ have bounded degrees and bounded distortion of edges and vertices (conditions (C1) and (C6) in 2.23). Then $(G_n)$ and $(G^T_n)$ are both uniform expander sequences (condition (C8) in 2.23).

**Proof.** We check that conditions (C1) and (C6) for $(G_n)$ together imply that $(G_n)$ is a uniform expander sequence, and then the same argument will apply to $(G^T_n)$ since (C1) and (C6) also hold for $(G^T_n)$.

Recall the following notation. Let $F : E_n \rightarrow [0,1]$ be given by $F(e) = \mu(e)$, where $\mu$ is the measure of maximal entropy on $X_{G_n}$. Also, $c_w(G_n)$ denotes the weighted Cheeger constant of $G_n$ (Definition 2.18). By the Standing Assumption 2.21, $Sp_x(G_n) = Sp_x(G_1)$ for each $n$. Therefore $\per(G_n)$ does not depend on $n$, and we let $p = \per(G_1)$. Let $G^{p,0}_n$ be an irreducible component of the $p$-th power graph of $G_n$, and let $g_n = g(G^{p,0}_n)$. Since $g_n$ only depends on the non-zero spectrum of $G_n$, which is constant in $n$ by the Standing Assumption 2.21, we have the $g_n$ is constant in $n$. Let $g = g_1$. By Lemma 2.26, there exists $b_n > 0$, depending only $g_n$ and $\per(G_n)$, such that $c_w(G_n) \geq b_n$. Since we have that $g_n = g$ and $\per(G_n) = p$ for all $n$, we may choose $b := b_1$, and we obtain that $c_w(G_n) \geq b > 0$ for all $n$. 


Now we relate $c_w(G_n)$ to $c(G_n)$ (Definition 2.18) using properties (C1) and (C6). For notation, let $m = m(G_n)$. Since $(G_n)$ satisfies conditions (C1) and (C6), there exists $K_1, K_2 > 0$ such that for every $n$ and every subset $S \subset V_n$,

$$K_1|S|\lambda^{-m} \leq F(S) \leq K_2|S|\lambda^{-m},$$

and

$$K_1|E_n(S, S)|\lambda^{-m} \leq F(S, S) \leq K_2|E_n(S, S)|\lambda^{-m}.$$

We already have that $c_w(G_n) \geq b$, which implies that for every $S$ such that $\emptyset \subseteq S \subseteq V_n$,

$$b \leq \frac{F(S, S)}{\min(F(S), F(S))} \leq \frac{K_2|E_n(S, S)|\lambda^{-m}}{\min(F(S), F(S))}.$$

Now assume $0 < |S| \leq |V_n|/2$. If $\min(F(S), F(S)) = F(S)$, then $\min(F(S), F(S)) = F(S) \geq K_1|S|\lambda^{-m}$. If $\min(F(S), F(S)) = F(S)$, then we have $\min(F(S), F(S)) = F(S) \geq K_1|S|\lambda^{-m} \geq K_1|S|\lambda^{-m}$. Combining these estimates gives that for all $S$ such that $0 < |S| \leq |V_n|/2$, we obtain that

$$|E_n(S, S)| \geq b \frac{K_1}{K_2}|S|,$$

which shows that $(G_n)$ is a uniform $(b \frac{K_1}{K_2})$-expander sequence. \hfill \square

**Lemma 2.28.** Let $(G_n)$ be a sequence of graphs satisfying the Standing Assumptions 2.21 and bounded distortion of weights (condition (C7) in 2.23). Then

1. there exists $K > 0$ such that for all $n$, $k$, and $S \subset B_k(G_n)$,

   $$K^{-1}|S| \leq \lambda^{m(G_n)+k}\mu(S) \leq |S|K;$$

2. there exists a $K > 0$ such that for all $n$, $k$, $e \in E_n$, and $S \subset B_k(G_n)$,

   $$K^{-1}|S \cap C_{e}^{m,k}| \leq \lambda^k\mu(S|C_{e}^{m,k}) \leq K|S \cap C_{e}^{m,k}|;$$

   where $C_{e}^{m,k} = \{b \in B_k(G_n) : b_1 = e\}$;

3. there exists $K > 0$ such that for all $n$, $k$, and $1 \leq s < t \leq k$, it holds that $\mu(A_{s,t}) \leq K\lambda^{-m(G_n)}$,

   where $A_{s,t} = \{b \in B_k(G_n) : b_s = b_t\}$;

4. there exists $K > 0$ such that for all $n$, $k > U(G_n)$, and $u \in V_n$, it holds that $\mu(\text{Per}_k(G_n)|C_{u}^{m,k}) \leq K\lambda^{-U(G_n)}$, where $C_{u}^{m,k} = \{b \in B_k(G_n) : i(b_1) = u\}$ and $U(G_n)$ was defined in Definition 2.16.
Proof. For notation, let $m = m(G_n)$ and $U = U(G_n)$.

Proof of (1). We have that

$$1 = \sum_{u \in V_n} \mu(u) = \sum_{u \in V_n} w_u^n v_u^n.$$ 

Then condition (C7) implies that there exists $K_1 > 0$ such that for each $n$ and $u$ in $V_n$,

$$K_1^{-1}|V_n|^{-1} \leq w_u^n v_u^n \leq K_1|V_n|^{-1}.$$ 

By the definition of $m$, there exists $K_2 > 0$ such that $K_2^{-1}|V_n|^{-1} \leq \lambda^{-m} \leq K_2|V_n|^{-1}$. It follows that there exists $K_3 > 0$ such that for each $n$ and $u$ in $V_n$,

$$K_3^{-1}\lambda^{-m} \leq w_u^n v_u^n \leq K_3\lambda^{-m}.$$ 

Then (C7) implies that there exists $K_4 > 0$ such that for any $n$ and any three vertices $u$, $u_1$ and $u_2$ in $V_n$,

$$K_4^{-1}w_{u_1}^n v_{u_2}^n \leq w_u^n v_u^n \leq K_4 w_{u_1}^n v_{u_2}^n.$$ 

Finally, we conclude that there exists $K_5 > 0$ such that for each $n$, $k$, and $b$ in $B_k(G_n)$, we have that

$$K_5^{-1}\lambda^{-(m+k)} \leq \mu(b) = w_{t(b)}^n \lambda^{-k} v_{t(b)}^n \leq K_5\lambda^{-(m+k)}.$$ 

The statement in (1) follows.

Proof of (2). The statement in (2) follows from the statement in (1) and the fact that $\mu(C_{e}^{n,k}) = \mu(e)$.

Proof of (3). Note that from (1) we have that there exists $K > 0$ such that

$$\mu(A_{s,t}) = \sum_{\gamma \in \text{Per}_{t-s}(G_n)} \mu(\gamma) \leq K\lambda^{-(m+t-s)}|\text{Per}_{t-s}(G_n)|.$$ 

Since $\text{Sp}_{\lambda}(A_n)$ does not depend on $n$ by our Standing Assumptions 2.21, we have that $|\text{Per}_{t-s}(G_n)|$ does not depend on $n$. Clearly, $|\text{Per}_{t-s}(G_n)|\lambda^{-(t-s)}$ is bounded as $t - s$ tends to infinity. Therefore there exists $K'$ such that

$$\mu(A_{s,t}) \leq K'\lambda^{-m},$$ 

as desired.

Proof of (4). By (2), we have that there exists $K_1 > 0$ such that for all $n$, $k > U$, and $u$ in $V_n$,

$$\mu(\text{Per}_k(G_n)|C_u^{n,k}) \leq K_1\lambda^{-k}|\text{Per}_k(G_n) \cap C_u^{n,k}|.$$
By (2), there exists $K_2 > 0$ such that for all $n$, $k > U$, and $u$ in $V_n$, 
\[
|B_{k-U}(G_n) \cap C_u^{m,k-U}| \leq K_2\lambda^{k-U}.
\]

By definition of the uniqueness parameter $U$, each path in $B_{k-U}(G_n) \cap C_u^{m,k-U}$ can be continued in at most one way to form a path in $\text{Per}_k(G_n) \cap C_u^{m,k}$. Therefore, with $K_3 = K_1K_2 > 0$, we have that for all $n$, $k > U$, and $u$ in $V_n$, 
\[
\mu(\text{Per}_k(G_n)|C_u^{m,k}) \leq K_1K_2\lambda^{-k}\lambda^{k-U} = K_3\lambda^{-U}.
\]

\begin{proposition}
Let $G_1$ be a graph with irreducible adjacency matrix $A_1$ having entries in \{0,1\} and spectral radius $\lambda > 1$. Let $G_n = G_1^{[n]}$ for $n \geq 2$. Then the sequence $(G_n)$ satisfies the Standing Assumptions 2.21 and conditions (C1)-(C8). Moreover,

(i) $d_{\text{max}}(G_n) = d_{\text{max}}(G_1)$ for all $n$;

(ii) there exists $C > 0$ such that $|m(G_n) - n| \leq C$ for all $n$;

(iii) $z(G_n) \geq \frac{1}{2}(n - 1)$ for all $n$;

(iv) $U(G_n) \geq n - 1$ for all $n$;

(v) $R(G_n) \leq n + R(G_1)$ for all $n$.
\end{proposition}

\textbf{Proof.} One may easily check from the definitions that each $A_n$ has entries in \{0,1\}, each $A_n$ is irreducible, and $\text{Sp}_x(G_n) = \text{Sp}_x(G_1)$. We show below that $m(G_n)$ tends to infinity as $n$ tends to infinity, which gives that $(G_n)$ satisfies the Standing Assumptions 2.21.

The set of in-degrees that appear in $G_n$ is constant in $n$, and so is the set of out-degrees that appear in $G_n$. Therefore $d_{\text{max}}(G_n) = d_{\text{max}}(G_1)$, which implies condition (C1).

By definition, $m(G_n) = [\log_\lambda |V_n|]$. Since $G_n = G_1^{[n]}$, we have that $|V_n| = |B_{n-1}(G_1)|$. By standard Perron-Frobenius theory, there exist constants $K_1$ and $K_2$ such that $K_1\lambda^n \leq |B_{n}(G_1)| \leq K_2\lambda^n$. It follows that there exists a constant $C > 0$ such that $|m(G_n) - n| \leq C$, and in particular, $m(G_n)$ tends to infinity.

Recall the higher-block coding map $\phi_n : X_{G_1} \to X_{G_n}$ (see Definition 2.1). If $x$ is a point in $X_{G_1}$, then let $V_n(x)$ be the set of vertices in $G_n$ traversed by $\phi_n(x)$. Let us show that $z(G_n) \geq (n - 1)/2$. Recall Fine and Wilf’s Theorem [26], which can be stated as follows. Let $x$ be a periodic sequence
with period $p$, and $y$ be a periodic sequence with period $q$. If $x[i+1, i+n] = y[i+1, i+n]$ for $n \geq p+q - \gcd(p, q)$ and $i$ in $\mathbb{Z}$, then $x = y$. It follows from this theorem that if $x$ and $y$ lie in distinct periodic orbits of $X_{G_1}$ and have periods less than or equal to $(n-1)/2$, then $V_n(x) \cap V_n(y) = \emptyset$. Thus $z(G_n) \geq (n-1)/2$, and in particular $(G_n)$ satisfies conditions (C2) and (C3).

Note that the map $\phi_n$ gives a bijection between $B_k(G_n)$ and $B_{k+n-1}(G_1)$ for all $k \geq 0$. Using this map, we check that $U(G_n) \geq n - 1$ as follows. For any two paths $b, c \in B_{n-1}(G_1)$, there is at most one path of length $2n - 2$ in $G_1$ of the form $bc$ (since every edge in such a path is specified by either $b$ or $c$). This fact implies that $U_1(G_n) \geq n - 1$. Now if $b$ is in $B_{n-1}(G_1)$ and $1 \leq s < t \leq n - 1$ are given, then there is at most one path $c$ in $B_{t+n-2}(G_1)$ such that $c[1, n-1] = b$ and $c[s, s+n-2] = c[t, t+n-2]$; indeed, if $c$ is such a path, then $c[1, n-1]$ is determined by $b$, and $c[n, t+n-1]$ is determined by the periodicity condition $c[s, s+n-2] = c[t, t+n-2]$. This fact implies that $U_2(G_n) \geq n - 1$, and thus we have that $U(G_n) \geq n - 1$, which, in particular, gives condition (C4).

Let us check that $R(G_n) \leq n + R(G_1)$, which will imply that $(G_n)$ satisfies condition (C5). The statement that $R(G_n) \leq n + R(G_1)$ is equivalent to the statement that for any two paths $b, c \in B_{n-1}(G_1)$, there exists a path $d$ in $G_1$ of length less than or equal to $R(G_1)$ such that $bdc$ is a path in $G_1$. In this formulation, the statement is clearly true, since, by the definition of $R(G_1)$, there is a path $d$ from $t(b)$ to $i(c)$ of length less than or equal to $R(G_1)$, and then the concatenation $bdc$ gives a path in $G_1$.

Let $w^1$ be a positive left (row) eigenvector for $A_1$ (corresponding to the eigenvalue $\lambda$), and let $v^1$ be a positive right (column) eigenvector for $A_1$ (corresponding to the the eigenvalue $\lambda$). Let $b \in B_{n-1}(G_1) = V_n$. Then let $w^n_b = w^1_{i(b)}$ and $v^n_b = v^1_{i(b)} \lambda^{-(n-1)}$. Then $w^n$ is a positive left eigenvector for $A_n$ and $v^n$ is a positive right eigenvector for $A_n$. It follows that $(G_n)$ satisfies conditions (C6) and (C7). In fact, to satisfy (C7), we may choose $K = \max(K_1, K_2)$, where $K_1 = (\max_i w_i^1)(\min_i w_i^1)^{-1}$ and $K_2 = (\max_i v_i^1)(\min_i v_i^1)^{-1}$.

Condition (C8) follows from the fact that $(G_n)$ satisfies condition (C7) (by applying Lemmas 2.24 and 2.27 in succession).

2.3. Probabilistic framework. Let $\Omega$ be the probability space consisting of the set $\{0, 1\}^n$ and the probability measure $\mathbb{P}_\alpha$, where $\mathbb{P}_\alpha$ is the product of the Bernoulli measures on each coordinate
with parameter $\alpha \in [0, 1]$. There is a natural partial order on $\Omega$, given by the relation $\omega \leq \tau$ if and only if $\omega_i \leq \tau_i$ for $i = 1, \ldots, n$. We say that a random variable $\chi$ on $\Omega$ is monotone increasing if $\chi(\omega) \leq \chi(\tau)$ whenever $\omega \leq \tau$. An event $A$ is monotone increasing if its characteristic function is monotone increasing. Monotone decreasing is defined analogously. Monotone random variables and events have been studied extensively \cite{27}; however, we require only a small portion of that theory. In particular, we will make use of the following proposition, a proof of which may be found in \cite{27}.

**Proposition 2.30 (FKG Inequality).** If $X$ and $Y$ are monotone increasing random variables on $\{0, 1\}^n$, then $E_\alpha(XY) \geq E_\alpha(X)E_\alpha(Y)$.

It follows easily from the FKG Inequality that if $\cap F_j$ is a finite intersection of monotone decreasing events, then $P_\alpha(\cap F_j) \geq \prod P_\alpha(F_j)$ (use induction and note that if $\chi_F$ is the characteristic function of the monotone decreasing event $F$, then $-\chi_F$ is monotone increasing). In fact, we only use this corollary, but we nonetheless refer to it as the FKG Inequality.

For a finite, directed graph $G$, we consider the discrete probability space on the set $\Omega_G = \{0, 1\}^E$, where $P_\alpha$ is the product of the Bernoulli($\alpha$) measures on each coordinate. The set $\Omega_G$ corresponds to the power set of $E$ in the usual way: $\omega$ in $\Omega_G$ corresponds to the set $F$ in $2^E$ such that $e$ is in $F$ if and only if $\omega(e) = 1$. Furthermore, $\Omega_G$ corresponds to the space of subgraphs of $G$: for $\omega$ in $\Omega_G$, define the subgraph $G(\omega)$ to have vertex set $V$ and edge set $E_\omega$, where an edge $e$ in $E$ is included in $E_\omega \subset E$ if and only if $\omega(e) = 1$. In the percolation literature, the edges $e$ such that $\omega(e) = 1$ are often called “open,” and the remaining edges are called “closed.” Since we are interested in studying edge shifts defined by graphs, we will refer to an edge $e$ as “allowed” when $\omega(e) = 1$ and “forbidden” when $\omega(e) = 0$. Finally, each $\omega$ in $\Omega_G$ can be associated to the SFT $X_\omega$ defined as the set of all bi-infinite, directed walks on $G$ that traverse only allowed edges (with respect to $\omega$). The probability measure $P_\alpha$ corresponds to allowing each edge of $G$ with probability $\alpha$, independently of all other edges. For the sake of notation, we suppress the dependence of $P_\alpha$ on the graph $G$.

**Definition 2.31.** In this work, we consider the following conjugacy invariants of SFTs. Let $E$ be the property containing only the empty shift. Let $Z$ be the property containing all SFTs with zero entropy. By convention, we let $E \subset Z$. For any SFT $X$, let $h(X)$ be the topological entropy, and let $I(X)$ be the number of irreducible components of $X$. If $X$ is non-empty, let $\beta(X)$ be defined by the equation $h(X) = \log(\beta(X))$. If $X$ is empty, let $\beta(X) = 0$. If $S$ is a property of SFTs and $G$
is a finite directed graph, then let $S_G \subset \Omega_G$ be the set of $\omega$ in $\Omega_G$ such that $X_\omega$ has property $S$. If $f$ is a function from SFTs to the real numbers and $G$ is a finite directed graph, then let $f_G : \Omega_G \to \mathbb{R}$ be the function $f_G(\omega) = f(X_\omega)$.

3. Emptiness. Recall that $\text{Sp}_X(G)$, $\zeta_G$, and $z(G)$ were defined in Definitions 2.9, 2.11, and 2.14, respectively.

**Theorem 3.1.** Let $(G_n)$ be a sequence of graphs such that $\text{Sp}_X(G_n) = \text{Sp}_X(G_1)$ for all $n$ and either (i) $\lambda = \lambda_{G_1} = 1$ or (ii) $\lambda = \lambda_{G_1} > 1$ and $z(G_n)$ tends to infinity as $n$ tends to infinity. Let $\zeta = \zeta_{G_1}$. Then

$$\lim_{n \to \infty} \mathbb{P}_\alpha(\mathcal{E}_{G_n}) = \begin{cases} (\zeta(\alpha))^{-1}, & \text{if } \alpha \in [0, 1/\lambda) \\ 0, & \text{if } \alpha \in [1/\lambda, 1]. \end{cases}$$

**Remark 3.2.** Theorem 1.1 can be obtained as a corollary of Theorem 3.1 by taking $(G_n)$ to be the sequence of $n$-block graphs of $X$. Indeed, if the SFT $X$ in Theorem 1.1 has zero entropy, then $\lambda = 1$, and the conclusion of Theorem 1.1 follows from case (i) in Theorem 3.1. If the SFT $X$ in Theorem 1.1 has positive entropy, then $\lambda > 1$ and $z(G_n)$ tends to infinity by the exact same argument in the proof of Proposition 2.29 (iii), and therefore the conclusion of Theorem 1.1 follows from case (ii) in Theorem 3.1.

In this section we provide a proof of Theorem 3.1. Before proceeding with the proof, we state a fact that will be useful in the investigations that follow. Recall that for a path $b$, we denote by $V(b)$ the set of vertices traversed by $b$.

**Lemma 3.3.** Suppose $G$ is a directed graph. Suppose $b$ is in $\text{Per}(G)$ such that $|V(b)| < \text{per}(b)$. Then there exists a path $c$ in $\text{Per}(G)$ such that $\text{per}(c) < \text{per}(b)$ and $V(c) \subset V(b)$.

**Proof.** Let $v$ be in $V(b)$. Then there exists a return path to $v$ following $b$, and we may choose a shortest return path $c$ to $v$ using only vertices in $V(b)$. Then $c$ is in $\text{Per}(G)$ and $\text{per}(c) < \text{per}(b)$, as desired. \qed

**Proof of Theorem 3.1.** Recall that an SFT is non-empty if and only if it contains a periodic point (see [40]).
First, assume that case (i) holds, which means that $\lambda = 1$. In this case, each $X_{G_n}$ contains finitely many orbits. Further, the number of periodic orbits of each period in $X_{G_n}$ is constant, and the probability of each periodic orbit being allowed in $X_\omega$ is constant. Therefore the conclusion follows immediately, since the sequence $\mathbb{P}_\alpha(E_{G_n})$ is constant.

Now assume that case (ii) holds. For the moment, consider a fixed natural number $n$. Let $\{\gamma_j\}_{j \in \mathbb{N}}$ be an enumeration of the periodic orbits of $X_{G_n}$ such that if $i \leq j$ then $\text{per}(\gamma_i) \leq \text{per}(\gamma_j)$. Let $p_i = \text{per}(\gamma_i) = |\gamma_i|$. Let $V_n(\gamma_j)$ be the vertices in $G_n$ traversed in the orbit $\gamma_j$ and let $E_n(\gamma_j)$ be the edges in $G_n$ traversed in the orbit $\gamma_j$.

Now for each $j$, let $A_j$ be the event that $\gamma_j$ is allowed, which is the event that all of the edges in $E_n(\gamma_j)$ are allowed. Let $F_j$ be the event that $\gamma_j$ is forbidden, which is $A_j^c$, the complement of $A_j$. Notice that $A_j$ is a monotone increasing event (if $\omega$ is in $A_j$ and $\omega \leq \omega'$, then $\omega'$ is in $A_j$), and $F_j$ is a monotone decreasing event. The fact that an SFT is non-empty if and only if it contains a periodic point implies that $E_{G_n} = \bigcap F_j$.

Combining the definition of $z(G_n)$ and Lemma 3.3, we obtain that if $\text{per}(\gamma_i) \leq z(G_n)$, then $|E_n(\gamma_i)| = p_i$. It follows that $\mathbb{P}_\alpha(F_i) = 1 - \alpha^{p_i}$ for each $i$ such that $p_i \leq z(G_n)$. Furthermore, the definition of $z(G_n)$ implies that the events $F_i$ such that $p_i \leq z(G_n)$ are all jointly independent. These observations give that

\begin{equation}
\mathbb{P}_\alpha(E_{G_n}) = \mathbb{P}_\alpha\left(\bigcap_{j \in \mathbb{N}} F_j\right) \leq \mathbb{P}_\alpha\left(\bigcap_{p_i \leq z(G_n)} F_i\right) = \prod_{p_i \leq z(G_n)} \mathbb{P}_\alpha(F_i) = \prod_{p_i \leq z(G_n)} (1 - \alpha^{p_i}).
\end{equation}

(3.1)

(3.2)

Using Lemma 3.3, we see that there is great redundancy in the intersection $\bigcap F_j$. Eliminating some of this redundancy, we obtain the following:

\begin{equation}
\bigcap_{j \in \mathbb{N}} F_j = \bigcap_{j : |E_n(\gamma_j)| = p_j} F_j.
\end{equation}

(3.3)

Then using Lemma 3.3 again and the fact that $|E_n(\gamma_j)| \leq |E_n|$, we see that the intersection on the right in Equation (3.3) is actually a finite intersection. Applying the FKG Inequality, we obtain
that
\begin{equation}
\mathbb{P}_\alpha(\mathcal{E}_{G_n}) = \mathbb{P}_\alpha\left(\bigcap_{j \in \mathbb{N}} F_j\right) = \mathbb{P}_\alpha\left(\bigcap_{j : |E_n(\gamma_j)| = p_j} F_i\right) \geq \prod_{j : |E_n(\gamma_j)| = p_j} \mathbb{P}_\alpha(F_i) \tag{3.4}
\end{equation}
\begin{equation}
= \prod_{j : |E_n(\gamma_j)| = p_j} (1 - \alpha^{p_j}) \geq \prod_{j : p_j \leq |E_n|} (1 - \alpha^{p_j}). \tag{3.5}
\end{equation}

Combining the inequalities in (3.1), (3.2), (3.4) and (3.5) gives that for each $n$,
\begin{equation}
\prod_{p_j \leq |E_n|} (1 - \alpha^{p_j}) \leq \mathbb{P}_\alpha(\mathcal{E}_{G_n}) \leq \prod_{p_j \leq z(G_n)} (1 - \alpha^{p_j}). \tag{3.6}
\end{equation}

By the Standing Assumption that $\text{Sp}_x(G_n) = \text{Sp}_x(G_1)$, we have that $|\text{Per}_p(G_n)|$ is independent of $n$. Since $z(G_n)$ and $|E_n|$ tend to infinity as $n$ tends to infinity, Equation (3.6) gives that
\[
\lim_{n \to \infty} \mathbb{P}_\alpha(\mathcal{E}_{G_n}) = \prod_{j=1}^{\infty} (1 - \alpha^{p_j}).
\]

Then Theorem 3.1 follows from the well-known product formula for $\zeta$ (see [40]), which may be stated as
\[
(\zeta(t))^{-1} = \prod_{j=1}^{\infty} (1 - t^{p_j}),
\]
along with the fact that $\zeta(t)$ converges for $t < 1/\lambda$ and diverges to $+\infty$ for $t \geq 1/\lambda$.

\hfill \Box

4. Subcritical phase. In this section we study random SFTs in the subcritical phase: $0 \leq \alpha < 1/\lambda$. The main result of this section is Theorem 4.2. Let us fix some notation for this section.

We consider a sequence of graphs $(G_n)$ such that $\text{Sp}_x(G_n) = \text{Sp}_x(G_1)$ and $z(G_n)$ tends to infinity as $n$ tends to infinity, with $\lambda = \lambda_{G_1} \geq 1$ and $\zeta = \zeta_{G_1}$. Since $\text{Sp}_x(G_n) = \text{Sp}_x(G_1)$, there exist shift-commuting bijections $\phi_n : \text{Per}(X_{G_1}) \to \text{Per}(X_{G_n})$. In other words, there exist bijections $\phi_n$ from the set of cyclic paths in $G_1$ to the set of cyclic paths in $G_n$ such that if $b$ is in $\text{Per}_p(G_1)$, then $\phi_n(b)$ is $\text{Per}_p(G_n)$. If $b$ is in $\text{Per}(G)$, then we refer to $\theta(b)$ (recall Definition 2.12) as a cycle. Using the fixed bijections $\phi_n$, we may refer to a cycle $\gamma$ as being in $G_n$ for any $n$. We fix an enumeration of the cycles in $G_1$, $\{\gamma_i\}_{i \in \mathbb{N}}$, and then since the bijections $\phi_n$ are fixed, this choice simultaneously gives enumerations of all the cycles in each $G_n$. For any $s$ in $\mathbb{N}$, let $p_s = \text{per}(\gamma_s)$. Let us begin with a lemma.
Lemma 4.1. Let \((G_n)\) be a sequence of graphs such that \(\text{Sp}_x(G_n) = \text{Sp}_x(G_1)\) and \(z(G_n)\) tends to infinity as \(n\) tends to infinity, with \(\lambda = \lambda_{G_1} \geq 1\) and \(\zeta = \zeta_{G_1}\). Given a non-empty, finite set \(S\) in \(\mathbb{N}\), let \(D_{G_n}(S)\) be the event that the set of allowed cycles is \(\{\gamma_s : s \in S\}\). Then

\[
\lim_{n \to \infty} P_{\alpha}(D_{G_n}(S)) = \begin{cases} 
(\zeta(\alpha))^{-1} \prod_{j \in S} \frac{\alpha_{pj}}{1-\alpha_{pj}}, & \text{if } \alpha \in [0, 1/\lambda) \\
0, & \text{if } \alpha \in [1/\lambda, 1],
\end{cases}
\]

The proof of Lemma 4.1 is an easy adaptation of the proof of Theorem 3.1, and we omit it for the sake of brevity.

Recall that \(I(X)\) denotes the number of irreducible components in the SFT \(X\), and for any graph \(G\), the random variable \(I_G : \Omega_G \to \mathbb{Z}_{\geq 0}\) is defined by the equation \(I_G(\omega) = I(X_\omega)\).

Theorem 4.2. Let \((G_n)\) be a sequence of graphs such that \(\text{Sp}_x(G_n) = \text{Sp}_x(G_1)\) and either (i) \(\lambda = \lambda_{G_1} = 1\) or (ii) \(\lambda = \lambda_{G_1} > 1\) and \(z(G_n)\) tends to infinity as \(n\) tends to infinity. Let \(\zeta = \zeta_{G_1}\). Then for \(0 \leq \alpha < 1/\lambda\),

1. \(\lim_{n \to \infty} P_{\alpha}(Z_{G_n}) = 1;\)
2. the sequence \((I_{G_n})\) converges in distribution to the random variable \(I_\infty\) such that \(P(I_\infty = 0) = (\zeta(\alpha))^{-1}\) and for \(k \geq 1\),

\[
P(I_\infty = k) = (\zeta(\alpha))^{-1} \sum_{S \subset \mathbb{N}, |S| = k} \prod_{s \in S} \frac{\alpha_{ps}}{1-\alpha_{pj}},
\]

where \(\{\gamma_i\}_{i=1}^\infty\) is an enumeration of the cycles in \(G_1;\)
3. the random variable \(I_\infty\) has exponentially decreasing tail and therefore finite moments of all orders.

Remark 4.3. One obtains Theorem 1.2 as a consequence of Theorem 4.2 by taking \((G_n)\) to be the sequence of \(n\)-block graphs of a non-empty SFT \(X\). Indeed, if the SFT \(X\) in Theorem 1.2 has zero entropy, then \(\lambda = 1\), and the conclusions of Theorem 1.2 follow from the case (i) in Theorem 4.2. If the SFT \(X\) in Theorem 1.2 has positive entropy, then \(\lambda > 1\) and \(z(G_n)\) tends to infinity by the exact same argument in the proof of Proposition 2.29 (iii), and therefore the conclusions of Theorem 1.2 follow from case (ii) in Theorem 4.2.

Proof of Theorem 4.2. Let \((G_n)\) be as above. Let \(0 \leq \alpha < 1/\lambda\).
First, assume that case (i) holds, which means that $\lambda = 1$. Conclusion (1) follows immediately, since for each $n$, we have that $P(\mathcal{Z}_{G_n}) = 1$ (the random SFT $X_\omega$ satisfies $0 = h(X_\omega) \leq h(X_{G_n}) = \log \lambda = 0$). Also, the fact that $\lambda = 1$ is equivalent to the fact that $G_1$ (and therefore $G_n$) contains only finitely many cycles. Then conclusions (2) and (3) also follow immediately, since the sequence $I_{G_n}$ is constant.

Now assume that case (ii) holds. Recall that we have an enumeration $\{\gamma_i\}_{i \in \mathbb{N}}$ of the cycles in $G_1$, which we refer to as an enumeration of the cycles in $G_n$, for any $n$, using the bijections $\phi_n$. Also recall that for any non-empty, finite set $S \subset \mathbb{N}$, we denote by $D_{G_n}(S)$ the event in $\Omega_{G_n}$ consisting of all $\omega$ such that the set of cycles in $G_n(\omega)$ is exactly $\{\gamma_s : s \in S\}$.

**Proof of Theorem 4.2** (1). Recall that an SFT has zero entropy if and only if it has at most finitely many periodic points [40]. Then we have that

\[
\mathcal{Z}_{G_n} = \mathcal{E}_{G_n} \cup \left( \bigcup_{S \subset \mathbb{N}} \mathcal{D}_{G_n}(S) \right).
\]

Also note that by the definition of $D_{G_n}(S)$, the union in (4.1) is a disjoint union. Thus we have that

\[
P_\alpha(\mathcal{Z}_{G_n}) = P_\alpha(\mathcal{E}_{G_n}) + \sum_{S \subset \mathbb{N}} P_\alpha(D_{G_n}(S)).
\]

Now let $S_1, \ldots, S_J$ be distinct, non-empty, finite subsets of $\mathbb{N}$. Then by Theorem 3.1 and Lemma 4.1 we have that

\[
\liminf_{n \to \infty} P_\alpha(\mathcal{Z}_{G_n}) \geq \liminf_{n \to \infty} P_\alpha(\mathcal{E}_{G_n}) + \sum_{j=1}^{J} \liminf_{n \to \infty} P_\alpha(D_{G_n}(S_j))
\]

\[= (\zeta(\alpha))^{-1} \left( 1 + \sum_{j=1}^{J} \prod_{s \in S_j} \frac{\alpha^{p_s}}{1 - \alpha^{p_s}} \right).\]

Since $J$ and $S_1, \ldots, S_J$ were arbitrary, we conclude that

\[
\liminf_{n \to \infty} P_\alpha(\mathcal{Z}_{G_n}) \geq (\zeta(\alpha))^{-1} \left( 1 + \sum_{S \subset \mathbb{N}} \prod_{0 < |S| < \infty \atop S \in S} \frac{\alpha^{p_s}}{1 - \alpha^{p_s}} \right).
\]

Using the facts that $\alpha^{p_s}/(1 - \alpha^{p_s}) = \sum_{k=1}^{\infty} (\alpha^{p_s})^k$ and $\alpha < 1/\lambda$ (which implies that the relevant infinite products and series converge uniformly), one may easily check that

\[
\left( 1 + \sum_{S \subset \mathbb{N}} \prod_{0 < |S| < \infty \atop S \in S} \frac{\alpha^{p_s}}{1 - \alpha^{p_s}} \right) = \zeta(\alpha).
\]
Thus we have shown that \( \liminf_n \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) \geq 1 \). Since \( \limsup_n \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) \leq 1 \), we conclude that \( \lim_n \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) = 1 \).

**Proof of Theorem 4.2 (2).** Since \( I_{G_n} \) takes values in \( \mathbb{Z}_{\geq 0} \), the sequence \( (I_{G_n}) \) converges in distribution to \( I_\infty \) if and only if \( \mathbb{P}_\alpha(I_{G_n} = k) \) converges to \( \mathbb{P}_\alpha(I_\infty = k) \) for each \( k \) in \( \mathbb{Z}_{\geq 0} \).

Note that \( I_{G_n}(\omega) = 0 \) if and only if \( \omega \) is in \( \mathcal{E}_{G_n} \), which implies that \( \mathbb{P}_\alpha(I_{G_n} = 0) = \mathbb{P}_\alpha(\mathcal{E}_{G_n}) \). Thus for \( \alpha < 1/\lambda \), Theorem 3.1 implies that \( \mathbb{P}_\alpha(I_{G_n} = 0) \) converges to \( (\zeta(\alpha))^{-1} \) as \( n \) tends to infinity.

Now let \( k \) be in \( \mathbb{N} \). Recall that \( \{\gamma_i\}_{i=1}^\infty \) is an enumeration of the cycles in \( G_1 \), and we have fixed bijections between these cycles and the cycles in each \( G_n \). By Theorem 4.2 (1), we have that \( \lim_n \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) = 1 \), and therefore \( \mathbb{P}_\alpha(I_{G_n} = k) = \mathbb{P}_\alpha(\{I_{G_n} = k\} \cap \mathcal{Z}_{G_n}) + \epsilon_n \), where \( \epsilon_n \) tends to 0 as \( n \) tends to infinity. Thus we need only focus on events of the form \( \{I_{G_n} = k\} \cap \mathcal{Z}_{G_n} \) for some \( k \).

Now if \( \omega \) is in \( \mathcal{Z}_{G_n} \), then \( I_{G_n}(\omega) \) is the number of periodic orbits in \( X_\omega \). Thus

\[
\mathbb{P}_\alpha(\{I_{G_n} = k\} \cap \mathcal{Z}_{G_n}) = \sum_{|S| = k} \mathbb{P}_\alpha(D_{G_n}(S)).
\]

For any \( n \) in \( \mathbb{N} \), let \( T_n^0 = \mathbb{P}_\alpha(\mathcal{E}_{G_n}) \). For \( k \) in \( \mathbb{N} \) and \( n \) in \( \mathbb{N} \), let

\[
T_n^k = \sum_{S \subset \mathbb{N}, |S| = k} \mathbb{P}_\alpha(D_{G_n}(S)).
\]

We have that \( \sum_{k=0}^\infty T_n^k = \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) \), and therefore \( \lim_n \sum_{k=0}^\infty T_n^k = 1 \) by Theorem 4.2 (1). Also, using Lemma 4.1, we have that \( \liminf_n T_n^k \geq T^k \), where \( T^0 = (\zeta(\alpha))^{-1} \) and for \( k \) in \( \mathbb{N} \),

\[
T^k = (\zeta(\alpha))^{-1} \sum_{S \subset \mathbb{N}, |S| = k} \prod_{s \in S} \frac{\alpha^{p_s}}{1 - \alpha^{p_s}}.
\]

Further, we have that \( \sum_{k=0}^\infty T^k = 1 \). It follows from these facts that \( \lim_n T_n^k = T^k \). Thus we have shown that for \( k \) in \( \mathbb{N} \),

\[
\lim_n \mathbb{P}_\alpha(I_{G_n} = k) = \lim_n \mathbb{P}_\alpha(\{I_{G_n} = k\} \cap \mathcal{Z}_{G_n}) = (\zeta(\alpha))^{-1} \sum_{S \subset \mathbb{N}, |S| = k} \prod_{s \in S} \frac{\alpha^{p_s}}{1 - \alpha^{p_s}},
\]

as desired.

**Proof of Theorem 4.2 (3).** For \( k \) in \( \mathbb{N} \), let

\[
T^k = \mathbb{P}_\alpha(I_\infty = k) = (\zeta(\alpha))^{-1} \sum_{S \subset \mathbb{N}, |S| = k} \prod_{s \in S} \frac{\alpha^{p_s}}{1 - \alpha^{p_s}}.
\]
We show that there for any real number \( \delta > 0 \), there exists \( k_0 \) such that \( T^{k+1} \leq \delta T^k \) for all \( k \geq k_0 \).

Let \( \delta > 0 \). Since \( \alpha < 1/\lambda \), we have that

\[
\sum_{i \in \mathbb{N}} \frac{\alpha^{p_i}}{1 - \alpha^{p_i}} < \infty.
\]

Now choose \( k_0 \) such that

\[
\sum_{i \geq k_0} \frac{\alpha^{p_i}}{1 - \alpha^{p_i}} < \delta.
\]

In the following sums, we will use that any set \( S \subset \mathbb{N} \) with \( |S| = j \) can be written as \( S = \{s_1, \ldots, s_j\} \), where \( s_1 < \cdots < s_j \). Note that in this case \( s_j \geq j \). Then for \( k \geq k_0 \) we have

\[
(\zeta(\alpha)) T^{k+1} = \sum_{S \subset \mathbb{N}} \prod_{i=1}^{k+1} \frac{\alpha^{p_i}}{1 - \alpha^{p_i}} = \sum_{S \subset \mathbb{N}} \prod_{i=1}^{k} \frac{\alpha^{p_i}}{1 - \alpha^{p_i}} \sum_{j > s_k} \frac{\alpha^{p_j}}{1 - \alpha^{p_j}} \leq \left( \sum_{S \subset \mathbb{N}} \prod_{i=1}^{k} \frac{\alpha^{p_i}}{1 - \alpha^{p_i}} \right) \delta = (\zeta(\alpha)) T^k \delta.
\]

Since \( \alpha < 1/\lambda \), we have that \( 0 < \zeta(\alpha) < \infty \), and we conclude that \( T^{k+1} \leq \delta T^k \) for all \( k \geq k_0 \).

We recognize the distribution of \( I_\infty \) as the sum of countably many independent Bernoulli trials, where the probability of success of trial \( i \in \mathbb{N} \) is given by \( \alpha^{p_i} \) for some enumeration \( \{\gamma_i\}_{i \in \mathbb{N}} \) of the cycles in \( G_1 \) (or any \( G_n \)). We record some facts about this distribution in the following corollary.

**Corollary 4.4.** With the same hypotheses as in Theorem 4.2, the characteristic function of \( I_\infty \) is given by

\[
\varphi_{I_\infty}(t) = (\zeta(\alpha))^{-1} \prod_{s} \left( 1 + e^{it} \frac{\alpha^{p_s}}{1 - \alpha^{p_s}} \right),
\]

where the product is over all periodic orbits in \( X \). It follows that the moment generating function of \( I_\infty \) is given by

\[
M_{I_\infty}(t) = (\zeta(\alpha))^{-1} \prod_{s} \left( 1 + e^{t} \frac{\alpha^{p_s}}{1 - \alpha^{p_s}} \right).
\]

**Remark 4.5.** In Theorems 3.1 and 4.2, we assert the existence of various limits to certain values. Beyond the bounds given in our proofs, we do not know at which rates these sequences converge to their limits.
5. Supercritical Phase. In this section we study random SFTs in the supercritical phase. The main results are Theorem 5.13 and Theorem 5.15. On a first reading, the reader may prefer to skip Section 5.1 and refer back to it as necessary. Our proof of Theorem 5.13 relies, in part, on showing that with large probability the number of allowed words of length \( k \) in a random SFT is close to \((\alpha \lambda)^k\), for a particular choice of \( k \). In our proof, we choose \( k \) to be polynomial in \( m = m(G_n) \) for two reasons. Firstly, we need \( k \) to dominate \( m \), so that the \( k \)-th root of the number of words of length \( k \) gives a good upper bound on the Perron eigenvalue of the random SFT. Secondly, \( k \) should be subexponential in \( m \), essentially because most paths in \( G_n \) with length subexponential in \( m \) are self-avoiding, and we need good bounds on the probability of paths of length \( k \) that exhibit “too-soon-recurrence.” For context, we recall a result of Ornstein and Weiss [45]. In fact, their result is quite general, but we only recall it in a very specific case. Let \( X \) be an irreducible SFT with measure of maximal entropy \( \mu \). For \( x \) in \( X \), let \( R_n(x) \) be the first return time (greater than 0) of \( x \) to the cylinder set \( x[1, n] \) under \( \sigma \). Then the result of Ornstein and Weiss implies that for \( \mu \)-a.e. \( x \) in \( X \), \( \lim_n n^{-1} \log R_n(x) = h(X) \). It follows from this result that for \( k \) polynomial in \( n \), the \( \mu \)-measure of the set of words of length \( k \) with a repeated \( n \)-word tends to 0. In the following lemmas, we give some quantitative bounds on the \( \mu \)-measure of the set of paths of length \( k \) in \( G_n \) with \( k - j \) repeated edges, where the important point for our purposes is that the bounds improve exponentially as \( j \) decreases. To get these bounds we employ some of the language and tools of information theory. After getting a handle on the \( \mu \)-measure of paths in \( G_n \) with certain self-intersection properties, our assumption that \((G_n)\) satisfies condition (C7) in 2.23 implies that \( \mu \)-measure on paths is the same as the counting measure up to uniform constants.

5.1. Information theory and lemmas. In keeping with the convention of information theory, \( \log(x) \) denotes the base 2 logarithm of \( x \).

**Definition 5.1.** A binary \( n \)-**code** on an alphabet \( A \) is a mapping \( C : A^n \to \{0, 1\}^* \), where \( \{0, 1\}^* \) is the set of all finite words on the alphabet \( \{0, 1\} \). We may refer to such mappings simply as codes. A code is **faithful** if it is injective. The function that assigns to each \( w \) in \( A^n \) the length of the word \( C(w) \) is called the **length function** of the code, and it will be denoted by \( \mathcal{L} \) when the code is understood. A code is a **prefix code** if \( w = w' \) whenever \( C(w) \) is a prefix of \( C(w') \). A **Shannon code** with respect to a measure \( \nu \) on \( A^n \) is a code such that \( \mathcal{L}(w) = [-\log \nu(w)] \).
We note that for a measure \( \nu \) on \( A^n \), there is a prefix Shannon code on \( A^n \) with respect to \( \nu \) [50]. We will also require the following two lemmas from information theory.

**Lemma 5.2 ([50]).** Let \( A \) be an alphabet. Let \( C_n \) be a prefix-code on \( A^n \), and let \( \mu \) be a shift-invariant Borel probability measure on \( A^Z \). Then

\[
\mu\left( \{ w \in A^n : L(w) + \log \mu(w) \leq -a \} \right) \leq 2^{-a}.
\]

**Proof.** Let \( B = \{ w \in A^n : L(w) + \log \mu(w) \leq -a \} \). Then for any \( w \) in \( B \), we have that \( \mu(w) \leq 2^{-L(w) - a} \). The Kraft inequality for prefix codes [50, p. 73] states that since \( L \) is a prefix code, \( \sum_{w \in A^n} 2^{-L(w)} \leq 1 \). Hence

\[
\mu(B) = \sum_{w \in B} \mu(w) \leq 2^{-a} \sum_{w \in B} 2^{-L(w)} \leq 2^{-a}.
\]

\( \square \)

**Lemma 5.3 ([50]).** There is a prefix code \( C : \mathbb{N} \to \{0, 1\}^* \) such that \( \ell(C(n)) = \log(n) + o(\log(n)) \), where \( \ell(C(n)) \) is the length of \( C(n) \).

**Definition 5.4.** A prefix code satisfying the conclusion of Lemma 5.3 is called an **Elias code**.

Recall that if \( b \) is a path in the graph \( G = (V, E) \), then we denote by \( E(b) \) the set of edges traversed by \( b \). Let \( (G_n) \) be a sequence of graphs satisfying our Standing Assumptions 2.21.

**Definition 5.5.** For each \( n, k \), and \( 1 \leq j \leq k - 1 \), let

\[
N_{n,k}^j = \{ b \in B_k(G_n) : |E_n(b)| \leq j \}.
\]

**Definition 5.6.** For each \( n, k \), and \( 1 \leq j \leq 2k - 1 \), let

\[
D_{n,k}^j = \{ (b, c) \in B_k(G_n) \times B_k(G_n) : E_n(b) \cap E_n(c) \neq \emptyset, |E_n(b) \cup E_n(c)| \leq j \}.
\]

**Definition 5.7.** For each \( n, k \), and \( 1 \leq j \leq k - 1 \), let

\[
Q_{n,k}^j = \{ b \in \text{Per}_k(G_n) : |E_n(b)| \leq j \}.
\]

**Definition 5.8.** For each \( n, k \), and \( 1 \leq j \leq 2k - 1 \), let

\[
S_{n,k}^j = \{ (b, c) \in \text{Per}_k(G_n) \times \text{Per}_k(G_n) : E_n(b) \cap E_n(c) \neq \emptyset, |E_n(b) \cup E_n(c)| \leq j \}.
\]
For any of the sets defined in Definitions 5.5 - 5.8, we use a “hat” to denote the set with “≤” replaced by “=” in the definition. For example,

$$\hat{N}_{n,k}^j = \{ b \in B_k(G_n) : |E_n(b)| = j \}.$$

The “hat” notation will only appear in the proof of Theorem 5.13. The following four lemmas find bounds on $|N_{n,k}^j|$, $|D_{n,k}^j|$, $|S_{2k-1}^j|$, and $|S_{n,k}^j|$.

The following lemma bounds the $\mu$-measure (and therefore the cardinality) of the set of paths of length $k$ in $G_n$ that traverse at most $j < k$ edges. The proof relies on a general principle in information theory (made precise by Lemma 5.2): a set of words that can be encoded “too efficiently” must have small measure. In order to use this principle, we find an efficient encoding of the paths of length $k$ in $G_n$ that traverse at most $j$ edges. The basic observation behind the coding is trivial: a path of length $k$ that only traverses $j < k$ edges must have $k - j$ repeated edges. Therefore, instead of encoding each of the $k - j$ repeated edges explicitly, we simply encode some combinatorial data that specifies when “repeats” happen and when the corresponding edges are first traversed.

**Lemma 5.9.** Let $(G_n)$ be a sequence of graphs satisfying the Standing Assumptions 2.21 and such that $(G_n)$ has local uniqueness of paths and bounded distortion of weights (conditions (C4) and (C7) in 2.23). Then there exists a polynomial $p_0(x)$ and $n_0$ such that for each $n \geq n_0$, $k > U(G_n)$ and $1 \leq j \leq k - 1$,

$$\mu(N_{n,k}^j) \leq p_0(k)^{\min(j - k, U(G_n))} \lambda^{-(m(G_n) + k - j)},$$

and

$$|N_{n,k}^j| \leq p_0(k)^{\min(j - k, U(G_n))} \lambda^j.$$

**Proof.** Consider $(G_n)$, $n$, $k$, and $j$ as in the hypotheses. Let $m = m(G_n)$ and $U = U(G_n)$. A path $b$ in $N_{n,k}^j$ from vertex $s$ to vertex $t$ contributes $w_s^n w_t^k \lambda^{-k}$ to $\mu(N_{n,k}^j)$. The condition (C7) gives a uniform constant $K$ such that $w_s^n w_t^k$ is bounded below by $(K^2 |V_n|)^{-1} = (K^2 \lambda^m)^{-1}$. Therefore the bound on $|N_{n,k}^j|$ follows from the bound on $\mu(N_{n,k}^j)$, since $|N_{n,k}^j| \leq K^2 \lambda^{m+k} \mu(N_{n,k}^j)$ (as in Lemma 2.28 (1)). We now proceed to show the bound on $\mu(N_{n,k}^j)$.

Let $r = k - j$. Consider $b$ in $N_{n,k}^j$. Then there exists $1 < t_1 < \cdots < t_r \leq k$ such that $b_{t_i} = b_{s_i}$ for some $1 \leq s_i < t_i$, for each $i = 1, \ldots, r$, where $s_i = \min\{ s \geq 1 : b_s = b_{t_i} \}$. Now we define a set
\( I \subset \{1, \ldots, r\} \) by induction. Let \( i_1 = 1 \) and \( \mathcal{I}_1 = \{i_1\} \). Assuming by induction that \( i_j \) and \( \mathcal{I}_j \) have been defined and that \( i_j < r \), we define \( i_{j+1} \) and \( \mathcal{I}_{j+1} \) as follows:

- if \( t_{i_{j+1}} - t_i > U \), let \( i_{j+1} = i_j + 1 \);
- otherwise, if \( t_{i_{j+1}} - t_i \leq U \), then let

\[
i_{j+1} = \max\{i_j < r : t_i - t_i \leq U\}.
\]

Let \( \mathcal{I}_{j+1} = \mathcal{I}_j \cup \{i_{j+1}\} \). This induction procedure terminates when \( i_j = r \) for some \( j \leq r \), and we denote this terminal \( j \) by \( j^* \). Let \( \mathcal{I} = \mathcal{I}_{j^*} \). Note that for each \( 0 \leq s \leq k - U \), we have that

\[
|\{i \in \mathcal{I} : s + 1 \leq t_i \leq s + U\}| \leq 2.
\]

It follows that \( |\mathcal{I}| \leq \min(r, 2k/U + 2) \).

Having defined the set \( \mathcal{I} \), we now decompose the integer interval \( \{1, \ldots, k\} \) into subintervals. First, let

\[
\mathcal{J} = \bigcup_{j=1}^{j^*} \{t_i\} \cup \{1 \leq s \leq k : \exists i_j, i_{j+1} \in \mathcal{I}, t_{i_{j+1}} - t_i \leq U \text{ and } t_i \leq s \leq t_{i_{j+1}}\}.
\]

Let \( \mathcal{J}_1, \ldots, \mathcal{J}_N \) be the maximal disjoint subintervals (with singletons allowed) of \( \{1, \ldots, k\} \) such that \( \mathcal{J} = \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_N \) and \( J_\ell < J_{\ell+1} \). Note that \( \sum_{\ell=1}^{N} |J_\ell| = |\mathcal{J}| \geq r \) and \( N \leq |\mathcal{I}| \). Then let \( \mathcal{I}_1, \ldots, \mathcal{I}_{N+1} \) be the maximal disjoint subintervals of \( \{1, \ldots, k\} \) such that

- \( \mathcal{I}_\ell \subset \{1, \ldots, k\} \setminus \mathcal{J} \) for each \( \ell = 1, \ldots, N + 1 \);
- \( \bigcup_{\ell=1}^{N+1} \mathcal{I}_\ell = \{1, \ldots, k\} \setminus \mathcal{J} \);
- and for each \( \ell = 1, \ldots, N \), we have that \( \mathcal{I}_\ell \) is non-empty and \( \mathcal{I}_\ell \subset \mathcal{I}_{\ell+1} \).

In summary, we have that \( \{1, \ldots, k\} = \mathcal{I}_1 \cup \mathcal{J}_1 \cup \cdots \cup \mathcal{I}_N \cup \mathcal{J}_N \cup \mathcal{I}_{N+1} \), and only \( \mathcal{I}_{N+1} \) may be empty.

For any \( 1 \leq s < t \leq k \), let \( A_{s,t} = \{b \in B_k(G_n) : b_s = b_t\} \). By Lemma 2.28 (3), there exists a uniform constant \( K_1 \) such that

\[
\mu(A_{s,t}) \leq K_1 \lambda^{-m}.
\]

For notation, if \( I \) is a subset of \( \{1, \ldots, k\} \), then \( b_I \) is \( b \) restricted to \( I \). Since \( \mu \) is 1-step Markov on \( X_{G_n} \), we have that

\[
\mu(b|A_{s_1,t_1}) = \mu(b_{I_1}|A_{s_1,t_1}) \prod_{\ell=1}^{N} \mu(b_{I_\ell}|A_{s_1,t_1} \cap b_{I_1} \ldots b_{I_{\ell-1}}) \prod_{\ell=2}^{N+1} \mu(b_{I_\ell}|A_{s_1,t_1} \cap b_{I_1} \ldots b_{I_{\ell-1}})
\]

\[
= \mu(b_{I_1}|A_{s_1,t_1}) \prod_{\ell=1}^{N} \mu(b_{I_\ell}|b_{I_1} \ldots b_{I_{\ell-1}}) \prod_{\ell=2}^{N+1} \mu(b_{I_\ell}|b_{I_1} \ldots b_{I_{\ell-1}})
\]
Given \( b \), we may form \( s_i, t_i, I_\ell \) and \( J_\ell \) as above, and then we encode \( b \) as follows

1. encode \( s_1 \) and \( t_1 \) using an Elias code;
2. encode \( b_{I_1} \) using a prefix Shannon code with respect to \( \mu(\cdot|A_{s_1,t_1}) \);
3. assuming \( b_{I_1,I_2} \) has been encoded, we encode \( b_{J_\ell} \) by encoding \( s_i \) and \( t_i \) for each \( i \) in \( I_\ell \) such that \( t_i \in J_\ell \), using an Elias code (and note that this information completely determines \( b_{J_\ell} \) by definition of \( U \) and construction of \( J \));
4. assuming \( b_{I_1,...,I_{\ell-1}} \) has been encoded, we encode \( b_{I_\ell} \) using a prefix Shannon code with respect to \( \mu(\cdot|b_{I_1,...,I_{\ell-1}}) \).

Now we analyze the performance of the code. Since the code is a concatenation of prefix codes, it is a prefix code. Since \( U \) tends to infinity as \( n \) tends to infinity (by (C4)) and \( k > U \), there exists \( n_0 \) such that for \( n \geq n_0 \) and \( 1 \leq s \leq k \), the length of the codeword in the Elias encoding of \( s \) is less than or equal to \( 2 \log k \). Then we have, neglecting bits needed to round up,

\[
L(b) \leq - \log \mu(b_{I_1}|A_{s_1,t_1}) + |I|(4 \log k) + \sum_{\ell=2}^{N+1} - \log \mu(b_{I_\ell}|b_{I_1,...,I_{\ell-1}}).
\]

Combining Equations (5.2), (5.3), and Equation (5.4), we have that

\[
L(b) + \log \mu(b) \leq |I|(4 \log k) + \log \mu(A_{s_1,t_1}) + \sum_{\ell=1}^{N} \log \mu(b_{I_\ell}|b_{I_1,...,I_{\ell-1}}).
\]

Now by Lemma 2.28 (2) and (3), there exist uniform constants \( K_2 \) and \( K_3 \) such that

\[
L(b) + \log \mu(b) \leq |I|(4 \log k) + K_2 - m \log \lambda + NK_3 - |J| \log \lambda
\]

\[
= |I|(4 \log k) + K_2 + NK_3 - (m + |J|) \log \lambda.
\]

By construction, \( |I| \leq \min(k - j, 2k/U + 2), N \leq |I|, \) and \( |J| \geq r = k - j. \) Then by Lemma 5.2, there exists a uniform constant \( K_4 > 0 \) such that

\[
\mu(N^{J_{n,k}}_{n,k}) \leq (K_4 k^4)^{\min(k-j,2k/U+2)} \lambda^{-(m+k-j)}.
\]

Letting \( p_0(x) = K_5 x^{12} \), for some uniform constant \( K_5 > 0, \) we obtain that

\[
\mu(N^{J_{n,k}}_{n,k}) \leq p_0(k)^{\min(k-j,k/U)} \lambda^{-(m+k-j)},
\]

which completes the proof. \( \square \)
The following lemma bounds the $\mu \times \mu$-measure (and therefore the cardinality) of the set of pairs paths of length $k$ in $G_n$ that share at least one edge and together traverse at most $j < 2k$ edges. The general strategy of encoding pairs of paths using combinatorial data and appealing to information theory is similar to that of Lemma 5.9. Lemma 5.10 involves the additional hypothesis that there exists a uniform bound $R$ such that for any pair of paths $(u, w)$ in $G_n$, there exists a path $uvw$ in $G_n$ with $|v| \leq R$. Using this hypothesis, one observes that pairs of paths can essentially be concatenated in $G_n$ and then treated as single paths as in Lemma 5.9.

**Lemma 5.10.** Let $(G_n)$ be a sequence of graphs satisfying the Standing Assumptions 2.21 and such that $(G_n)$ has local uniqueness of paths, small diameter, and bounded distortion of weights (conditions (C4), (C5) and (C7) in 2.23). Then there exists a polynomial $p_1(x)$ and $n_1$ such that for $n \geq n_1$, $k > R(G_n)$ and $1 \leq j \leq 2k - 1$,

$$\mu \times \mu(D_{n,k}^j) \leq p_1(k)^{\min(2k-j,k/U(G_n))} \lambda^{-(m(G_n)+2k-j)},$$

and

$$|D_{n,k}^j| \leq p_1(k)^{\min(2k-j,k/U(G_n))} \lambda^{j+m(G_n)}.$$

**Proof.** Consider $(G_n)$, $n$, $k$, and $j$ as in the hypotheses. Let $m = m(G_n)$, $U = U(G_n)$, and $R = R(G_n)$. Note that the bound on $|D_{n,k}^j|$ follows from the bound on $\mu \times \mu(D_{n,k}^j)$, since condition (C7) implies that there exists a uniform constant $K$ such that $|D_{n,k}^j| \leq K \lambda^{2m+2k} \mu \times \mu(D_{n,k}^j)$ (as in Lemma 2.28 (1)). We now proceed to show the bound on $\mu \times \mu(D_{n,k}^j)$.

By the definition of $R$, for every pair $(b, c) \in B_k(G_n) \times B_k(G_n)$, there exists a path $d_1$ in $G_n$ such that $|b| \leq R$ and $bcd$ is in $B_{2k+|d_1|}(G_n)$. We choose a single such $d_1$ for each pair $(b, c)$, and we choose a (possibly empty) path $d_2$ such that $bd_1cd_2$ is in $B_{2k+R(G_n)}$ (whose existence is guaranteed by the fact that $G_n$ is irreducible). If $(b, c) \in D_{n,k}^j$, then $bd_1cd_2$ is in $N_{n,2k+R}^{j+R}$. Using condition (C5), we have that $R \leq m+C$ for a uniform constant $C$. Then we have that there exist uniform constants $K_1$, $K_2$, and $K_3$ such that for each $n$, each $k$, and each pair $(b, c)$ in $B_k(G_n) \times B_k(G_n)$,

$$\mu \times \mu((b, c)) \leq K_1 \lambda^{-(2m+2k)} \leq K_2 \lambda^{-(m+R+2k)} \leq K_3 \mu(bd_1cd_2).$$

Thus Lemma 5.9 implies that there exists a polynomial $p_0(x)$ and $n_0$ such that for $n \geq n_0$,

$$\mu \times \mu(D_{n,k}^j) \leq K_3 \mu(N_{n,2k+R}^{j+R}) \leq K_3 p_0(2k + R)^{\min(2k-j,(2k+R)/U)} \lambda^{-(m+2k-j)}.$$
With \( n_1 = n_0 \) and \( p_1(x) = K_4p_0(3x)^3 \) for a uniform constant \( K_4 \), we have
\[
\mu \times \mu(D_{n,k}^j) \leq p_1(k)^{\min(2k-j,k/U)}\lambda^{-(m+2k-j)},
\]
which completes the proof.

The following two lemmas (Lemmas 5.11 and 5.12) give bounds on the \( \mu \times \mu \) measure (and therefore the cardinality) of the set of pairs of periodic paths in \( G_n \) with certain overlap properties. The general ideas are similar to those in Lemmas 5.9 and 5.10, but in order to get precise bounds on the relevant sets, we exploit the fact that these sets consist of pairs of periodic paths. In other words, when we encode paths using their the pattern of “repeats,” we also take into account their assumed periodicity.

**Lemma 5.11.** Let \((G_n)\) be a sequence of graphs satisfying the Standing Assumptions 2.21 and bounded distortion of weights (condition (C7) in 2.23). Then there exists a polynomial \( p_2(x) \) and \( n_2 \) such that for each \( n \geq n_2 \) and \( k > U(G_n) \),
\[
\mu \times \mu(S_{n,k}^{2k-1}) \leq p_2(k)\lambda^{-(2m(G_n)+U(G_n))},
\]
and
\[
|S_{n,k}^{2k-1}| \leq p_2(k)\lambda^{2k-U(G_n)}.
\]

**Proof.** Consider \((G_n), n, \) and \( k \) as in the hypotheses. Let \( m = m(G_n) \) and \( U = U(G_n) \). Note that the bound on \( |S_{n,k}^{2k-1}| \) follows from the bound on \( \mu \times \mu(S_{n,k}^{2k-1}) \), since condition (C7) implies that there exists a uniform constant \( K \) such that \( |S_{n,k}^{2k-1}| \leq K\lambda^{2m+2k}\mu \times \mu(S_{n,k}^{2k-1}) \) (as in Lemma 2.28 (1)). We now proceed to show the bound on \( \mu \times \mu(S_{n,k}^{2k-1}) \).

Let \( b \) be in \( \text{Per}_k(G_n) \). Let \( e \) be in \( E_n(b) \). For \( i = 1, \ldots, k \), let \( C_i \subset B_k(G_n) \) be the set of paths \( c \) of length \( k \) in \( G_n \) such that \( c_i = e \). Then Lemma 2.28 (parts (1) and (4)) implies that there exist uniform constants \( K_1 \) and \( K_2 \) such that
\[
\mu(\text{Per}_k(G_n) \cap C_1) = \mu(C_1)\mu(\text{Per}_k(G_n)|C_1) \leq K_1\lambda^{-m}\mu(\text{Per}_k(G_n)|C_1)
\]
\[
\leq K_2\lambda^{-(m+U)}.
\]
Let \( C \) be the set of paths \( c \) of length \( k \) in \( G_n \) such that \( e \in E_n(c) \). Then \( C = \bigcup_{i=1}^k C_i \), and by shift-invariance of \( \mu \),
\[
\mu(\text{Per}_k(G_n) \cap C) \leq \sum_{i=1}^k \mu(\text{Per}_k(G_n) \cap C_i) \leq K_2k\lambda^{-(m+U)}.
\]
Since $e \in E_n(b)$ was arbitrary, it follows from inequality (5.11) that
\[
\mu(\{c \in \text{Per}_k(G_n) : E_n(c) \cap E_n(b) \neq \emptyset\}) \leq \sum_{e \in E_n(b)} \mu(\{c \in \text{Per}_k(G_n) : e \in E_n(c)\}) \leq K_2 \sum_{e \in E_n(b)} k \lambda^{-(m+U)} \leq K_2 k^2 \lambda^{-(m+U)}.
\]

Since $b \in \text{Per}_k(G_n)$ was arbitrary, we conclude that there exists a uniform constant $K_3$ such that
\[
\mu \times \mu(S_{n,k}^{2k-1}) \leq K_2 \mu(\text{Per}_k(G_n)) k^2 \lambda^{-(m+U)} \leq K_3 k^2 \lambda^{-(2m+U)},
\]
where the last inequality follows from Lemma 2.28 (4). This inequality completes the proof. \qed

**Lemma 5.12.** Let $(G_n)$ be a sequence of graphs satisfying the Standing Assumptions 2.21 and such that $(G_n)$ has local uniqueness of paths, small diameter, and bounded distortion of weights (conditions (C4), (C5) and (C7) in 2.23). Then there exists a polynomial $p_3(x)$ and $n_3$ such that for $n \geq n_3$, $k > U(G_n)$, and $1 \leq j \leq 2k - 1$,
\[
\mu \times \mu(S_{n,k}^j) \leq p_3(k)^{k/U(G_n)} \lambda^{-(m(G_n)+U(G_n)+2k-j)},
\]
and
\[
|S_{n,k}^j| \leq p_3(k)^{k/U(G_n)} \lambda^{j+m(G_n)-U(G_n)}.
\]

**Proof.** Consider $(G_n)$, $n$, $k$, and $j$ as in the hypotheses. Let $m = m(G_n)$, $U = U(G_n)$, and $R = R(G_n)$. Note that the bound on $|S_{n,k}^j|$ follows from the bound on $\mu \times \mu(S_{n,k}^j)$, since condition (C7) implies that there exists a uniform constant $K$ such that $|S_{n,k}^j| \leq K \lambda^{2m+2k} \mu \times \mu(S_{n,k}^j)$ (as in Lemma 2.28 (1)). We now proceed to show the bound on $\mu \times \mu(S_{n,k}^j)$.

Let $e$ be in $E_n$ and let $C_1$ be the set of paths $b$ of length $k$ in $G_n$ such that $b_1 = e$. Then it follows from Lemma 2.28 (4) that there exists a uniform constant $K_1$ such that
\[
(5.12) \quad \mu(\text{Per}_k(G_n)|C_1) \leq K_1 \lambda^{-U}.
\]

To each pair $(b,c)$ in $S_{n,k}^j$, let us associate a particular path of length $2k + R$ in $G_n$, which we construct as follows. Let $(b,c)$ be in $S_{n,k}^j$. By definition of $S_{n,k}^j$, there is at least one edge $e$ in $E_n(b) \cap E_n(c)$. Let $\tau$ be the cyclic permutation of $\{1, \ldots, k\}$ of order $k$ given by $(12 \ldots k)$. Let $\tau$ act on periodic paths of length $k$ in $G_n$ by permuting the indices: $\tau(b_1 \ldots b_k) = b_{\tau(1)} \ldots b_{\tau(k)}$. Then let $b'$ be in $\{\tau(b) : \ell \in \{1, \ldots, k\}, \tau(b)_k = e\}$. Similarly, let $c'$ be in $\{\tau(c) : \ell \in \{1, \ldots, k\}, \tau(c)_1 = e\}$. 


Now choose a path $d_1$ in $G_n$ such that $|b'd_1c'| \leq R$ and $b'd_1c'$ is a path in $G_n$ (the existence of such a path $d_1$ is guaranteed by the definition of $R$). By irreducibility of $G_n$ we also choose a (possibly empty) path $d_2$ in $G_n$ such that $b'd_1c'd_2$ is in $B_{2k+R}$. We associate the path $b'd_1c'd_2$ to the pair $(b,c)$, and note that there exist uniform constants $K_2$, $K_3$, and $K_4$ (by Lemma 2.28 (1) and condition (C5)) such that

$$
\mu \times \mu((b,c)) \leq K_2 \lambda^{-(2m+2k)} \leq K_3 \lambda^{-(m+R+2k)} \leq K_4 \mu(b'd_1c'd_2).
$$

Now we use the same construction as in the proof of Lemma 5.9 with only slight modification. We encode the words $b'd_1c'd_2$ as follows.

1. Construct $I$, $J$, and the partition of $\{1, \ldots, 2k + R\}$ as in the proof of Lemma 5.9, with the additional condition that $J \cap \{k+1, \ldots, k+R\} = \emptyset$. (In other words, we ignore any “repeats” introduced by $d$.)

2. Encode $b'$ as in the proof of Lemma 5.9.

3. To encode the path $d_1$, we first encode the fact that $b'_k = c'_1$ (by encoding $k$ and $k+|d_1|$ using an Elias code), and then encode $d_1$ using a prefix Shannon code with respect to $\mu(|A_{k,k+|d_1|} \cap b'|$).

4. Encode $c'$ as in the proof of Lemma 5.9.

5. Encode $d_2$ using a prefix Shannon code with respect to $\mu(|b'd_1c'|$).

For large $n$, encoding the fact that $b'_k = c'_1$ adds less than $4 \log(2k + R)$ to $L(b'd_1c'd_2)$. On the other hand, we have that there is a uniform constant $K_5 > 0$ such that $\mu(A_{k,k+|d_1|}|b') \leq K_5 \lambda^{-U}$, by Lemma 2.28 (4). Thus, there exists $n_3$ and a uniform constant $K_6$ such that for $n \geq n_3$, we have

$$
L(b'd_1c'd_2) + \log \mu(b'd_1c'd_2) \leq (|I| + 1)(4 \log(2k + R)) + NK_6 - (m + U + |J|) \log \lambda,
$$

with $|I| \leq 2k/U + 2$, $N \leq |I|$, and $|J| \geq 2k - j - 1$. Then by Lemma 5.2 there is a polynomial $p_4(x)$ such that for $n \geq n_3$,

$$
\mu(\{b'd_1c'd_2 : (b,c) \in S_{n,k}^{(j)}\}) \leq p_4(k)^{k/U} \lambda^{-(m+U+2k-J)}.
$$

Note that the number of pairs $(b,c)$ associated to the a path $b'd_1c'd_2$ is at most $k^2$, and hence

$$
\mu \times \mu(S_{n,k}^{(j)}) \leq k^2 p_4(k)^{k/U} \lambda^{-(m+U+2k-J)}.
$$

Now let $p_4(x) = x^2 p_4(x)$, and the proof is complete. □
5.2. Entropy. Recall that if $G$ is a graph, then $\beta_G$ is the random variable such that $\beta_G(\omega)$ is the spectral radius of the adjacency matrix of $G(\omega)$.

Theorem 5.13. Let $(G_n)$ be a sequence of graphs that satisfies the Standing Assumptions 2.21 and such that $(G_n)$ has local uniqueness of paths, small diameter, and bounded distortion of weights (conditions (C4), (C5), and (C7) in 2.23). Then for $1/\lambda < \alpha \leq 1$ and $\epsilon > 0$,

$$\lim_{n \to \infty} P_\alpha (|\beta_{G_n} - \alpha \lambda| \geq \epsilon) = 0,$$

and the convergence to the limit is exponential in $m(G_n)$.

Remark 5.14. If we assume that $X$ is irreducible in the statement of Theorem 1.3, then Theorem 1.3 is a direct corollary of Theorem 5.13, obtained by choosing $(G_n)$ to be the sequence of $n$-block graphs of an irreducible SFT with positive entropy (and using the fact that such a sequence satisfies the hypotheses of Theorem 5.13 by Proposition 2.29). In the case when $X$ is reducible, $X$ has a finite number of irreducible components of positive entropy, $X_1, \ldots, X_r$, and there exist $i$ such that $h(X_i) = h(X)$. For all large $n$, we have that $B_n(X_i) \cap B_n(X_j) = \emptyset$ for $i \neq j$, which means that the entropies of the random subshifts appearing inside each of these components are mutually independent. Applying Theorem 5.13 to each of these components, we obtain Theorem 1.3 for reducible $X$.

Proof of Theorem 5.13. Let $\alpha$ be in $(1/\lambda, 1]$. Let $m = m(G_n)$ and $U = U(G_n)$. Let $b$ be a path in $G_n = (V_n, E_n)$. Let $\xi_b : \Omega_n \to \mathbb{R}$ be the random variable defined by

$$\xi_b(\omega) = \begin{cases} 1, & \text{if } b \text{ is allowed in } G_n(\omega) \\ 0, & \text{else.} \end{cases}$$

Now let

$$\phi_{n,k} = \sum_{b \in B_k(G_n)} \xi_b, \quad \text{and} \quad \psi_{n,k} = \frac{1}{|V_n|} \sum_{b \in \text{Per}_k(G_n)} \xi_b.$$

For each $n$ and $k$, we have that $\psi_{n,k} \leq \beta_n^k \leq \phi_{n,k}$. Indeed, $\psi_{n,k}$ is the average number of loops of length $k$ based at a vertex in $G_n$. Thus there is at least one vertex $v$ with at least $\psi_{n,k}$ loops of length $k$ based at $v$, and it follows that $k^{-1} \log \psi_{n,k} \leq \log \beta_n$ since these loops may be concatenated freely. Also, it follows from subadditivity that $\log \beta_n = \lim_{k} k^{-1} \log \phi_{n,k} = \inf_k k^{-1} \log \phi_{n,k}$, which implies that $\beta_n^k \leq \phi_{n,k}$ for all $n$ and $k$. 
Fix $0 < \nu < 1$, and let $k = \lceil m^{1+\nu} \rceil + i$, where $i$ is chosen such that $0 \leq i \leq \per(G_1) - 1$ and \per(G_1) divides $k$. Recall that if $(G_n)$ is the sequence of $n$-block graphs of a fixed graph $G$, then by Proposition 2.29 we have that $m$ and $n$ differ by at most a uniform constant, and thus $k \sim n^{1+\nu}$.

We will show below that as $n$ tends to infinity,

(I) $(\mathbb{E}_\alpha \phi_{n,k})^{1/k}$ tends to $\alpha \lambda$;

(II) $(\mathbb{E}_\alpha \psi_{n,k})^{1/k}$ tends to $\alpha \lambda$;

(III) there exists $K_1 > 0$ and $\rho_1 > 0$ such that $\frac{\Var(\phi_{n,k})}{(\mathbb{E}_\alpha \phi_{n,k})^2} \leq K_1 e^{-\rho_1 m}$;

(IV) there exists $K_2 > 0$ and $\rho_2 > 0$ such that $\frac{\Var(\psi_{n,k})}{(\mathbb{E}_\alpha \psi_{n,k})^2} \leq K_2 e^{-\rho_2 m}$.

Recall Definitions 5.5 - 5.8, as well as the modification of these definitions using "hats." Notice that

$$\mathbb{E}_\alpha \phi_{n,k} = \sum_{b \in B_k(G_n)} \mathbb{E}_\alpha \xi_b = \sum_{b \in B_k(G_n)} \alpha^{\lvert E_n(b) \rvert} = \sum_{j=1}^k \alpha^j \lvert \hat{N}_{n,k}^j \rvert.$$  

Also,

$$\lvert V_n \rvert \mathbb{E}_\alpha \psi_{n,k} = \sum_{b \in \Per_k(G_n)} \mathbb{E}_\alpha \xi_b = \sum_{b \in \Per_k(G_n)} \alpha^{\lvert E_n(b) \rvert} = \sum_{j=1}^k \alpha^j \lvert \hat{Q}_{n,k}^j \rvert.$$  

Regarding variances, we have

$$\Var(\phi_{n,k}) = \sum_{(b,c) \in B_k(G_n)^2} \alpha^{\lvert E_n(b) \cup E_n(c) \rvert} \left(1 - \alpha^{\lvert E_n(b) \cap E_n(c) \rvert}\right) \leq \sum_{j=1}^{2k-1} \alpha^j \lvert \hat{D}_{n,k}^j \rvert;$$

and

$$\lvert V_n \rvert^2 \Var(\psi_{n,k}) = \sum_{(b,c) \in \Per_k(G_n)^2} \alpha^{\lvert E_n(b) \cup E_n(c) \rvert} \left(1 - \alpha^{\lvert E_n(b) \cap E_n(c) \rvert}\right) \leq \sum_{j=1}^{2k-1} \alpha^j \lvert \hat{S}_{n,k}^j \rvert.$$  

For the remainder of this proof, we use the following notation: if $(x_n)$ and $(y_n)$ are two sequences, then $x_n \sim y_n$ means that the limit of the ratio of $x_n$ and $y_n$ tends to 1 as $n$ tends to infinity.

**Proof of (I).** By Lemma 2.28 (1), there exists a uniform constant $K_1 > 0$ such that

$$(5.17) \ \mathbb{E}_\alpha \phi_{n,k} = \sum_{j=1}^k \alpha^j \lvert \hat{N}_{n,k}^j \rvert \geq \alpha^k \sum_{j=1}^k \lvert \hat{N}_{n,k}^j \rvert = \alpha^k \lvert B_k(G_n) \rvert \geq K_1 \alpha^k \lambda^{m+k}.$$  

Taking $k$-th roots, letting $n$ tend to infinity, and using that $m/k \sim m^{-\nu}$ tends to 0, we obtain that

$$\liminf_n \left(\mathbb{E}_\alpha \phi_{n,k}\right)^{1/k} \geq \alpha \lambda.$$  

By Lemma 2.28 (1) and Lemma 5.9, we have that there exists $n_0$, a polynomial $p_0(x)$, and a
uniform constant $K_2 > 0$ such that for $n \geq n_0$,

\[
\mathbb{E}_{\alpha} \phi_{n,k} = \sum_{j=1}^{k} \alpha^j |N_{n,k}^j|
\]

\[
\leq \sum_{j=1}^{k-1} \alpha^j |N_{n,k}^j| + \alpha^k |B_k(G_n)|
\]

\[
\leq (p_0(k))^{k/U}\left(\sum_{j=1}^{k-1} (\alpha \lambda)^j\right) + K_2 \alpha^k \lambda^{k+m}
\]

\[
\leq (\alpha \lambda)^k \lambda^m\left(\frac{1}{\alpha \lambda - 1} p_0(k)^{k/U} \lambda^{-m} + K_2\right).
\]

By condition (C4) and the fact that $k \sim m^{1+\nu}$, we have that

- $m$ tends to infinity as $n$ tends to infinity by the Standing Assumptions 2.21;
- $m/k \sim m^{-\nu}$, which tends to zero as $n$ tends to infinity;
- $U \geq m - C$, which tends to infinity as $n$ tends to infinity.

Thus, taking $k$-th roots and letting $n$ tend to infinity, we have that $\limsup_n (\mathbb{E}_{\alpha} \phi_{n,k})^{1/k} \leq \alpha \lambda$, which concludes the proof of (I).

**Proof of (II).** Let $p = \text{per}(G_1) = \text{per}(G_n)$. Note that since $p$ divides $k$, there exists a uniform constant $K_3 > 0$ such that $|\text{Per}_k(G_n)| \geq K_3 \lambda^k$ for large enough $k$. We choose $n$ large enough so that this inequality is satisfied. Then we have that

\[
\mathbb{E}_{\alpha} \psi_{n,k} = |V_n|^{-1} \sum_{j=1}^{k} \alpha^j |\hat{Q}_{n,k}^j|
\]

\[
\geq |V_n|^{-1} \alpha^k \sum_{j=1}^{k} |\hat{Q}_{n,k}^j|
\]

\[
= |V_n|^{-1} \alpha^k |\text{Per}_k(G_n)|
\]

\[
\geq K_3 \lambda^{-m} \alpha^k \lambda^k.
\]

Taking $k$-th roots, letting $n$ tend to infinity, and using that $m/k \sim m^{-\nu}$ tends to 0, we get that $\liminf_n (\mathbb{E}_{\alpha} \psi_{n,k})^{1/k} \geq \alpha \lambda$. Recall that $0 \leq \psi_{n,k} \leq \phi_{n,k}$. Therefore it follows from (I) that $\limsup_n (\mathbb{E}_{\alpha} \psi_{n,k})^{1/k} \leq \alpha \lambda$. Thus we have shown (II).

**Proof of (III).** For $j \leq 2k-1$, Lemma 5.10 implies that there is $n_1$ and a polynomial $p_1$ such that $|D_{n,k}^j| \leq p_1(k)^{k/U} \lambda^{j+m}$ and $|D_{n,k}^{2k-1}| \leq p_1(k) \lambda^{2k+m}$ for $n \geq n_1$. Now using that $\mathbb{E}_{\alpha} \phi_{n,k} \geq K_1 \alpha^k \lambda^{m+k}$
(see Equation (5.17)), we obtain that there exists a uniform constant $K_5 > 0$ such that

$$\frac{\text{Var} \Phi_{n,k}}{(\mathbb{E}_\alpha \Phi_{n,k})^2} \leq \frac{\sum_{j=1}^{2k-1} \alpha^j |\hat{D}_{n,k}^j|}{K_7^2 \alpha^{2k} \lambda^{2m+2k}}$$

$$= \frac{\sum_{j=1}^{2k-1-m} \alpha^j |\hat{D}_{n,k}^j| + \sum_{j=2k-m}^{2k-1} \alpha^j |\hat{D}_{n,k}^j|}{K_7^2 \alpha^{2k} \lambda^{2m+2k}}$$

$$\leq \frac{\sum_{j=1}^{2k-1-m} \alpha^j |\hat{D}_{n,k}^j| + \alpha^{2k-m} \sum_{j=2k-m}^{2k-1} |\hat{D}_{n,k}^j|}{K_7^2 \alpha^{2k} \lambda^{2m+2k}}$$

$$\leq \frac{p_1(k) \lambda^{m} \sum_{j=1}^{2k-1-m} (\alpha \lambda)^j + \alpha^{2k-m} |\hat{D}_{n,k}^{2k-1}|}{K_7^2 \alpha^{2k} \lambda^{2m+2k}}$$

$$\leq \frac{K_5 p_1(k) \lambda^{m} (\alpha^m \lambda^{2k-m} + \alpha^{2k-m} p_1(k) \lambda^{2k+m})}{K_7^2 \alpha^{2k} \lambda^{2m+2k}}$$

$$\leq \frac{K_5}{K_7^2} \frac{p_1(k) \lambda^{m}}{(\alpha \lambda)^m} + \frac{p_1(k)}{K_7^2 (\alpha \lambda)^m}$$

$$\leq \left( \frac{p_1(k) \lambda^{m}}{(\alpha \lambda)^m} \right)^m + \frac{p_1(k)}{K_7^2 (\alpha \lambda)^m}. $$

Using the facts that $U \geq m - C$ and $k \sim m^{1+\nu}$, we have that $k/U m$ is asymptotically bounded above by $2 m^{\nu-1}$. Since $\nu - 1 < 0$, it holds that $p_1(k) \lambda^{m}$ tends to 1. Thus we obtain that for any $0 < \rho_1 < \ln \alpha$, there exists $K_6 > 0$ and $n_2$ such that for $n \geq n_2$, it holds that

$$\text{Var} \Phi_{n,k}(\mathbb{E}_\alpha \Phi_{n,k})^{-2} \leq K_6 e^{-\rho_1 m},$$

which proves (III).

**Proof of (IV).** For $j \leq 2k - 1$, Lemma 5.12 together with (C4) implies that there is $n_3$ and a polynomial $p_3$ such that $|S_{n,k}^j| \leq p_3(k) \lambda^j$ for $n \geq n_3$. Also, Lemma 5.11 implies that there is $n_4$ and a polynomial $p_2$ such that $|S_{n,k}^{2k-1}| \leq p_2(k) \lambda^{2k-U}$ for $n \geq n_4$. Now using that
\(|V_n| \mathbb{E}_\alpha \psi_{n,k} \geq K_3 \alpha^k \lambda^k\), we obtain that there exists \(K_7 > 0\) such that, with \(K := K_3,\)
\[
\frac{\text{Var} \psi_{n,k}}{(\mathbb{E}_\alpha \psi_{n,k})^2} \lesssim \frac{\sum_{j=1}^{2k-U-1} \alpha^j |\hat{S}_{n,k}^j|}{K^2 \alpha^{2k} \lambda^{2k}}
\]
\[
= \frac{\sum_{j=1}^{2k-U-1} \alpha^j |\hat{S}_{n,k}^j| + \sum_{j=2k-U}^{2k-1} \alpha^j |\hat{S}_{n,k}^j|}{K^2 \alpha^{2k} \lambda^{2k}}
\]
\[
\lesssim \frac{\sum_{j=1}^{2k-U-1} \alpha^j |\hat{S}_{n,k}^j| + \alpha^{2k-U} \sum_{j=2k-U}^{2k-1} |\hat{S}_{n,k}^j|}{K^2 \alpha^{2k} \lambda^{2k}}
\]
\[
\lesssim \frac{p_3(k)^{k/U} \sum_{j=1}^{2k-U-1} (\alpha \lambda)^j + \alpha^{2k-U} |\hat{S}_{n,k}^{2k-1}|}{K^2 \alpha^{2k} \lambda^{2k}}
\]
\[
\lesssim \frac{K_7 p_3(k)^{k/U} (\alpha \lambda)^{2k-U} + \alpha^{2k-U} p_2(k) \lambda^{2k-U}}{K^2 \alpha^{2k} \lambda^{2k}}
\]
\[
\lesssim \frac{K_7}{K^2} \left( \frac{p_3(k)^{k/U^2}}{(\alpha \lambda)^U} \right) + \frac{p_2(k)}{K^2 (\alpha \lambda)^U}.
\]

Using the facts that \(U \geq m - C\) and \(k \sim m^{1+\nu}\), we have that \(k/U^2\) is asymptotically bounded above by \(2m^{\nu-1}\). Since \(\nu - 1 < 0\), it holds that \(p_3(k)^{k/U^2}\) tends to 1. Thus we obtain that for any \(0 < \rho_2 < \log \alpha\lambda\), there exists \(K_8 > 0\) and \(n_5\) such that for \(n \geq n_5,\)
\[
\frac{\text{Var} \phi_{n,k}}{(\mathbb{E}_\alpha \phi_{n,k})^2} \leq K_8 e^{-\rho_2 m},
\]
which proves (IV).

**Proof of Theorem 5.13 using (I)-(IV).** Recall that \(\psi_{n,k} \leq \beta_n^k \leq \phi_{n,k}\). Let \(\varepsilon > 0\). Since \(\alpha \lambda > 1\), we may assume without loss of generality that \(\alpha \lambda - \varepsilon > 1\). Then
\[
P_\alpha \left( |\beta_n - \alpha \lambda| \geq \varepsilon \right) = P_\alpha \left( \beta_n \geq \alpha \lambda + \varepsilon \right) + P_\alpha \left( \beta_n \leq \alpha \lambda - \varepsilon \right)
\]
\[
= P_\alpha \left( \beta_n^k \geq (\alpha \lambda + \varepsilon)^k \right) + P_\alpha \left( \beta_n^k \leq (\alpha \lambda - \varepsilon)^k \right)
\]
\[
\leq P_\alpha \left( \phi_{n,k} \geq (\alpha \lambda + \varepsilon)^k \right) + P_\alpha \left( \psi_{n,k} \leq (\alpha \lambda - \varepsilon)^k \right).
\]

We will bound each of the two terms in Equation (5.20). Notice that
\[
P_\alpha \left( \phi_{n,k} \geq (\alpha \lambda + \varepsilon)^k \right) = P_\alpha \left( \phi_{n,k} - \mathbb{E}_\alpha \phi_{n,k} \geq (\alpha \lambda + \varepsilon)^k - \mathbb{E}_\alpha \phi_{n,k} \right)
\]
\[
= P_\alpha \left( \phi_{n,k} - \mathbb{E}_\alpha \phi_{n,k} \geq \mathbb{E}_\alpha \phi_{n,k} \left( \left( \frac{\alpha \lambda + \varepsilon}{\mathbb{E}_\alpha \phi_{n,k}} \right)^{1/k} - 1 \right) \right).
\]
Let $d_{n,k}^1 = (\text{Var}(\phi_{n,k}))^{1/2} / \mathbb{E}_\alpha \phi_{n,k}$. Then by Chebychev’s Inequality,

\begin{align}
\P_{\alpha} \left( \phi_{n,k} \geq (\alpha \lambda + \epsilon)^k \right) &= \P_{\alpha} \left( \phi_{n,k} - \mathbb{E}_\alpha \phi_{n,k} \geq (\text{Var}(\phi_{n,k}))^{1/2} \frac{1}{d_{n,k}^1} \left( \frac{\alpha \lambda + \epsilon}{(\mathbb{E}_\alpha \phi_{n,k})^{1/k}} - 1 \right) \right) \\
&= \P_{\alpha} \left( \phi_{n,k} - \mathbb{E}_\alpha \phi_{n,k} \geq (\text{Var}(\phi_{n,k}))^{1/2} \frac{1}{d_{n,k}^1} \left( \frac{\alpha \lambda + \epsilon}{(\mathbb{E}_\alpha \phi_{n,k})^{1/k}} - 1 \right) \right) \leq \left( \frac{d_{n,k}^1}{\left( \frac{\alpha \lambda + \epsilon}{(\mathbb{E}_\alpha \phi_{n,k})^{1/k}} - 1 \right)} \right)^2.
\end{align}

The denominator in the right-hand side of (5.23) might be 0 for finitely many $n$, but by properties (I) and (III), there exists $K_9 > 0$ such that for large enough $n$,

\[ \P_{\alpha} \left( \phi_{n,k} \geq (\alpha \lambda + \epsilon)^k \right) \leq \left( \frac{d_{n,k}^1}{\left( \frac{\alpha \lambda + \epsilon}{(\mathbb{E}_\alpha \phi_{n,k})^{1/k}} - 1 \right)} \right)^2 \leq K_9 e^{-\rho_1 m}. \]

Similarly, we let $d_{n,k}^2 = (\text{Var}(\psi_{n,k}))^{1/2} / \mathbb{E}_\alpha \psi_{n,k}$, and then Chebychev’s Inequality gives that

\begin{align}
\P_{\alpha} \left( \psi_{n,k} \leq (\alpha \lambda - \epsilon)^k \right) &= \P_{\alpha} \left( \psi_{n,k} - \mathbb{E}_\alpha \psi_{n,k} \leq (\text{Var}(\psi_{n,k}))^{1/2} \frac{1}{d_{n,k}^2} \left( \frac{\alpha \lambda - \epsilon}{(\mathbb{E}_\alpha \psi_{n,k})^{1/k}} - 1 \right) \right) \\
&= \P_{\alpha} \left( \psi_{n,k} - \mathbb{E}_\alpha \psi_{n,k} \leq (\text{Var}(\psi_{n,k}))^{1/2} \frac{1}{d_{n,k}^2} \left( \frac{\alpha \lambda - \epsilon}{(\mathbb{E}_\alpha \psi_{n,k})^{1/k}} - 1 \right) \right) \leq \left( \frac{d_{n,k}^2}{\left( \frac{\alpha \lambda - \epsilon}{(\mathbb{E}_\alpha \psi_{n,k})^{1/k}} - 1 \right)} \right)^2,
\end{align}

Again, the denominator in the right-hand side might be 0 for finitely many $n$, but by properties (II) and (IV), there exists $K_{10} > 0$ such that for large enough $n$,

\[ \P_{\alpha} \left( \psi_{n,k} \leq (\alpha \lambda - \epsilon)^k \right) \leq \left( \frac{d_{n,k}^2}{\left( \frac{\alpha \lambda - \epsilon}{(\mathbb{E}_\alpha \psi_{n,k})^{1/k}} - 1 \right)} \right)^2 \leq K_{10} e^{-\rho_2 m}. \]

In conclusion, we obtain that there exists $K_{11} > 0$ such that for large enough $n$,

\[ \P_{\alpha} (|\beta_n - \alpha \lambda| \geq \epsilon) \leq K_{11} e^{-\min(\rho_1, \rho_2) m}. \]

\[ \square \]

5.3. Irreducible components of positive entropy.

**Theorem 5.15.** Let $(G_n)$ be a sequence of graphs that satisfies the Standing Assumptions 2.21, with $p = \text{per}(G_1) = \text{per}(G_n)$, and such that

- $(G_n)$ has bounded degrees (condition (C1) in 2.23),
• \((G_n)\) has fast separation of periodic points (condition (C3) in 2.23),
• \((G_n)\) has uniform forward and backward expansion (condition (C8) in 2.23).

Let \(U_{G_n}\) be the event in \(\Omega_{G_n}\) that \(G_n(\omega)\) contains a unique irreducible component \(C\) of positive entropy. Also, let \(W_{G_n}\) be the event (contained in \(U_{G_n}\)) that the induced edge shift on \(C\) has period \(p\). Then there exists \(c > 0\) such that for \(1 - c < \alpha \leq 1\),

\[
\lim_{n \to \infty} P_\alpha(U_{G_n}) = 1, \quad \text{and} \quad \lim_{n \to \infty} P_\alpha(W_{G_n}) = 1,
\]

and the convergence to these limits is exponential in \(m(G_n)\).

**Remark 5.16.** Theorem 1.4 is a corollary of Theorem 5.15: if \(X\) is an irreducible SFT of positive entropy, then the sequence of \(n\)-block graphs for \(X\) satisfies the hypotheses of Theorem 5.15 by Proposition 2.29. In fact, if \(X\) is a reducible SFT, we may apply Theorem 1.4 to each irreducible component independently, which allows us to conclude the following. Let \(X\) be a reducible SFT with irreducible components \(X_1, \ldots, X_r\) such that \(p_i = \text{per}(X_i)\) for each \(i\). Let \(W_n\) be the event in \(\Omega_n\) that \(X_\omega\) has exactly \(r\) irreducible components with periods \(p_1, \ldots, p_r\). Then there exists \(c > 0\) such that for \(\alpha \in (1 - c, 1]\), we have that \(\lim_n P_\alpha(W_n) = 1\), with exponential (in \(n\)) convergence to the limit.

**Definition 5.17.** Let \(G\) be a directed graph. For each vertex \(v\) in \(G\), and for each \(\omega\) in \(\Omega_G\), let \(\Gamma^+_\omega(v)\) be the union of \(\{v\}\) and the set of vertices \(u\) in \(G\) such that there is an allowed path from \(v\) to \(u\) in \(G(\omega)\). Similarly, for each vertex \(v\) in \(G\) and each \(\omega\) in \(\Omega_G\), let \(\Gamma^-_\omega(v)\) be the union of \(\{v\}\) and the set of vertices \(u\) in \(G\) such that there is an allowed path from \(u\) to \(v\) in \(G(\omega)\). Also, let \(I_\omega(v) = \Gamma^+_\omega(v) \cap \Gamma^-_\omega(v)\), which is the vertex set of the irreducible component containing \(v\) in \(G(\omega)\).

The proof of the following proposition is an adaptation of the proof of Lemma 2.2 in [2].

**Proposition 5.18.** Let \((G_n)\) be a sequence of graphs satisfying the Standing Assumptions 2.21 and such that \((G_n)\) has bounded degrees and uniform forward and backward expansion (conditions (C1) and (C8) in 2.23). Let \(r_n\) be a sequence of integers such that \(r_n \geq am(G_n)\), for some \(a > 0\), for all large \(n\). Let \(C^+_G_{G_n}\) be the event in \(\Omega_{G_n}\) consisting of all \(\omega\) such that there exists a vertex \(v\) in \(G_n\) with \(r_n \leq \Gamma^+_\omega(v) \leq |V_n|/2\). Then there exists \(c > 0\) such that for \(\alpha > 1 - c\),

\[
\lim_{n \to \infty} P_\alpha(C^+_G_{G_n}) = 0,
\]
and the convergence of this limit is exponential in $m(G_n)$. Furthermore, the same statement holds with "+" replaced by "−".

**Proof.** Let $m = m(G_n)$. Let $b > 0$ be such that both $(G_n)$ and $(G^T_n)$ are $b$-expander sequences (where the existence of such a $b$ is guaranteed by condition (C8)). We use the notation in Definition 5.17. For any $v$ in $V_n$ and any $\omega$ in $\Omega_{G_n}$, the set $\Gamma^+_\omega(v)$ has the property that all edges in $E_n(\Gamma^+_\omega(v), \Omega_{G_n})$ are forbidden (by $\omega$). Then the fact that $G_n$ is a $b$-expander implies that for a particular subset $S$ of $V_n$, the probability that $S = \Gamma^+_\omega(v)$ for some $v$ is bounded above by $(1 - \alpha)^{|S|}$. The number of subsets $S$ of $V_n$ with $|S| = r$ that could appear as $\Gamma^+_\omega(v)$ for some $v$ is bounded above by $(\Delta e)^r$, where $e$ is the base of the natural logarithm [2, Lemma 2.2] (see also [1, Lemma 2.1] or [37, p. 396, Exercise 11]). Then for $\alpha$ such that $\Delta e (1 - \alpha) < 1$, we have that for any $0 \leq r_n \leq |V_n|/2$, 

\begin{align*}
\mathbb{P}_\alpha(C^+_\omega) = \mathbb{P}_\alpha(\exists v \text{ such that } r_n \leq |\Gamma^+_\omega(v)| \leq \frac{|V_n|}{2}) \\
\leq \sum_{r=r_n}^{\lfloor V_n \rfloor} |V_n|(\Delta e)^r (1 - \alpha)^{br} \\
\leq |V_n|(\Delta e (1 - \alpha)^b)^{r_n} \frac{1}{1 - \Delta e (1 - \alpha)} \\
\leq (\lambda^{1/a} \Delta e (1 - \alpha)^b)^{am} \frac{1}{1 - \Delta e (1 - \alpha)}.
\end{align*}

Thus there is a $c > 0$ (depending only on $a$, $b$, $\lambda$, and $\Delta$) such that if $\alpha > 1 - c$, then the right-hand side of the inequality in (5.31) tends to zero exponentially in $m(G_n)$ as $n$ tends to infinity. In particular, we may take 

$$c = \left(\frac{1}{\lambda}\right)^{1/ab} \left(\frac{1}{\Delta e}\right)^{1/b}.$$ 

Since $(G^T_n)$ is also a uniform $b$-expander, the same estimates hold with $C^-_{G_n}$ in place of $C^+_\omega$. 

**Proof of Theorem 5.15.** Let $(G_n)$ be as in the statement of Theorem 5.15. Let $m = m(G_n)$, $z = z(G_n)$, and $p = \text{per}(G_1) = \text{per}(G_n)$. We use the notation in Definition 5.17. Consider the following events:

$$F^+_n = \{\omega \in \Omega_n : \forall v \in V_n, \Gamma^+_\omega(v) \leq z(G_n) - 2p \text{ or } \Gamma^+_\omega(v) > |V_n|/2\}$$
$$F^-_n = \{\omega \in \Omega_n : \forall v \in V_n, \Gamma^-_\omega(v) \leq z(G_n) - 2p \text{ or } \Gamma^-_\omega(v) > |V_n|/2\}$$
$$F_n = F^+_n \cap F^-_n.$$
Recall that condition (C3) gives $a > 0$ such that $z \geq am$. Note that Proposition 5.18 gives $c > 0$ such that for $1 - c < \alpha \leq 1$, there exists $K_1, K_2 > 0$ and $\rho_1, \rho_2 > 0$ such that for large $n$, 

$$\mathbb{P}_\alpha(\Omega_n \setminus F_n^+) \leq K_1 e^{-\rho_1 m} \quad \text{and} \quad \mathbb{P}_\alpha(\Omega_n \setminus F_n^-) \leq K_2 e^{-\rho_2 m}.$$ 

Fix such an $\alpha$, and note that for all large enough $n$, we have the following estimate: $\mathbb{P}_\alpha(\Omega_n \setminus F_n) \leq 2 \max(K_1, K_2) e^{-\min(\rho_1, \rho_2) m}$.

Consider $\omega$ in $F_n$. Suppose that there exists $v_1$ and $v_2$ in $V_n$ such that $|I_\omega(v_1)| > z - 2p$ and $|I_\omega(v_2)| > z - 2p$. Then by definition of $F_n$, we must have that $\Gamma_\omega^+(v_1) \cap \Gamma_\omega^-(v_2) \neq \emptyset$ and $\Gamma_\omega^-(v_1) \cap \Gamma_\omega^+(v_2) \neq \emptyset$. It follows that there is a path from $v_1$ to $v_2$ in $G_n(\omega)$, and there is a path from $v_2$ to $v_1$ in $G_n(\omega)$. Thus $I_\omega(v_1) = I_\omega(v_2)$. We have shown that for $\omega$ in $F_n$, there is at most one irreducible component of cardinality greater than $z - 2p$. Note that this argument implies that for $\omega$ in $F_n$, all allowed periodic orbits $\gamma$ such that $|V_\gamma(\gamma)| > z - 2p$ must lie in the same irreducible component.

By definition of $z$, if $I_\omega$ is an irreducible component of $G_n(\omega)$ with positive entropy, then $|I_\omega| > z$ (since it must contain at least two periodic orbits with overlapping vertex sets). We deduce that for $\omega$ in $F_n$, there is at most one irreducible component of $G_n(\omega)$ with positive entropy.

We now show that there exists an irreducible component of positive entropy with probability tending exponentially to 1. Let $z_1 = z - i$, where $i$ is chosen (for each $n$) such that $0 \leq i \leq p - 1$ and $p$ divides $z_1$. Then let $z_2 = z_1 - p$. Consider the following sequences of random variables:

$$f_n = \sum_{b \in \text{Per}_{z_1}(G_n)} \xi_b, \quad \text{and} \quad g_n = \sum_{b \in \text{Per}_{z_2}(G_n)} \xi_b.$$ 

Note that by the definition of $z$ and Lemma 3.3, we have that $|E_n(b)| = |b|$ for any periodic path $b$ with period less than or equal to $z$. Furthermore, any two such paths are disjoint. Therefore the random variables $\{\xi_b\}_{b \in \text{Per}_{z_1}(G_n)}$ are jointly independent, and the random variables $\{\xi_b\}_{b \in \text{Per}_{z_2}(G_n)}$ are also jointly independent. Thus

$$\mathbb{E}_\alpha f_n = \sum_{b \in \text{Per}_z(G_n)} \alpha^{z_1} = \alpha^{z_1} |\text{Per}_{z_1}(G_n)|,$$

$$\mathbb{E}_\alpha g_n = \sum_{b \in \text{Per}_{z_2}(G_n)} \alpha^{z_2} = \alpha^{z_2} |\text{Per}_{z_2}(G_n)|,$$

$$\text{Var}(f_n) = \sum_{b \in \text{Per}_{z_1}(G_n)} \alpha^{z_1} (1 - \alpha^{z_1}) = \alpha^{z_1} (1 - \alpha^{z_1}) |\text{Per}_{z_1}(G_n)|$$

$$\text{Var}(g_n) = \sum_{b \in \text{Per}_{z_2}(G_n)} \alpha^{z_2} (1 - \alpha^{z_2}) = \alpha^{z_2} (1 - \alpha^{z_2}) |\text{Per}_{z_2}(G_n)|.$$
As \( n \) tends to infinity, \( z \) tends to infinity since \( z \geq am \) and \( m \) tends to infinity. Then by the Standing Assumptions 2.21 (in particular, we use that \( \text{Sp}_x(G_n) = \text{Sp}_x(G_1) \)) and the fact that \( p \) divides \( z_1 \) and \( z_2 \), we have that each of of the sequences \( \lambda^{-z_1}\left|\text{Per}_{z_1}(G_n)\right| \) and \( \lambda^{-z_2}\left|\text{Per}_{z_2}(G_n)\right| \) tends to a finite, non-zero limit as \( n \) tends to infinity (and in fact the limit is \( p \)).

For two sequences \( x_n \) and \( y_n \) of positive real numbers, let \( x_n \sim y_n \) denote the statement that their ratio tends to a finite, non-zero limit as \( n \) tends to infinity. Then we have that \( \mathbb{E}_\alpha f_n \sim (\alpha \lambda)^{z_1} \sim \text{Var}(f_n) \) and \( \mathbb{E}_\alpha g_n \sim (\alpha \lambda)^{z_2} \sim \text{Var}(g_n) \). Note that \( \mathbb{E}_\alpha f_n \geq \text{Var}(f_n) \) and \( \mathbb{E}_\alpha g_n \geq \text{Var}(g_n) \). A simple application of Chebychev’s inequality implies that

\[
\mathbb{P}_\alpha(f_n \leq 0) \leq \mathbb{P}_\alpha\left(f_n - \mathbb{E}_\alpha f_n \leq - \text{Var}(f_n)\right) \\
\leq \left(\frac{1}{\text{Var}(f_n)^{1/2}}\right)^2 \sim \left(\frac{1}{\alpha \lambda}\right)^{z_1} \leq \left(\frac{1}{\alpha \lambda}\right)^{am-i},
\]

and

\[
\mathbb{P}_\alpha(g_n \leq 0) \leq \mathbb{P}_\alpha\left(g_n - \mathbb{E}_\alpha g_n \leq - \text{Var}(g_n)\right) \\
\leq \left(\frac{1}{\text{Var}(g_n)^{1/2}}\right)^2 \sim \left(\frac{1}{\alpha \lambda}\right)^{z_2} \leq \left(\frac{1}{\alpha \lambda}\right)^{am-i-p}.
\]

We have shown that the probability that there is no periodic orbit of period \( z_1 \) tends to 0 exponentially in \( m \) as \( n \) tends to infinity, and the probability that there exists no periodic orbit of period \( z_2 \) tends to 0 exponentially in \( m \) as \( n \) tends to infinity.

In summary, we have shown that the following events occur with probability tending to 1 exponentially in \( m \) as \( n \) tends to infinity:

- there exists a periodic point of period \( z - i \);
- there exists a periodic point of period \( z - i - p \);
- any two periodic points of period greater than \( z - 2p \) lie in the same irreducible component (of necessarily positive entropy);
- there is at most one irreducible component of positive entropy.

We conclude that with probability tending to 1 exponentially in \( m \) as \( n \) tends to infinity, there exists a unique irreducible component of positive entropy, and the induced edge shift on that component has period \( p \).

Remark 6.1. The proofs of Theorems 3.1 and 4.2 do not require all of the Standing Assumptions 2.21. In fact, these proofs only use that \( \text{Sp}_\chi(G_n) = \text{Sp}_\chi(G_1) \) for each \( n \) and that \( z(G_n) \) tends to infinity as \( n \) tends to infinity.

Remark 6.2. Theorem 3.1 states that at the critical threshold \( \alpha = 1/\lambda \), the probability of emptiness tends to zero. Using the fact that entropy is a monotone increasing random variable (as defined in Section 2.3), one may deduce from Theorem 5.13 that for \( \alpha = 1/\lambda \), the probability that the random SFT has zero entropy tends to 1. It might be interesting to know more about the behavior of typical random SFTs at the critical threshold.

Remark 6.3. We have considered only random \( \mathbb{Z} \)-SFTs, but one may also consider random \( \mathbb{Z}^d \)-SFTs for any \( d \) in \( \mathbb{N} \) by adapting the construction of \( \Omega_n \) and \( P_\alpha \) in the obvious way. It appears that most of the proofs presented above may not be immediately adapted for \( d > 1 \), but there is one exception, which we state below. Let \( X \) be a non-empty \( \mathbb{Z}^d \)-SFT. For \( d > 1 \), there are various zeta functions for \( X \) (for a definition distinct from ours, see [39]); we consider

\[
\zeta_X(t) = \exp \left( \sum_{p=1}^{\infty} \frac{N_p t^p}{p} \right),
\]

where \( N_p \) is the number of periodic points \( x \) in \( X \) such that the number of points in the orbit of \( x \) divides \( p \). The function \( \zeta_X \) has radius of convergence \( 1/\rho \), where \( \log \rho = \lim \sup p^{-1} \log(N_p) \). For example, for a full \( \mathbb{Z}^d \) shift on \( a \) symbols, \( \rho = a \), regardless of \( d \). Using exactly the same proof as presented in Section 3, we obtain that

\[
\lim \sup_n \mathbb{P}_\alpha(\mathcal{E}_n) \leq \begin{cases} 
(\zeta_X(\alpha))^{-1}, & \text{if } \alpha \in [0, 1/\rho) \\
0, & \text{if } \alpha \in [1/\rho, 1].
\end{cases}
\]

For \( \alpha \geq 1/\rho \), this bound implies that the limiting probability of emptiness is 0. In this context, we note that there is no algorithm, which, given a \( \mathbb{Z}^d \)-SFT \( X \) defined by a finite list of finite forbidden configurations, will decide whether \( X \) is empty [6]. Nonetheless, we may be able to compute the limiting probability of emptiness. For example, if \( X \) is a full shift on \( a \) symbols, then for \( \alpha \geq 1/a \), we have that the limiting probability of emptiness is 0.
Remark 6.4. One may also consider more general random subshifts. Recall that a set $X \subset \mathcal{A}^\mathbb{Z}$ is a subshift if it is closed and shift-invariant. For a non-empty subshift $X$ and a natural number $n$, we may consider the (finite) set of subshifts obtained by forbidding words of length $n$ from $X$. After defining a probability measure $P_\alpha$ on this space as in Section 2, we obtain random subshifts of $X$. Now we may investigate the asymptotic probability of properties of these random subshifts. Recall that any subshift $X$ can be written as $\bigcap X_n$, where $(X_n)$ is a sequence of SFTs (called the Markov approximations of $X$) and $\lim_n h(X_n) = h(X)$. A subshift $X$ is called almost sofic [48] if there exists a sequence $(X_n)$ of irreducible SFTs such that $X_n \subset X$ and $\lim_n h(X_n) = h(X)$. Using this inner and outer approximation by SFTs, the conclusion of Theorem 1.3 still holds if the system $X$ is only assumed to be an almost sofic subshift.

Remark 6.5. Theorem 5.15 asserts the existence of a constant $c > 0$, but we are left with several questions about this constant. Fix a sequence $(G_n)$ satisfying the hypotheses of Theorem 5.15. Let $\alpha_* = \inf\{\alpha > 0 : \lim_n P_\alpha(U_n) = 1\}$. What is $\alpha_*$? What is $\alpha_*$ in the case that $(G_n)$ is the sequence of $n$-block graphs of a mixing SFT of positive entropy (or even a full shift)?

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References.


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