Equivariant dendroidal sets and simplicial operads

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Abstract
We establish a Quillen equivalence between the homotopy theories of equivariant Segal operads and equivariant simplicial operads with norm maps. Together with previous work, we further conclude that the homotopy coherent nerve is a right-Quillen equivalence from the model category of equivariant simplicial operads with norm maps to the model category structure for equivariant-∞-operads in equivariant dendroidal sets.

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1 Introduction

This paper is the last in a series of four (after [Per18, BPa, BPc]) and concludes a project to establish a homotopy theoretical equivalence between equivariant colored simplicial operads with norm maps and $G$-$\infty$-operads in equivariant dendroidal sets, thus generalizing the analogous Cisinski-Moerdijk project [CM13a, CM13b, CM11] in the non-equivariant setting.

The key novelty (and difficulty) faced in the equivariant setting is that the homotopy theory of operads then needs to account for an extra piece of structure, the so called norm maps, which we now briefly recall (for a more extensive discussion, see the introductions to any of [Per18],[BPb],[BPc]).

For simplicity, let us focus on the category $sOp^*_\ast$ of single colored simplicial operads. Letting $G$ be a fixed finite group, a $G$-equivariant (single colored simplicial) operad is a $G$-object $O \in sOp^G_\ast$. Crucially, note that the $n$-th level $O(n)$ then admits commuting actions by $\Sigma_n$ and $G$ or, equivalently, a $G \times \Sigma_n$ action. One upshot of Blumberg and Hill’s work in [BH15] is that the preferred notion of equivalence in $sOp^G_\ast$ is that of graph equivalence, by which we mean maps $O \to P$ in $sOp^G_\ast$ such that the fixed point maps

$$O(n)^\Gamma \sim P(n)^\Gamma$$

for $\Gamma \leq G \times \Sigma_n$ such that $\Gamma \cap \Sigma_n = \ast$ (1.1)

are Kan equivalences. The term graph is motivated by a description of such $\Gamma$: they are necessarily of the form $\Gamma = \{(h, \phi(H))| h \in H\}$ for some subgroup $H \leq G$ and homomorphism $\phi: H \to \Sigma_n$. Such fixed points $O(n)^\Gamma$ are called spaces of norm maps since, for $X$ an $O$-algebra, the algebra multiplication maps on the left below

$$O(n) \times X^n \to X \quad O(n)^\Gamma \times N_{\Gamma}X \to X$$

induce $H$-equivariant maps as on the right above, where $N_{\Gamma}X$ is a so called norm object, which denotes $X^n$ together with the $H$-action induced by the identification $H \simeq \Gamma$.

The cornerstone of this project was the discovery by the authors of a category $\Omega_G$ of $G$-trees whose objects encode compositions of norm maps in a $G$-operad $O$, and which extends the Moerdijk-Weiss category $\Omega$ of trees (whose objects encode composition in an operad). This category $\Omega_G$ then allowed us to build model structures on the categories $dSet^G = Set^\Gamma^{op} \times G$ of equivariant dendroidal sets [Per18] and $sOp^G$ of equivariant colored simplicial operads [BPc], where in both cases the notion of weak equivalence depends on (a colored variant of) the norm map data as in (1.1). Our main result in this paper is then the following, generalizing [CM13b, Thm. 8.15].

**Theorem I.** There is a Quillen equivalence

$$W: dSet^G \rightleftarrows sOp^G: hcN$$

(1.2)

between equivariant dendroidal sets and equivariant simplicial operads with norm maps.

Here the right adjoint $hcN$ is a variant of the nerve that accounts for homotopy information, called the homotopy coherent nerve, while the left adjoint $W$ is a “fattened operadification” which is related to the Boardman-Vogt resolution of operads.

We note that while, at the level of categories, the adjunction in Theorem I is obtained from that in the non-equivariant analogue result [CM13b, Thm. 8.15], Theorem I is not a formal consequence of [CM13b, Thm. 8.15] since none of our model structures are built formally from those in [CM13b, Thm. 8.15].

Similarly, slicing over the terminal operad $Comm$ in Theorem I recovers the equivalence of homotopy theories between equivariant simplicial categories and equivariant quasicategories from
However, Bergner’s results rely upon the axiomatic framework from [Ste16], and as noted in e.g. the intro to [BPc], the operadic case is intrinsically outside the bounds of this formalism.

The conclusion to the proof of Theorem I is given at the end of §4.3. However, this proof requires some background, which we now recall. Just as in [CM13b], we make use of two additional categories, the category $sdSet^G = Set^G × Δ^{op} × G$ of equivariant dendroidal simplicial sets and its subcategory $PreOp^G$ of equivariant preoperads, which fit into a diagram

$$
\begin{array}{ccc}
PreOp^G & \xleftarrow{N} & sOp^G \\
\gamma^* & \downarrow & \downarrow b,cN \\
sdSet^G & \xleftarrow{c} & dSet^G
\end{array}
$$

(1.3)

where $N$ is the nerve functor and $\gamma^*$, $c$ are the natural inclusions.

The model structures on the categories featured in (1.3) were built in previous work: more specifically [Per18, Thm. 2.1] provides the model structure on $dSet^G$, [BPc, Thm. 1] provides the model structure on $sOp^G$, [BPa, Def. 4.22] gives the model structure on $sdSet^G$, and [BPa, Thm. 4.39] provides the model structure on $PreOp^G$. These model structures generalize those in the work of Cisinski-Moerdijk in the non-equivariant operadic context, which in turn generalize corresponding model structures in the categorical context. The following table summarizes the relevant model structures, along with the nomenclature for the fibrant objects.

<table>
<thead>
<tr>
<th>“categories up to htpy”</th>
<th>“operads up to htpy”</th>
<th>“equivariant operads up to htpy”</th>
</tr>
</thead>
<tbody>
<tr>
<td>simplicial sets $sSet$</td>
<td>Joyal model structure $\infty$-categories</td>
<td>$dSet^G$ model str. from [CM11]</td>
</tr>
<tr>
<td>$sSet$</td>
<td>Joyal model structure $\infty$-operads</td>
<td>$dSet^G$ model structure from [Per18] $G$-$\infty$-operads</td>
</tr>
<tr>
<td>bisimplicial sets $ssSet$</td>
<td>Rezk model structure complete Segal spaces</td>
<td>simp. dend. sets $sdSet^G$</td>
</tr>
<tr>
<td>$ssSet$</td>
<td>Rezk model structure complete Segal spaces</td>
<td>model str. from [CM13a] complete dend. Segal spaces</td>
</tr>
<tr>
<td>Segal precategories $SeCat$</td>
<td>Segal categories</td>
<td>Segal preoperads $PreOp^G$</td>
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<td>$SeCat$</td>
<td>Segal categories</td>
<td>model str. from [CM13a] equiv. Segal operads</td>
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<tr>
<td>simplicial categories $sCat$</td>
<td>Bergner model structure</td>
<td>simplicial operads $sOp$</td>
</tr>
<tr>
<td>$sCat$</td>
<td>Bergner model structure</td>
<td>model str. from [CM13b] equiv. simplicial operads $sOp^G$</td>
</tr>
</tbody>
</table>

Table 1: A summary of models for $\infty$-categories, $\infty$-operads, and $G$-$\infty$-operads.

Considering now the functors in (1.3), we have previously established that $c$ and $\gamma^*$ are both left adjoints in a Quillen equivalence [BPa, Thms. 4.30 and 4.41]. The proof strategy for establishing Theorem I can then be summarized as follows.

First, the $(W, hcN)$ adjunction is shown to be Quillen (cf. Proposition 4.47).

Second, the square (1.3) is shown to commute at the level of homotopy categories (this is what is actually shown at the end of §4.3, by establishing the zigzag of weak equivalences in (4.52)).

Third and last, it thus suffices to show that the top horizontal functor $N$ in (1.3) induces an equivalence of homotopy categories. As in [CM13b], this last step requires some care. The functor $N$ preserves all weak equivalences (see e.g. the proof of Theorem 4.39) and is thus already a derived functor, but is not quite right Quillen due to $PreOp^G$ not having enough fibrant objects or, dually, having too many cofibrant objects. To address this, we show in §3.4 that $PreOp^G$ admits an alternative model structure, called the tame model structure (cf. Theorem 3.39), with the same weak equivalences but less cofibrant objects. Using this alternative model structure, it
can then be shown (Theorem 4.39) that \(N\) becomes the right adjoint in a Quillen equivalence, concluding the argument.

1.1 Outline

First, in §2 we mostly recall some previous notions that are use throughout. Namely, §2.1 and §2.2 recall the key properties of the categories \(\Omega\) of trees and \(\Omega_G\) of \(G\)-trees needed throughout, while §2.3 recalls the category \(d\text{Set}^G\) of equivariant dendroidal sets.

The overall goal of §3 is to produce the alternative tame model structure on \(\text{PreOp}^G\). The content of §3.1 and §3.2 are again expository in nature, recalling the model structures on \(sd\text{Set}^G\), \(\text{PreOp}^G\). In §3.3 we introduce a somewhat novel construction, called the fibered tensored product \(\otimes_e\), which is then used in §3.4 to describe and build the tame model structure on \(\text{PreOp}^G\). One advantage of framing our description of the tame model structure in terms of \(\otimes_e\) is that the model structure on \(s\text{Op}^G\) can be described using an analogous tensor product, thus simplifying the task of showing that the \(\tau: \text{PreOp}^G_{tame} \rightleftarrows s\text{Op}^G: N\) adjunction is Quillen.

The goal of §4 is to prove Theorem I (up to Lemma 4.28, whose proof is postponed to §5). First, §4.1 recalls the model structure on \(s\text{Op}^G\) from [BP]. Then, in §4.2 we establish Theorem 4.39, showing that the top functor \(N\) in (1.3) induces an equivalence of homotopy categories. Lastly, §4.3 concludes the proof of Theorem I by showing that (1.3) commutes in a homotopical sense.

Our last main section §5 is dedicated to the rather technical proof of Lemma 4.28, which examines the homotopical properties of certain pushouts in \(\text{Op}^G\) after applying the nerve functor \(N: \text{Op}^G \rightarrow d\text{Set}^G\), and is at the core of the proof of Theorem 4.39 and thus also of Theorem I.

In Appendix A we provide an explicit description of the left adjoint in the \(W: d\text{Set} \rightleftarrows \text{Op}: h\text{cN}\) adjunction, extending work of Dugger-Spivak [DS11] from the categorical context to the operadic context. This description plays a minor role in our proofs in Section 4.3, being used to describe \(W\) when applied to an inner horn, cf. Lemma 4.45 (this description is essentially left as an exercise to the reader in the proof of [CM13b, Prop. 4.5]). Nonetheless, we believe our description is of intrinsic interest, since our approach is rather different from that in [DS11], being focused on formal properties of the category \(\Omega\) of trees.

Lastly, in Appendix B we give an explicit description of the discretization of a \(G\)-∞-operad \(X \in d\text{Set}^G\), adapting the similar non-equivariant description in [MW09, §6]. This then allows us to show that, for a fibrant operad \(\Sigma \in s\text{Op}^G\), the natural discretizations of \(h\text{cN}(\Sigma) \in d\text{Set}^G\) and \(N(\Sigma) \in sd\text{Set}^G\) coincide, cf. Proposition B.12, thus generalizing the non-equivariant analogue result [CM13b, Prop. 4.8]. Just as in the non-equivariant story, Proposition B.12 is closely related to the proof of Proposition 4.47, which shows that \(W: d\text{Set} \rightleftarrows \text{Op}: h\text{cN}\) is a Quillen adjunction, though Proposition 4.47 does not require the full strength of Proposition B.12. A more detailed discussion can be found in Remark 4.48.

2 Equivariant trees and dendroidal sets

In this mostly expository section, we recall the categories of trees, as well as the associated presheaf categories, which will be needed throughout the paper. A more detailed discussion can be found in [Per18], [BP].
2.1 Trees and forests

We start by recalling the Moerdijk-Weiss category $\Omega$ of trees [MW07]. First, each object of $\Omega$ can be encoded by a (rooted) tree diagram $T$ as below.

![Tree Diagram](image)

(2.1)

Edges with no vertices $\circ$ above them are called leaves, the unique bottom edge is called the root, and edges that are neither are called inner edges. In the example above, $a$ and $b$ are leaves, $r$ is the root, and $c$ and $e$ are inner edges. The sets of edges, inner edges, and vertices of a tree $T$ are denoted $E(T)$, $E_i(T)$, and $V(T)$ respectively.

Describing the maps in $\Omega$ requires some care. To do so, we recall the algebraic notion of a broad poset, originally due to Weiss [Wei12] and further developed in [Per18]. For each edge $t$ in a tree topped by a vertex $\circ$, we write $t^\uparrow$ for the tuple of edges immediately above $t$. In the example (2.1) one has $r^\uparrow = cde$, $c^\uparrow = ab$, and $e^\uparrow = \epsilon$, where $\epsilon$ is the empty tuple. We then encode each vertex symbolically as $t^\uparrow \leq t$, which we call a generating broad relation. This notation is motivated by a form of transitivity. For example, in (2.1) the relations $cde \leq r$ and $ab \leq c$ generate, under broad transitivity, the relation $abde \leq r$, and one may similarly obtain relations $cd \leq r$ and $abd \leq r$. These relations, together with identity relations $t \leq t$, then form the broad poset associated with $T$.

A map of trees $\varphi: S \to T$ in $\Omega$ is then an underlying map of edge sets $\varphi: E(U) \to E(V)$ which preserves broad relations.

If an edge $t$ is pictorially above (or equal to) an edge $s$, we write $t \leq_d s$; equivalently, there exists a broad relation $s_1 \ldots s_n \leq s$ such that $t = s_i$ for some $i$.

Additionally, we will find it convenient to assume that each tree is equipped with a planar structure. Informally, this means that each object $T \in \Omega$ has a preferred planar representation as in (2.1) (a more formal definition of planar structures as suitable extensions of the partial order $\leq_d$ to a total order can be found in [BPb, §3.1]). Moreover, we will then prefer to work with a model of $\Omega$ for which the subcategory of planar maps is skeletal, i.e. such that the only planar isomorphisms are the identities.

Notation 2.2. We write $\eta$ for the stick tree, the unique tree with a single edge and no vertices.

Example 2.3. The labels on the edges of the tree diagrams below indicate a map to $E(T)$ with $T$ defined as in (2.1). The maps determined by four leftmost trees are maps of trees, while the rightmost is not.

![Tree Diagrams](image)

A map of trees $\varphi: S \to T$ is called:

- a tall map if $\varphi(l_S) = l_T$ and $\varphi(r_S) = r_T$, with $l_{(-)}$ and $r_{(-)}$ denoting the tuple of leaf edges and the root edge;
• a face map if it is injective on edges; an inner face if it is also tall; and an outer face if it does not factor through a non-isomorphism inner face map;

• a degeneracy if it is surjective on edges and preserves leaves (and is thus tall).

Pictorially, an inner face map \( U \to V \) removes some edges in \( V \) (while merging the vertices adjacent to those edges), outer face maps remove some vertices of \( V \), and degeneracies collapse some of the unary vertices of \( U \).

**Example 2.4.** In Example 2.3, the first map is an inner face, the second an outer face, the third is a face that is neither inner nor outer, and the fourth is a degeneracy.

**Notation 2.5.** We will label a map in \( \Omega \) by the letters \( d/i/o/t/f/p \) to indicate that the map is a degeneracy/inner face/outer face/tall/face/planar.

**Proposition 2.6** ([BPa, Prop. 2.2]). A map of trees \( \varphi: U \to V \) has a factorization, unique up to unique isomorphisms,

\[
U \xrightarrow{d} U' \xrightarrow{i} U'' \xrightarrow{o} V
\]

as indicated: a degeneracy followed by an inner face map followed by an outer face map.

**Remark 2.8.** A map \( \psi: U \to V \) is tall (resp. a face) iff in the decomposition (2.7) the component labeled \( o \) (resp. \( d \)) is an isomorphism. As such, by combining the first two (resp. last two) maps in (2.7) one recovers the “tall-outer face” (resp. “degeneracy-face”) factorization of the map \( \varphi \) [BPb, Prop. 3.31], [Per18, Prop. 5.37].

**Remark 2.9.** Following the previous remark, it is natural to consider the class of maps \( \psi: U \to V \) such that the factor labeled \( i \) in (2.7) is an isomorphism; Equivalently, \( \psi \) is convex precisely if the “tall-outer face” and “degeneracy-face” factorizations coincide.

Notably, it follows from this characterization that convex maps are closed under composition.

When accounting for planar structures, one has the following refinement of (2.7).

**Proposition 2.10** (cf. [BPa, Prop. 2.2]). A map of trees \( \varphi: U \to V \) has a strictly unique factorization

\[
U \xrightarrow{z} U_p \xrightarrow{d,p} \phi(U) \xrightarrow{i,p} \phi(U) \xrightarrow{o,p} V
\]

as indicated: an isomorphism followed by a planar degeneracy, a planar inner face, and a planar outer face.

**Remark 2.12.** The notation \( \phi(U) \) is motivated by the fact that this tree has edge set \( E(\phi(U)) = \phi(E(U)) \), while the \( \phi(U) \) notation is an instance of the outer closure of an inner face notation in [BPa, Not. 2.14].

**Remark 2.13.** Generalizing Remarks 2.8 and 2.9, one has that for any subset of the symbols \( \{ z, -, i, o \} \) in (2.11), the type of maps such that the corresponding factors are identities is closed under composition.

For example, the maps \( \psi \) such that the factors labeled \( z \) and \( i, p \) are identities are the planar convex maps, while those maps such that the factors labeled \( d, p \) and \( o, p \) are identities are the (not necessarily planar) inner face maps. Both of these kinds of maps are closed under composition.
A corolla is a tree with a single vertex. We note that the subcategory of $\Omega$ spanned by corollas and isomorphisms is isomorphic to the category $\Sigma$ of standard finite ordered sets and (non-ordered) isomorphisms.

**Notation 2.14.** For each $U \in \Omega$, there exists a unique corolla $lr(U) \in \Sigma$ equipped with a planar tall map $lr(U) \rightarrow U$, which we call the leaf-root of $U$.

Next, we consider categories of (colored) forests (cf. [BPc, Defn. 2.56]).

**Definition 2.15.** The category $\Phi$ of forests is the coproduct completion of the category $\Omega$ of trees: objects are formal coproducts $F = \sqcup_{i \in I} F_i$ with $F_i \in \Omega$, and an arrow $\varphi : \sqcup_{i \in I} F_i \rightarrow \sqcup_{j \in J} F'_j$ is given by a map $\varphi : I \rightarrow J$ and maps $\varphi_i : F_i \rightarrow F'_\varphi(i)$ in $\Omega$ for each $i \in I$.

The sets of edges, inner edges, vertices of a forest $F = \sqcup_i F_i$ are defined in the obvious way as

$$E(F) = \sqcup_i E(F_i), \quad E^i(F) = \sqcup_i E^i(F_i), \quad V(F) = \sqcup_i V(F_i).$$

**Definition 2.16.** Let $\mathcal{C}$ be a set of colors. The category $\Phi_\mathcal{C}$ of $\mathcal{C}$-forests has

- objects pairs $\tilde{F} = (F, \mathcal{C})$ with $F \in \Phi$ a forest and $\mathcal{C}(E(F)) \rightarrow \mathcal{C}$ a coloring of its edges,
- arrows $\tilde{F} = (F, \mathcal{C}) \rightarrow (F', \mathcal{C}') = \tilde{F}'$ maps $\varphi : F \rightarrow F'$ in $\Phi$ such that $\mathcal{C} = \mathcal{C}' \varphi$.

These naturally assemble into a category $\Phi_\mathcal{C} \rightarrow \text{Set}$ fibered over $\text{Set}$. Moreover, we denote by $\Phi_\mathcal{C}^i \subseteq \Phi_\mathcal{C}$ the wide subcategory of those arrows whose maps on uncolored forests are outer faces.

Moreover, we write $\Omega_\mathcal{C} \subseteq \Phi_\mathcal{C}$, which we call the category of $\mathcal{C}$-trees, for the full subcategory spanned by the objects whose underlying forest is a tree, and $\Sigma_\mathcal{C} \subseteq \Omega_\mathcal{C}$, which we call the category of $\mathcal{C}$-corollas, for the further subcategory of objects whose underlying tree is a corolla and whose maps are isomorphisms.

### 2.2 Equivariant trees

We now recall the category $\Omega_G$ of equivariant trees, which encodes the combinatorics of compositions of norm maps; a thorough discussion can be found in [Per18, §5] or [BPa, §2].

We begin with an explicit example. Let $G = D_4 = \{1, \rho, \rho^2, \rho^3, \sigma, \rho\sigma, \rho^2\sigma, \rho^3\sigma\}$ be the dihedral group with 8 elements, and $L \leq K \leq H \leq G$ denote the subgroups $H = \langle \rho^2, \sigma \rangle$, $K = \langle \rho^2 \rangle$, $L = \langle 1 \rangle$. Then we have a $G$-tree $T$ as below, with the expanded representation given by the two trees on the left below, and the orbital representation given by the single decorated tree on the right.

![Diagram](image)

(2.17)

The expanded representation is a planar representation of the $G$-sets of edges and vertices, equipped with names for the edges which indicate the group action on the set of edges (and hence also vertices). In the orbital representation, each edge indicates an orbit of edges from the expanded representation, and is labeled by the appropriate transitive $G$-set.

In general, we have the following.
Definition 2.18. Let $\Phi^G$ denote the category of forests with $G$-action. The category $\Omega_G$ of $G$-trees is the full subcategory of $\Phi^G$ spanned by those $G$-forests whose $G$-set $I$ of tree components has a transitive $G$-action.

Remark 2.19. We can describe $G$-trees in $\Omega_G$ by $T = u_iT_i$ with $I$ a transitive $G$-set and $T_i$ trees. Alternatively, any choice of component $T_i$ has a stabilizer $H \leq G$, and then we have a decomposition $T \cong G \cdot H \cdot T_i$. Finally, if $\Gamma \leq G \times \text{Aut}(T_i)$ is the graph subgroup (cf. (1.1)) $\{(h, \phi)h \mid h \in H\}$ associated to the homomorphism $\Phi : H \to \text{Aut}(T_i)$ encoding the $H$-action on $T_i$, we have $T \cong G \cdot T_i/\Gamma$.

For $T \in \Omega_G$, we write $E_G(T) = E(T)/G$, $E'_G(T) = E'(T)/G$, $V_G(T) = V(T)/G$ for the set of edge orbits, inner edge orbits, and $G$-vertices, respectively.

Remark 2.20. $\Omega_G$ is fibered over the orbit category $O_G$ of transitive $G$-sets and $G$-maps, by the functor which sends a $G$-tree $T = u_iT_i$ to its $G$-set of tree components $I$.

Remark 2.21. The two representations of $G$-trees from (2.17) play complementary roles in our analysis: the expanded representation displays all of the relevant edge information, and thus is often useful for describing maps between $G$-trees; conversely, the orbital representation compactly displays the relevant data for composing norm maps of operads (see e.g., [BPa, Remark 3.39]).

As in $\Omega$, maps $\varphi : S \to T$ between $G$-trees can be built from a few basic types of maps. Maps in each fiber over $O_G$ are called rooted, as they give a planar isomorphism of $G$-sets on the set of roots, i.e., tree components. A rooted map is called a degeneracy/inner face/outer face/tall face/planar map if $\varphi_j : S_j \to T_{\varphi(j)}$ is so for some (and thus all) $j \in J$.

The Cartesian arrows are a new type of map, dubbed quotients, which act as twisted fold maps on the forest components; explicitly, $\varphi$ is a quotient if $\varphi_j : S_j \to T_{\varphi(j)}$ is an isomorphism for some (and thus all) $j \in J$. We give a minimal example of a quotient below. Further discussion and examples can be found in [Per18, Rem. 5.49].

Example 2.22. Let $G = \mathbb{Z}/2\mathbb{Z}$ be the cyclic group with two elements. We have the following quotient map on $G$-corollas, where $\alpha \mapsto a$, $\alpha^- \mapsto -a$, $\beta \mapsto b$, $\gamma \mapsto c$.

Corollary 2.23 (cf. Prop. 2.10, [Per18, Rem. 5.49]). A map $\Phi : S \to T$ in $\Omega_G$ has a strictly unique factorization

$$S \xrightarrow{d,r} \phi_S \xrightarrow{1,p} \phi_S \xrightarrow{\alpha,p} \phi^*T \xrightarrow{q} T$$

as indicated: a rooted degeneracy followed by a planar inner face, a planar outer face, and a quotient map $q$. 

8
**Definition 2.24.** We denote by $\Omega^0_G \subseteq \Omega_G$ the wide subcategory of $G$-trees and quotient maps. Further, we write $\Sigma_G \subseteq \Omega^0_G$ for the full subcategory spanned by $G$-corollas and quotient maps, where $C \in \Omega_G$ is a $G$-corolla if $|V_G(C)| = 1$.

**Notation 2.25.** As in Notation 2.14, for any $T = u_i T_i \in \Omega_G$ there exists a unique $G$-corolla $lr(T) \in \Sigma_G$ equipped with a planar tall map $lr(T) \to T$ called the leaf-root of $T$. Explicitly, $lr(T) = u_i lr(T_i)$.

Finally, contrasting with the above, we define the faces of a $G$-trees non-equivariantly.

**Definition 2.26.** Let $T = u_i T_i \in \Omega_G$ be a $G$-tree. An (outer) face of $T$ is an (outer) face map $U \to T_i$ from some $U \in \Omega$ to a component $T_i$ of $T$. A face of $T$ is called planar if the map $U \to T_i$ preserves the planar order.

Let $\text{Face}(T)$ denote the $G$-poset of planar faces of $T$, and let $\text{Face}_0(T) \subseteq \text{Face}(T)$ denote the subposet spanned by the planar outer faces of $T$ with no inner edges.

### 2.3 Equivariant dendroidal sets

Recall that the category of dendroidal sets is the presheaf category $\text{dSet} = \text{Set}^{\Omega^{op}}$. Further, for a (finite) group $G$, the category of $G$-equivariant dendroidal sets is the $G$-object category $\text{dSet}^G = \text{Set}^{G \times \Omega^{op}}$.

One key subtlety when working with $\text{dSet}^G$ is that for each equivariant dendroidal set $X \in \text{dSet}^G$ its levels $X(U)$ are indexed by non-equivariant trees $U \in \Omega$, while the key classes of maps in $\text{dSet}^G$ are defined in terms of $G$-trees $T \in \Omega_G$.

To make this precise, we first extend the Yoneda embedding $\Omega[-] : \Omega \to \text{dSet}$ notation for the representable functors $\Omega(U)(V) = \Omega(V, U)$ to obtain extended Yoneda embeddings (here the right embedding is simply obtained from the left embedding by taking $G$-objects)

\[
\begin{array}{ccc}
\Phi & \xrightarrow{\Omega[-]} & \text{dSet} \\
\downarrow u_i F_i & & \downarrow u_i \Omega[F_i] \\
\Phi^G & \xrightarrow{\Omega[-]} & \text{dSet}^G \\
\downarrow u_i F_i & & \downarrow u_i \Omega[F_i]
\end{array}
\quad (2.27)
\]

Next, note that since $G$-trees $\Omega_G$ are defined as a subcategory of $G$-forests $\Phi^G$, the right side of (2.27) defines representables $\Omega[T] \in \text{dSet}^G$ for $T \in \Omega_G$. These representable presheaves $\Omega[T]$ are then the basis for generalizing several fundamental presheaves in dendroidal sets $\text{dSet}$ (cf. [CM13a, §2]) to equivariant dendroidal sets $\text{dSet}^G$ (cf. [Per18, §6]), as follows.

**Definition 2.28.** For $T = u_i T_i$, a $G$-tree, we define the boundary $\partial \Omega[T] \subseteq \Omega[T]$ by

\[
\partial \Omega[T] = \bigsqcup_i \partial \Omega[T_i] = \colim_{U \in \text{Face}(T), U \neq T} \Omega[U] = \bigcup_{U \in \text{Face}(T), U \neq T} \Omega[U].
\]

Next, for $\emptyset \neq E \subseteq E(T)$ a non-empty $G$-subset of inner edges, we write $E_i = E \cap E(T_i)$ and define the associated $G$-inner horn by

\[
\Lambda^E[T] = \bigsqcup_i \Lambda^{E_i}[T_i] = \colim_{U \in \text{Face}(T), (T_i, E_i) \neq U} \Omega[U] = \bigcup_{U \in \text{Face}(T), (T_i, E_i) \neq U} \Omega[U].
\]

Finally, the Segal core of $T$ is

\[
\Sigma[T] = \bigsqcup_i \Sigma[T_i] = \colim_{U \in \text{Face}_0(T)} \Omega[U] = \bigcup_{U \in \text{Face}_0(T)} \Omega[U].
\]
Remark 2.29. For $T \in \Omega_G$, a decomposition $T \simeq G \cdot_H T_*$ with $T_* \in \Omega^H$ yields
\[
\Omega[T] \simeq G \cdot_H \Omega[T_*], \quad \partial \Omega[T] \simeq G \cdot_H \partial \Omega[T_*], \quad \Lambda^E[T] \simeq G \cdot_H \Lambda^E[T_*], \quad \Sigma[T] \simeq G \cdot_H \Sigma[T_*].
\]

Mimicking the non-equivariant story, the maps in Definition 2.28 are then the basis for a model structure on $dSet^G$.

In the following, we recall that a class of maps is called saturated if it is closed under pushouts, retracts, and transfinite compositions.

**Definition 2.30.** The class of $G$-normal monomorphisms in $dSet^G$ is the saturation of the boundary inclusions $\partial \Omega[T] \to \Omega[T]$ for $T \in \Omega_G$.

The class of $G$-inner anodyne extensions in $dSet^G$ is the saturation of the $G$-inner horn inclusions $\Lambda^E[T] \to \Omega[T]$ for $T \in \Omega_G$ and $G$-subset $\emptyset \neq E \subseteq E'(T)$.

**Definition 2.31.** A map $X \to Y$ in $dSet^G$ is called a $G$-inner fibration if it has the right lifting property with respect to all $G$-inner horn inclusions $\Lambda^E[T] \to \Omega[T]$.

Moreover, if $X \to *$ is a $G$-inner fibration then $X \in dSet^G$ is called a $G$-$\infty$-operad.

Informally, $G$-$\infty$-operads can be thought of as “operads with weak composition laws for norm maps”.

To recall the model structure on $dSet^G$, we need two more ingredients. First, we write
\[
\nu: sSet \rightleftarrows dSet: \nu^* \quad \nu^*: \text{Cat} \rightleftarrows \text{Op}: \nu
\]
for the adjunctions where the left adjoints $\nu_!$ are the natural inclusions. Note that the left adjunction is induced by the natural inclusion $\nu: \Delta \to \Omega$. Second, we write
\[
\tau: sSet \rightleftarrows \text{Cat}: \tau^* \quad \tau^*: dSet \rightleftarrows \text{Op}: \tau
\]
for the adjunctions where the right adjoints are the nerve functors described by $(\mathcal{N}C)(u) = \text{Cat}([u], \mathcal{C})$ for $\mathcal{C} \in \text{Cat}$ and $(\mathcal{N}O)(T) = \text{Op}(\Omega(T), O)$ for $O \in \text{Op}$ and $\Omega(T)$ the colored operad (of sets) generated by $T$ (cf. [MW07, §3], [Per18, Rem. 4.4, Ex. 4.6] or (1.13)). Recall [MW09, Prop. 5.3 and Thm. 6.1] that the nerve functors $\tau$ are fully faithful, with their essential image characterized as those presheaves that satisfy a Segal condition.

**Theorem 2.34.** [Per18, Thm 2.1] There exists a left proper (cf. [Per18, Prop. 8.8]) model structure on $dSet^G$ such that:

- the cofibrations are the $G$-normal monomorphisms;
- the fibrant objects are the $G$-$\infty$-operads;
- the fibrations between $G$-$\infty$-operads are the $G$-inner fibrations $X \to Y$ such that the induced maps on fixed-point homotopy categories $\tau \nu^*(X^H \to Y^H)$ are isofibrations of categories for all $H \leq G$;
- the weak equivalences are the smallest class of maps closed under 2-out-of-3 which contains the $G$-inner anodyne extensions and the trivial fibrations (i.e. those maps with the right lifting property against the $G$-normal monomorphisms).

In addition to the category $dSet^G = \text{Set}^{G^{\text{op}}}$ of equivariant dendroidal sets, there is also a category $dSet_G = \text{Set}_G^{G^{\text{op}}}$, which we call the category of genuine dendroidal sets. It turns out that some natural constructions in the non-equivariant setting generalize to produce objects in $dSet_G$.
rather than in \( \mathsf{dSet}^G \) (e.g. \( \mathsf{dSet}_G \)) is essential to establishing the characterization of the fibrant objects in \( \mathsf{dSet}_G \), cf. [Per18, §8.2]), so we next recall the connection between the two categories.

Let \( \mathcal{O}_G \) denote the orbit category of the group \( G \), i.e. the category of transitive \( G \)-sets \( G/H \) for \( H \leq G \) and \( G \)-set maps. Regarding the group \( G \) as a single object category, one then has a fully faithful inclusion \( \iota : \mathsf{G} \to \mathcal{O}_G \) sending the object of \( \mathsf{G} \) to the free \( G \)-set \( G/e \). In addition, there is a fully faithful inclusion \( \Omega \times \mathcal{O}_G \to \Omega_G \) given by \( (T, G/H) \mapsto G/H \cdot T \). Altogether, one obtains a commutative diagram of fully faithful inclusions as follows.

\[
\begin{array}{ccc}
\Delta \times \mathcal{O}_G & \xrightarrow{\iota} & \Omega \times \mathcal{O}_G \\
\downarrow \iota & & \downarrow \iota \\
\Omega \times \mathcal{O}_G & \xrightarrow{\iota} & \Omega_G
\end{array}
\]

The connection between \( \mathsf{dSet}^G \) and \( \mathsf{dSet}_G \) is then given by the right adjunction

\[
\begin{array}{ccc}
\mathsf{dSet}^G & \xleftarrow{\iota^*} & \mathsf{sSet} \xrightarrow{\iota_*} \mathsf{dSet}_G \\
\downarrow N & & \downarrow N \\
\mathsf{dSet}^G & \xleftarrow{\iota^*} & \mathsf{dSet}_G
\end{array}
\]

where we note that the right adjoints \( \iota_* \), \( \iota^* \) are fully faithful inclusions. Explicitly, \( \iota_* \mathcal{O}_G(\iota(T), X) \simeq \mathcal{O}_G(T, X) \) for \( T \simeq G/H \cdot T \) with \( T \in \Omega_H \).

The fully faithful functors appearing in (2.32), (2.33), (2.35) then fit into commutative diagrams as below where \( \mathsf{Op}_G \) is the category of genuine equivariant operads. These are an extension of the notion of operad which in the single colored context were first defined in [BPb] via algebraic means. However, to sidestep the technical work needed to extend the definition in [BPb] to the colored context, here we regard \( \mathsf{Op}_G \) simply as the full subcategory of \( \mathsf{dSet}_G \) of those objects satisfying a Segal condition as in [BPa, Defn. 3.35], so that the fact that \( N_G \) is fully faithful is tautological.

\[
\begin{array}{ccc}
\mathsf{Cat}^G & \xrightarrow{\iota^*} & \mathsf{Op}^G \\
N & & N \\
\mathsf{sSet}^G & \xrightarrow{\iota^*} & \mathsf{dSet}^G
\end{array}
\]

\[
\begin{array}{ccc}
\mathsf{Cat}^G & \xrightarrow{\iota^*} & \mathsf{Op}^G \\
N & & N \\
\mathsf{sSet}^G & \xrightarrow{\iota^*} & \mathsf{dSet}^G
\end{array}
\]

(2.36)

By taking left adjoints of the vertical nerve functors above we obtain the following diagram where those squares that include natural transformations do not commute. Here the existence of the dashed left adjoint \( \tau_G \) to \( N_G \) requires justification, with a full discussion of \( \tau_G \) being the objective of Appendix B.

\[
\begin{array}{ccc}
\mathsf{sSet}^G & \xrightarrow{\tau^*} & \mathsf{dSet}^G \\
\downarrow N & & \downarrow N \\
\mathsf{Cat}^G & \xrightarrow{\tau^*} & \mathsf{Op}^G
\end{array}
\]

\[
\begin{array}{ccc}
\mathsf{sSet}^G & \xrightarrow{\tau^*} & \mathsf{dSet}^G \\
\downarrow N & & \downarrow N \\
\mathsf{Cat}^G & \xrightarrow{\tau^*} & \mathsf{Op}^G
\end{array}
\]

(2.37)

3 The tame model structure on preoperads

Our goal in this section is to build the alternative tame model structure \( \mathsf{PreOp}^G_{\text{tame}} \) on preoperads that is needed for the nerve functor \( N : \mathsf{sOp}^G \to \mathsf{PreOp}^G \) in (1.3) to be right Quillen.
First, in §3.1 and §3.2 we review the model structures on simplicial dendroidal sets $sdSet^G$ and preoperads $PreOp^G$ built in [BPa]. Then, in §3.3 we build an auxiliary construction, the fibered simplicial tensoring $\otimes$, which plays a key role in our description of the tame model structure in §3.4.

### 3.1 Equivariant dendroidal Segal spaces

We recall the several model structures on the category of equivariant simplicial dendroidal sets $sdSet^G = \Set^{\Delta^{op} \times \Omega^* \times G}$ introduced in [BPa].

We first recall some notation for objects in $sdSet^G$.

**Notation 3.1.** For $X \in sdSet^G$, we write $X_n(U)$ for the evaluation at $n \in \Delta$ and $U \in \Omega$, and refer to $n$ and $U$ as the simplicial and dendroidal directions. More generally, we write

$$X(-): sSet \to dSet^G, \quad X(-): dSet^G \to sSet$$

for the colimit-preserving functors such that $X_{\Delta[n]} = X_n$ and $X(G \cdot \Omega[U]) = X(U)$ for $n \geq 0$, $U \in \Omega$. Explicitly, $X_K(U) = sSet(K, X(U))$ and $X(A)_n = dSet^G(A, X_n)$.

Additionally, we have natural fully-faithful inclusions (as presheaves that are constant along one of the directions)

$$c_0: dSet^G \longrightarrow sdSet^G, \quad c_1: sSet \longrightarrow sdSet^G$$

and we will often identify presheaves with their images under $c_1$. Lastly, for $A \in dSet^G$ and $K \in sSet$ we write $A \times K$ for the presheaf $(A \times K)_n(U) = A(U) \times K_n$.

The various model structures on $sdSet^G$ arise from the theory of (generalized) Reedy categories. First, since $\Delta^{op}$ is a Reedy category, the identification $sdSet^G = (dSet^G)^{\Delta^{op}}$ together with the model structure on $dSet^G$ from [Per18] (cf. Theorem 2.34) yields the simplicial Reedy model structure on $sdSet^G$. Note that the weak equivalences in this model structure are the dendroidal equivalences, i.e. the maps $f: X \to Y$ such that $X_n \to Y_n$ is a weak equivalence in $dSet^G$ for all $n \geq 0$.

Second, as discussed in [BPa, Ex. A.7], $\Omega^{op} \times G$ is a generalized Reedy category and the family of graph subgroups of (1.1) is Reedy admissible in the sense of [BPa, Ex. A.2]. Hence, by [BPa, Thm. A.8], the identification $sdSet^G = sSet^{\Delta^{op} \times G}$ together with the Kan model structure on $sSet$ yields the (equivariant) dendroidal Reedy model structure on $sdSet^G$. The weak equivalences in this model structure are the simplicial equivalences, i.e. the maps $f: X \to Y$ such that $X(T) \to Y(T)$ are Kan equivalences in $sSet$ for each $T \in \Omega_G$.

Third, as the simplicial and dendroidal Reedy model structures in $sdSet^G$ above have the same cofibrations, the joint left Bousfield localization framework in [BPa, §4.1] yields the following.

**Theorem 3.2.** The simplicial and dendroidal Reedy model structures on $sdSet^G$ have a smallest\(^1\) common left Bousfield localization, which we call the joint Bousfield localization. Moreover:

(i) the joint Bousfield localization is left proper;

(ii) both the dendroidal and simplicial equivalences in $sdSet^G$ are also joint equivalences;

(iii) $X$ is joint fibrant iff $X$ is both simplicial and dendroidal Reedy fibrant;

(iv) if $X, Y$ are joint fibrant then a map $X \to Y$ is a joint equivalence iff it is a simplicial equivalence iff it is a dendroidal equivalence;

\(^1\) Here “smallest” means that the class of weak equivalences is as small as possible.
(v) if $X, Y$ are dendroidal fibrant then a map $X \rightarrow Y$ is a joint equivalence iff it is a dendroidal equivalence iff $X_0 \rightarrow Y_0$ is an equivalence in $d\text{Set}^G$;

(vi) if $X \rightarrow Y$ is a joint (co)fibration, the level maps $X_n \rightarrow Y_n, n \geq 0$ are (co)fibrations in $d\text{Set}^G$ and the maps $X(\Omega[T]) \rightarrow Y(\Omega[T]), T \in \Omega_G$ are (co)fibrations in $s\text{Set}$.

In particular, if $X$ is joint fibrant then $X_n \in d\text{Set}^G$ and $X(\Omega[T]) \in s\text{Set}$ are fibrant.

**Proof.** The existence of a smallest common left Bousfield localization is an application of [BPa, Prop. 4.1] with the hypothesis that the dendroidal/simplicial Reedy model structures admit localizations being guaranteed by Hirschhorn’s existence result [Hir03, Thm. 4.1.1]. (i) then follows from [Hir03, Thm. 4.1.1(3)]. (ii) holds by definition. (iii) and (iv) are [BPa, Prop. 4.1(i)(ii)]. (v) is [BPa, Cor. 4.29(iii)]. Lastly, (vi) follows from [BPa, Lemmas A.27(i), A.29(i)].

Fourth (and last), one has the (equivariant) dendroidal Segal space model structure on $s\text{dSet}^G$, which is the localization of the dendroidal Reedy model structure with respect to the Segal core inclusions

$$Sc[T] \hookrightarrow \Omega[T], \quad T \in \Omega_G.$$

**Lemma 3.3.** The weak equivalences in the dendroidal Reedy, dendroidal Segal space, and joint Reedy model structures on $s\text{dSet}^G$ are closed under filtered colimits.

**Remark 3.4.** More explicitly, weak equivalences are closed under filtered colimits if a map of filtered colimits\footnote{We note that this is commonly understood as a map of simplicial sets of the form $\colim_i C_i \rightarrow \colim_i D_i$.} is a weak equivalence if all $C_i \rightarrow D_i$ are. Notably, this is readily shown to be equivalent to the claim that filtered colimits of simplicial sets are homotopy colimits.

**Proof.** Weak equivalences in the dendroidal Reedy model structure are simplicial equivalences, so in that case the result is inherited from the analogous claim for $s\text{Set}$. The result for the latter two model structures follows since they are left Bousfield localizations of the dendroidal Reedy model structure (as the alternative condition in Remark 3.4 is clearly preserved under localization).

In what follows we will often make reference to the 0-(co)skeleton of some $X \in s\text{dSet}^G$ in the dendroidal Reedy structure. To avoid confusion with the 0-(co)skeleta for the simplicial Reedy structure, and noting that $\eta$ is the only tree in $\Omega$ of degree 0, we introduce the following notation.

**Notation 3.5.** Let $X \in s\text{dSet}^G$. We write $sk_\eta X, csk_\eta X \in s\text{dSet}^G$ for the (co)skeleta described by

$$(sk_\eta X)(U) = \coprod_{E(U)} X(\eta), \quad (csk_\eta X)(U) = \prod_{E(U)} X(\eta).$$

### 3.2 Segal preoperads

In this section we recall the normal model structure on preoperads $\text{PreOp}^G$ introduced and studied in [BPa, §4 and §5].

**Definition 3.6.** The category of (equivariant) preoperads $\text{PreOp}^G$ is the full subcategory of $s\text{dSet}^G$ spanned by those $X$ such that $X(\eta)$ is a discrete simplicial set.

**Definition 3.7.** Given $X \in \text{PreOp}^G$ we call $X(\eta)$ the color set of $X$ and denote the associated color set functor as follows.

$$\begin{array}{ccc}
\text{PreOp}^G & \xrightarrow{\mathcal{C}} & \text{Set}^G \\
X & \xmapsto{\mathcal{C}_X = X(\eta)} &
\end{array}$$

(3.8)
Further, for each fixed $C \in \text{Set}^G$ we write $\text{PreOp}^G_C \subset \text{PreOp}^G$ for the fiber subcategory of (3.8) over $C$, consisting of those $X$ such that $X(\eta) = C$ and maps that are the identity on colors.

Lastly, for $f: \mathcal{C} \to \mathcal{D}$ a map of $G$-sets of colors we define adjoint functors

$$f_*: \text{PreOp}^G_\mathcal{C} \rightleftarrows \text{PreOp}^G_\mathcal{D} : f^*$$

via the pushout and pullback squares below (note that $sk_\eta f_! A = \coprod_{\mathcal{D}} \Omega [\eta]$ depends only on $\mathcal{C}$ while $csk_\eta f^* X = \prod_{\mathcal{D}} \Omega [\eta]$ depends only on $\mathcal{D}$)

$$\begin{array}{ccc}
sk_\eta A & \longrightarrow & sk_\eta f_! A \\
\downarrow & & \downarrow \\
A & \longrightarrow & f_! A
\end{array}$$

$$\begin{array}{ccc}
f^* X & \longrightarrow & X \\
\downarrow & & \downarrow \\
csk_\eta f^* X & \longrightarrow & csk_\eta X
\end{array}$$

The inclusion $\gamma^*: \text{PreOp}^G \to \text{sdSet}^G$ admits a left adjoint $\gamma_!$ and a right adjoint $\gamma^*$ described by the following pushout and pullback squares.

$$\begin{array}{ccc}
sk_\eta X & \longrightarrow & \pi_0 sk_\eta X \\
\downarrow & & \downarrow \\
X & \longrightarrow & \gamma_! X
\end{array}$$

$$\begin{array}{ccc}
\gamma_* X & \longrightarrow & X \\
\downarrow & & \downarrow \\
csk_\eta X_0 & \longrightarrow & csk_\eta X
\end{array}$$

(3.9)

More explicitly: $\gamma_! X(U) = X(U)$ if $U \notin \Delta$ is not linear while $\gamma_! X([n])$ for $[n] \in \Delta$ linear is given by the pushout on the left below; $\gamma_* X(U)$ is given by the pullback on the right below.

$$\begin{array}{ccc}
X(\eta) & \longrightarrow & \pi_0 X(\eta) \\
\downarrow & & \downarrow \\
X([n]) & \longrightarrow & \gamma_! X([n])
\end{array}$$

$$\begin{array}{ccc}
\gamma_* X(U) & \longrightarrow & X(U) \\
\downarrow & & \downarrow \\
\prod_{\pi_{\mathcal{E}(U)} X_0(\eta)} & \longrightarrow & \prod_{\pi_{\mathcal{E}(U)}} X(\eta)
\end{array}$$

By largely formal arguments, the joint model structure on $\text{sdSet}^G$ in Theorem 3.2 induces a model structure on $\text{PreOp}^G$, as follows. We say a map $X \to Y$ in $\text{PreOp}^G$ is a joint equivalence (resp. normal monomorphism) if $\gamma^* X \to \gamma^* Y$ is a joint equivalence (resp. normal monomorphism) in $\text{sdSet}^G$. The following is then [BPa, Thms. 4.39 and 4.42], with the additional “moreover” claims inherited from the analogous conditions in $\text{sdSet}^G$ in Theorem 3.2(i) and Lemma 3.3.

**Theorem 3.10.** There is a model structure on $\text{PreOp}^G$, called the normal model structure, such that weak equivalences (resp. cofibrations) are the joint equivalences (normal monomorphisms).

Moreover, this model structure is left proper and weak equivalences are closed under filtered colimits.

Lastly, the adjunction $\gamma^*: \text{PreOp}^G \rightleftarrows \text{sdSet}^G : \gamma_*$ is a Quillen equivalence between the normal model structure and the joint model structure on $\text{sdSet}^G$.

For our purposes, we also need to recall a convenient “Dwyer-Kan” description of the joint equivalences between fibrant objects in $\text{PreOp}^G$. For this purpose, we first introduce the following new notation, which extends notation in [BPa, Def. 5.7] and will simplify our discussion of the nerve functor (see, e.g. (4.15)).
Notation 3.11. Let $X \in \text{sSet}^G$, $A \in \text{dSet}^G$, and $c: A(\eta) \to X(\eta)$ be a $G$-equivariant map.

We define $X_c(A) \in \text{sSet}$ as the pullback below (here the two squares are identical, providing only different descriptions of the bottom-right corner).

$$
\begin{array}{ccc}
X_c(A) & \to & X(A) \\
\downarrow & & \downarrow \\
X(\sk_d A) & \to & \left(\prod_{A(\eta)} X(\eta)\right)^G \\
\end{array}
$$

Further, when $A = G \cdot \Omega[U]$ for $U \in \Omega$, we abbreviate $X_c(G \cdot \Omega[U])$ as $X_c(U)$.

Note that one thus has a coproduct decomposition

$$
X(A) \cong \coprod_{c: A(\eta) \to X(\eta)} X_c(A). 
$$

(3.12)

Remark 3.13. Our primary examples of Notation (3.11) occur when $A = \Omega[T]$ for $T \in \Omega_G$, in which case $\Omega[T](\eta) = \mathcal{E}(T)$ so that $c: \Omega[T](\eta) \to X(\eta)$ can be regarded as a coloring of the edges $\mathcal{E}(T)$ of $T$ by the colors $X(\eta)$ of the preoperad $X$.

Remark 3.14. Specifying Remark 3.13 to the case of $T = C$ a $G$-corolla, the coloring $c: \Omega[C](\eta) \to X(\eta)$ is tantamount to a map $c: \partial \Omega[C] \to X$, i.e. to a $G$-profile in the sense of [BPa, Def. 5.6]. Further, since there is an identification $X(\partial \Omega[C]) = \prod_{i \leq \mathcal{E}(C)} X(\eta)^{H_i}$ with $H_i \leq G$ the isotropy of the edge $e_i$, the data of the coloring $c$ is equivalent to a choice of $x_i \in X(\eta)^{H_i}$.

As such, one has an identification

$$
X_c(\Omega[C]) = X(x_1, \cdots, x_n; x_0)
$$

where the mapping space $X(x_1, \cdots, x_n; x_0)$ is as defined in [BPa, Defn. 5.7]. Further, the decomposition in (3.12) then extends the decomposition in [BPa, Rem. 5.14].

Remark 3.15. Fix $\mathcal{C} \in \text{Set}^G$ and consider the fiber subcategory $\text{PreOp}_G \subset \text{PreOp}^G$.

For $U \in \Omega$, the decomposition $X(U) = \prod_{c: \mathcal{E}(U) \to \mathcal{C}} X_c(U)$ in (3.12) (note that we are using the abbreviated notation at the end of Notation 3.11) then induces an equivalence of categories

$$
\begin{array}{ccc}
\text{PreOp}_G & \xrightarrow{\cong} & \text{Fun}_*(G \times \Omega_{\mathcal{C}}^\text{op}, \text{sSet}) \\
(U \mapsto X(U)) & \longmapsto & ((U, c) \mapsto X_c(U)) \\
\end{array}
$$

(3.16)

where $\Omega_\mathcal{C}$ denotes the $\mathcal{C}$-colored trees of Definition 2.16 and $\text{Fun}_*(G \times \Omega_{\mathcal{C}}^\text{op}, \text{sSet}) \subset \text{Fun}(G \times \Omega_{\mathcal{C}}^\text{op}, \text{sSet})$ is the subcategory of pointed functors, i.e. functors $Y$ such that $Y(\eta_c) = \ast$, where $c \in \mathcal{C}$ is a color and $\eta_c$ denotes the stick tree colored by $c$.

Remark 3.17. Given the alternative notation $\tilde{\mathcal{U}} = (U, c)$ for $\mathcal{C}$-trees and the equivalence (3.16), it seems natural to abbreviate $X_c(U)$ as just $X(\tilde{U})$. However, we will have significant need for the notation $\Omega_{\mathcal{C}}(\Omega[T])$ in Remarks 3.13,3.14, but this latter notation is not readily recovered from (3.16). As such, when dealing with preoperads we work only with the $X_c(A), X_c(U)$ notations, reserving $\Omega(\mathcal{C})$ style notations for the context of operads (see §4.1).

Remark 3.18. Let $X \in \text{PreOp}^G$. For any $G$-tree $T \in \Omega_G$ and coloring $c: \mathcal{E}(T) \to X(\eta)$ one has

$$
X_c(\text{Sc}[T]) \cong \prod_{v \in \text{V}_G(T)} X_c(\Omega[T_v])
$$

where $c_v$ denotes the restricted coloring given by the composite $\mathcal{E}(T_v) \to \mathcal{E}(T) \xrightarrow{c} X(\eta)$.

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We can now recall the notion of Segal operad, cf. [CM13b, Def. 5.5], [BPa, Def. 4.40].

Definition 3.19. A preoperad \( X \in \text{PreOp}^G \) is called a (equivariant) Segal operad if \( X(\Omega[T]) \to X(\Sigma(T)) \) is a Kan equivalence for each \( T \in \Omega_G \). Equivalently, by (3.12) and Remark 3.18, this means that the natural maps

\[
X_t(\Omega(T)) \xrightarrow{\sim} \prod_{v \in V_G(T)} X_{\epsilon_v}(\Omega(T_v))
\]

are Kan equivalences for all \( T \in \Omega_G \) and \( G \)-equivariant coloring \( c : E(T) \to \mathcal{E}_X = X(\eta) \).

Further, a Segal operad \( X \) is additionally called a Reedy fibrant Segal operad if \( \gamma^*X \) is dendroidal Reedy fibrant in \( \text{sdSet}^G \). Equivalently ([BPa, Remark 4.41]), this means that \( \gamma^*X \) is fibrant in the dendroidal Segal space model structure on \( \text{sdSet}^G \).

Remark 3.21. Since discrete simplicial sets \( X(\eta) \) are Kan complexes, for any preoperad \( X \in \text{PreOp}^G \) one can form a certain strict fibrant replacement \( X \to \tilde{X} \) in \( \text{sdSet}^G \) such that \( X(\eta) = \tilde{X}(\eta) \), so that \( \tilde{X} \) is again a preoperad. Moreover, since the maps \( X(\Omega[T]) \to \tilde{X}(\Omega[T]) \) for \( T \in \Omega_G \) are Kan equivalences, (3.12) implies that so are the maps \( X_t(\Omega(T)) \to \tilde{X}_t(\Omega(T)) \) for any coloring \( c : E(T) \to X(\eta) \).

Note that (3.20) then implies that \( X \) is a Segal operad iff \( \tilde{X} \) is a Reedy fibrant Segal operad.

We will show that joint equivalences between Segal operads admit a Dwyer-Kan type description in terms of fully faithfulness and essential surjectivity conditions (cf. Theorem 4.8). To describe essential surjectivity, we need to recall a discrete algebraic structure associated to a Segal preoperad. In the following, we make use of the category \( \text{dSet}_G = \text{Set}^G_{\Omega} \) of genuine dendroidal sets discussed in §2.3 as well as its obvious generalization \( \text{sdSet}_G = \text{sdSet}^G \).

Definition 3.22. Given a Segal operad \( X \in \text{PreOp}^G \), define its homotopy genuine operad \( \text{ho}(X) \in \text{dSet}_G \) by

\[
\text{ho}(X) = \pi_0(\nu_{\gamma}^*X)
\]

with \( \nu_{\gamma} : \text{sdSet}^G \to \text{sdSet}_G \) and \( \pi_0 : \text{sdSet}_G \to \text{dSet}_G \) defined in the natural way.

Remark 3.23. The “genuine operad” moniker for \( \text{ho}(X) \in \text{dSet}_G \) refers to the fact that this presheaf satisfies a certain strict Segal condition, as shown in [BPa, Prop. 5.9] (technically the cited result only covers the case of \( X \) Reedy fibrant, but it is immediate that for \( X, \tilde{X} \) as in Remark 3.21 it is \( \text{ho}(X) \approx \text{ho}(\tilde{X}) \)).

However, for our immediate purposes we will not need the full strength of this statement, but only a more familiar consequence. Writing \( i_G : \Delta \times \Omega_G \to \Omega_G \) for the inclusion \((\{n\}, G/H) \to G/H \cdot [n] \), one has \( i_G^* \text{ho}(X) \in \text{sSet}^G_{\Omega} \) and the Segal condition for \( \text{ho}(X) \) implies that \( i_G^* \text{ho}(X) \) is a coefficient system of nerves of categories [BPa, Rem. 5.11].

Definition 3.24. A map \( f : X \to Y \) of Segal operads in \( \text{PreOp}^G \) is called

(i) fully-faithful if for all \( G \)-corollas \( C \) and all \( G \)-equivariant coloring \( c : E(C) \to \mathcal{E}_X = X(\eta) \) the induced map

\[
X_t(\Omega[C]) \to Y_t(\Omega[C])
\]

is a Kan equivalence in \( \text{sSet} \).

(ii) essentially surjective if the map \( i_G^* \text{ho}(X) \to i_G^* \text{ho}(Y) \) of \( G \)-coefficient systems of categories is levelwise essentially surjective.

(iii) a Dwyer-Kan equivalence if it is both fully-faithful and essentially surjective.
The following then summarizes [BPa, Remark 4.41, Thms. 5.51 and 5.48] with the additional fact that the “further” claim holds for all Segal operads, rather than just the Reedy fibrant ones, following from Remark 3.21.

**Theorem 3.25.** The fibrant objects in the normal model structure on $\text{PreOp}^G$ are precisely the Reedy fibrant Segal operads.

Further, a map between Segal operads is a joint equivalence iff it is a Dwyer-Kan equivalence.

### 3.3 Fibered simplicial tensor and cotensor

In this section we introduce an auxiliary simplicial tensoring on $\text{PreOp}^G$ that will play a major role in our definition of the tame model structure $\text{PreOp}^G_{	ext{tame}}$ in §3.4, as well as streamline the comparison between $\text{PreOp}^G_{	ext{tame}}$ and $\text{sOp}^G$ in §4.2.

We first define the adjoint simplicial cotensoring, which admits a very simple description in terms of the $X_c(U)$ construction introduced in Notation 3.11 and the identification (3.16).

**Definition 3.26.** Given $X \in \text{PreOp}^G_C$ and $K \in \text{sSet}$ we define their fiber cotensor $\{K, X\}_c \in \text{PreOp}^G_C$ by

$$\{K, X\}_c(U) = X_c(U)^K.$$  
for $U \in \Omega$ a tree and $c : E(T) \to C = X(\eta)$ a coloring.

Alternatively, $\{K, X\}_c$ is given by the pullback in $\text{sdSet}^G$ (where the left square simply evaluates the right square at $U \in \Omega$)

$$\begin{array}{ccc}
\{K, X\}_c(U) & \longrightarrow & X(U)^K \\
\downarrow & & \downarrow \\
\Pi_{E(U)}X(\eta) & \longrightarrow & (\Pi_{E(U)}X(\eta))^K
\end{array}$$

and

$$\begin{array}{ccc}
\text{csk}_\eta X & \longrightarrow & (\text{csk}_\eta X)^K \\
\downarrow & & \downarrow \\
K \longrightarrow & & K
\end{array}$$

**Definition 3.28.** Given $X \in \text{PreOp}^G_C$ and $K \in \text{sSet}$ we define their fiber tensor $X \otimes \_c \in \text{PreOp}^G_C$ by the pushout in $\text{sdSet}^G$

$$\begin{array}{ccc}
(\text{sk}_\eta X) \times K & \longrightarrow & \text{sk}_\eta X \\
\downarrow & & \downarrow \\
X \times K & \longrightarrow & X \otimes \_c K
\end{array}$$

Moreover, one has $(X \otimes \_c, K)(U) = X(U) \times K$ whenever $U$ is a non-linear tree (equivalently, $\Omega(U, \eta) = \emptyset$) and that $(X \otimes \_c, K)([n])$ is given by the following pushout when $U = [n]$ is linear.

$$\begin{array}{ccc}
X(\eta) \times K & \longrightarrow & X(\eta) \\
\downarrow & & \downarrow \\
X([n]) \times K & \longrightarrow & (X \otimes \_c, K)([n])
\end{array}$$

**Remark 3.30.** In [CM13b, §7.1] the objects $\Omega[T] \otimes \_c K$ were denoted $\Omega[\_c K, T]$ and built by hand.

**Remark 3.31.** If $K \in \text{sSet}$ is connected, comparing the left square in (3.9) with the square (3.29) yields an identification $\gamma ! (X \times K) = X \otimes \_c K$.  

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Remark 3.32. For each fixed $K \in \mathsf{sSet}$ the fiber tensor and cotensor determine an adjunction as on the left below.

\[
\begin{array}{c}
\text{PreOp}^G \\
\downarrow \downarrow \downarrow \downarrow \\
\text{PreOp}_K^G \end{array}
\quad \text{and} \quad
\begin{array}{c}
\text{PreOp}^G \\
\downarrow \downarrow \downarrow \downarrow \\
\text{PreOp}_K^G
\end{array}
\]

Moreover, this adjunction is fibered over the color set functor $\mathfrak{C}_* : \text{PreOp}^G \to \text{Set}^G$, which in particular means that for each fixed set of colors $\mathfrak{C}$ one has a restricted adjunction as on the right above.

Remark 3.33. Remark 3.32 implies that the fiber cotensor $X \otimes_{\mathfrak{C}_*} K$ preserves colimits on the $X$ variable. However, some care is needed when dealing with the $K$ variable. For each fixed color set $\mathfrak{C}$, one has that the functor

\[
\text{PreOp}^G \times \mathsf{sSet} \xrightarrow{(-) \otimes_{\mathfrak{C}_*} (-)} \text{PreOp}_K^G
\]

is part of a two-variable adjunction, which in particular means that $X \otimes_{\mathfrak{C}_*} (-) : \mathsf{sSet} \to \text{PreOp}_K^G$ (where $\mathfrak{C} = X(\eta)$) preserves colimits. On the other hand, this means that $X \otimes_{\mathfrak{C}_*} (-) : \mathsf{sSet} \to \text{PreOp}_K^G$ only preserves those colimits which coincide in $\text{PreOp}^G$ and $\text{PreOp}_K^G$, namely the connected colimits.

On the other hand, for coproducts one instead has that the canonical map $u_i : X \otimes_{\mathfrak{C}_*} K_i \to X \otimes_{\mathfrak{C}_*} (u_i K_i)$ is a cocartesian arrow over the fold map $u_i : \mathfrak{C} \to \mathfrak{C}$.

Remark 3.34. Let $X \to Y$ be any map in $\text{PreOp}^G$ which is the identity on colors and $K \in \mathsf{sSet}$. Then the top horizontal maps in (3.29) for $X,Y$ coincide, and likewise for the left square in (3.9) for $X \times K, Y \times K$. It thus follows that the squares below are pushout squares in $\mathsf{sdSet}^G$.

\[
\begin{array}{c}
X \times K \xrightarrow{\gamma} (X \times K) \xrightarrow{\eta} X \otimes_{\mathfrak{C}_*} K \\
\downarrow \downarrow \downarrow \downarrow \\
Y \times K \xrightarrow{\gamma} (Y \times K) \xrightarrow{\eta} Y \otimes_{\mathfrak{C}_*} K
\end{array}
\]

Lemma 3.35. Let $f : X \to Y$ be a map in $\text{PreOp}^G$ and $k : K \to L$ be a map in $\mathsf{sSet}$. Then $\gamma^* (f \square_{\mathfrak{C}_*} k)$ is a pushout in $\mathsf{sdSet}^G$ of the map $$(f : X \to Y) \square (K \to L).$$

Proof. Since the left square below is a pushout square

\[
\begin{array}{c}
X \otimes_{\mathfrak{C}_*} K \xrightarrow{\gamma} f : X \otimes_{\mathfrak{C}_*} K \xrightarrow{\eta} Y \otimes_{\mathfrak{C}_*} K \\
\downarrow \downarrow \downarrow \downarrow \\
X \otimes_{\mathfrak{C}_*} L \xrightarrow{\gamma} f : X \otimes_{\mathfrak{C}_*} L \xrightarrow{\eta} Y \otimes_{\mathfrak{C}_*} L
\end{array}
\]

one has $(X \to Y) \square_{\mathfrak{C}_*} k = (f : X \to Y) \square_{\mathfrak{C}_*} k$. Since $f : X \to Y$ is the identity on colors, Remark 3.34 then says that the center squares below are pushout squares.

\[
\begin{array}{c}
f : X \times K \xrightarrow{f \times K} f : X \times L \xrightarrow{f \times L} f : X \otimes_{\mathfrak{C}_*} L \\
\downarrow \downarrow \downarrow \downarrow \\
Y \times K \xrightarrow{\gamma} Y \times L \xrightarrow{\gamma} Y \otimes_{\mathfrak{C}_*} L
\end{array}
\]

A standard argument (see, e.g. [RV14, Obs. 5.1]) then shows that the pushout map for the right square above is a pushout of the pushout map for the left square, finishing the proof.

\[\square\]
3.4 Definition and existence of the tame model structure

**Definition 3.36.** The *tame cofibrations* in $\text{PreOp}^G$ are the saturation of the following maps

- (TC1) $G/H \cdot (\emptyset \to \Omega[\eta])$ for $H \leq G$;
- (TC2) $\Omega[C] \otimes e_* (\partial \Delta[n] \to \Delta[n])$ for $C \in \Sigma_G$, $n \geq 0$;
- (TC3) $(\text{Sc}[T] \to \Omega[T]) \sqcap e_* (\partial \Delta[n] \to \Delta[n])$ for $T \in \Omega_G$, $n \geq 0$.

**Definition 3.37.** $I \in \text{PreCat} \simeq \text{PreOp} \downarrow \Omega[n]$ is called a *pseudo-interval* if $I(\eta) = \{0, 1\}$, the map $\Omega[\eta] \sqcup \Omega[\eta] = \text{sk}_n I \to I$ is a tame cofibration and the map $I \to \Omega[\eta]$ is a weak equivalence.

**Definition 3.38.** The *tame anodyne cofibrations* in $\text{PreOp}^G$ are the saturation of the following maps

- (TA1) $G/H \cdot (\Omega[\eta] \to I)$ for $H \leq G$ and $I \in \text{PreCat}$ a countable pseudo-interval;
- (TA2) $\Omega[C] \otimes e_* (\Lambda^i[n] \to \Delta[n])$ for $C \in \Sigma_G$, $0 \leq i \leq n$, $1 \leq n$;
- (TA3) $(\text{Sc}[T] \to \Omega[T]) \sqcap e_* (\partial \Delta[n] \to \Delta[n])$ for $T \in \Omega_G$, $n \geq 0$.

We can now state the main result of this section.

**Theorem 3.39** (cf. [CM13b, Thm. 7.19]). There is a model structure on $\text{PreOp}^G$, called the tame model structure, such that:

- the weak equivalences are the complete equivalences (i.e. detected by inclusion into $\text{sdSet}^G$);
- the generating cofibrations are the maps (TC1),(TC2),(TC3);
- $X \in \text{PreOp}^G$ is fibrant iff the map $X \to *$ has the right lifting property against (TA1),(TA2),(TA3);
- a map $X \to Y$ between fibrant objects is a fibration iff it has the right lifting property against (TA1),(TA2),(TA3).

Moreover, the identity adjunction $\text{PreOp}^G_{\text{tame }} \simeq \text{PreOp}^G_{\text{normal }}$ is a Quillen equivalence.

Before proving Theorem 3.39, we collect a few lemmas.

**Lemma 3.40.** Tame cofibrations (resp. tame anodyne cofibrations) are cofibrations (resp. trivial cofibrations) in the normal model structure on $\text{PreOp}^G$.

**Proof.** It suffices to check the given claims for the generating maps in Definitions 3.36, 3.38.

The (TC1) case is immediate. For (TC2),(TC3),(TA2),(TA3) we apply Lemma 3.35 (note that for (TC2),(TA2) the map $f: X \to Y$ is $\emptyset \to \Omega[C]$, so that $f_* X \to Y$ is the inclusion $\partial \Omega[C] \to \Omega[C]$) and in all such cases it is straightforward that the corresponding map $(f_* X \to Y) \sqcap k$ is a (trivial) cofibration in $\text{sdSet}^G$. (TA1) follows by definition and the (TC1),(TC2),(TC3) cases. $\square$

**Lemma 3.41.** Any map $X \to Y$ which has the right lifting property against (TC1),(TC2),(TC3) is a weak equivalence in $\text{PreOp}^G$.

**Proof.** Writing $f: \mathcal{C} \to \mathcal{D}$ for the underlying map of colors, consider the factorization $X \to f^* Y \to Y$. Noting that lifting problems against (TC1) depend only on objects and both of (TC2) and (TC3) consist of maps which are identities on objects, we see that $X \to Y$ has the right lifting property against (TC1) iff $f^* Y \to Y$ does and the right lifting property against (TC2),(TC3) iff $X \to f^* Y$ does. We argue separately that $f^* Y \to Y$ and $X \to f^* Y$ are joint equivalences.
Consider first the map \( f^*Y \to Y \). Note now that \( f^*Y \to Y \) has the right lifting proper against all maps \( (\partial \Omega[T] \to \Omega[T]) \times \Delta[n] \). Indeed, if \( T \simeq G/H \eta \) is a stick \( G \)-tree, this is precisely the lifting condition against (TC1), and otherwise it follows automatically since \( (\partial \Omega[T] \to \Omega[T]) \times \Delta[n] \) is the identity on objects. Therefore, the levels \( (f^*Y)_n \to Y_n \) are trivial fibrations in \( \text{dSet}^G \), showing that \( f^*Y \to Y \) is a dendroidal equivalence, and thus a joint equivalence.

Consider now the map \( X \to f^*Y \). The lifting property against (TC2) together with the decompositions in (3.12) then say that the maps \( X_\ast(\Omega[C]) \to (f^*Y)_\ast(\Omega[C]) \) are trivial Kan fibrations for all \( G \)-corollas \( C \in \Sigma_{\Omega} \) and colorings \( c : E(C) \to \mathcal{C} \). Now consider a \( G \)-tree \( T \) and coloring \( c : E(T) \to \mathcal{C} \) and consider the following diagram.

\[
\begin{array}{ccc}
X_\ast(\Omega[T]) & \longrightarrow & X_\ast(Sc[T]) \\
\downarrow & & \downarrow \cong \\
(f^*Y)_\ast(\Omega[T]) & \longrightarrow & (f^*Y)_\ast(Sc[T]) \\
\downarrow & & \downarrow \cong \\
\prod_{v \in \mathcal{V}_G(T)} X_\ast(\Omega[T]) & \longrightarrow & \prod_{v \in \mathcal{V}_G(T)} (f^*Y)_\ast(\Omega[T])
\end{array}
\]

The discussion above shows that the rightmost map is a trivial Kan fibration, and thus so is \( X(Sc[T]) \to f^*Y(Sc[T]) \). But it now follows from the lifting property against (TC3) that the maps \( X(\Omega[T]) \to f^*Y(\Omega[T]) \) are trivial Kan fibrations for all \( G \)-trees, showing that \( X \to f^*Y \) is a simplicial equivalence, and thus a joint equivalence. \( \square \)

**Remark 3.42.** Tame cofibrant replacement in \( \text{PreOp}^G \) can be performed without changing objects. Indeed, given any \( A \in \text{PreOp}^G \), one has that \( sk_\Omega A = \prod_{A(\eta)} \Omega[\eta] \) is tame cofibrant by (TC1). Thus, the small object argument for (TC2),(TC3) applied to the map \( sk_\Omega A \to A \) gives a factorization \( sk_\Omega A \to \overline{A} \to A \) where: \( sk_\Omega A \to \overline{A} \) is in the saturation of (TC2),(TC3), so that \( \overline{A} \) is tame cofibrant; \( \overline{A} \to A \) has the right lifting property against (TC2),(TC3) by construction and against (TC1) since it is the identity on objects, and is thus a weak equivalence by Lemma 3.41.

**Lemma 3.43.** Let \( X \in \text{PreOp}^G \) be a Segal operad and \( \Omega[1] \to X^H \) be a \( H \)-equivalence for some \( H \leq G \). Then there exists a countable pseudo-interval \( I \) and factorization \( \Omega[1] \to I \to X^H \).

**Proof.** Recall that one can find a simplicial equivalence \( X \to \overline{X} \) with \( \overline{X} \) fibrant in the normal model structure on \( \text{PreOp}^G \). Since Reedy fibrant preoperads are in particular Segal spaces, it is then well known that, for \( J \) for the nerve of the contractible groupoid with two objects, one has a dashed arrow as on the left below (this is originally due to Rezk [Rez01, Thm. 6.2]; alternatively, see [BPa, Prop. 5.26(iv)]).

\[
\begin{array}{ccc}
\Omega[1] & \longrightarrow & X^H \\
\downarrow & & \downarrow \cong \\
J & \longrightarrow & \overline{X}^H
\end{array}
\]

Writing \( J \to \overline{J} \to \overline{X}^H \) for the trivial cofibration followed by fibration in the normal model structure on \( \text{PreOp} \), we now form the right diagram above, where the square is a pullback. Here we note that \( X^H \to \overline{X}^H \) is a simplicial equivalence, i.e. the maps \( X^H(T) \to \overline{X}^H(T) \) are Kan equivalences for each \( T \in \Omega \), while the maps \( J(T) \to \overline{X}^H(T) \) are Kan fibrations. Hence, since \( s\text{Set} \) is proper, \( J' \to J \) is again a simplicial equivalence.

By construction, the canonical map \( J' \to \Omega[\eta] \) is a simplicial equivalence, but to obtain the required countability and the tame cofibrancy condition in Definition 3.37 we will need to replace \( J' \). Firstly, a countable replacement can be obtained by adapting either the argument between Lemmas 4.2 and 4.3 of [Ber07a] or the more refined argument in the proof of [HSS00, ...]
Lemma 5.1.7. Briefly, since the spaces $J'(\mathcal{T})$ are all contractible, one may build nested countable subpresheaves $I^t, n \in J'$ as in

$$\Omega[1] = I^t, 0 \subseteq I^t, 1 \subseteq I^t, 2 \subseteq \ldots \subseteq J'$$

such that all maps $I^t, n(\mathcal{T}) \to I^t, n+1(\mathcal{T})$ are nullhomotopic (informally, and given a countable $I^t, n$, one needs only countably many simplices of $J'$ to kill of the homotopy groups of $I^t, n$; hence by adding those simplices and closing under the presheaf operations one obtains $I^t, n+1$). Thus, setting $I' = \bigcup_n I^t, n$ we still have that $I' \to \Omega[\eta]$ is a simplicial equivalence, but $I'$ is now countable.

Lastly, the small object argument for (TC2), (TC3) applied to $\Omega[1] \to I'$ gives a factorization $\Omega[1] \to I \to I'$ where the map $I \to I'$ is a complete equivalence (see the argument in Remark 3.42), and the countable preoperad $I$ now has the tame cofibrancy property required by Definition 3.37.

Proof of Theorem 3.39. We first note that, assuming the existence of the tame model structure, the “moreover” claim follows immediately from the fact that weak equivalences in the two model structures coincide, together with Lemma 3.40, which provides the remaining claim that tame cofibrations are normal cofibrations.

To show the existence claims, we will verify conditions C1, C2, C3, C4, C5 in [Sta14, Prop. 2.3], which is a variation of J. Smith’s theorem [Bek00, Thm. 1.7] that includes a further criterion for detecting fibrations between fibrant objects.

$PreOp^G$ is certainly locally presentable, as it is a presheaf category. That the weak equivalences in $PreOp^G$ are accessible follows since they are the preimage by $\gamma^*$ of the weak equivalences in $sdSet^G$ (see [Lur09, Cor. A.2.6.5] and [Lur09, Cor. A.2.6.6]).

Conditions C1 and C3 therein are equivalent to the 2-out-of-6 condition for weak equivalences, and are thus inherited from $sdSet^G$. Moreover, C2 has already been verified in Lemma 3.41.

We next check C4. Note first that the maps in $I\text{-col} \cap W$ are closed under pushout and transfinite composition, as they are trivial cofibrations in the normal model structure in $PreOp^G$, so that C4 needs only be checked for the maps in (TA1), (TA2), (TA3) themselves, rather than their saturation. The case of maps in (TA1) is tautological. The fact that the maps in (TA2), (TA3) are in the saturation of (TC2), (TC3) is clear, and the fact that these maps are weak equivalences follows from Lemma 3.40.

Lastly, we check C5. The lifting condition against (TA3) says that $J$-fibrant objects are such that the maps $X(\Omega[T]) \to X(Sc[T])$ are trivial fibrations, and thus that such $X$ are Segal operads. Therefore, by the “further” statement in Theorem 3.25 it suffices to check that $J$-fibrations between Segal operads which are also DK equivalences have the right lifting property against the maps in (TC1), (TC2), (TC3). Given $X \to Y$ a $J$-fibration with $J$-fibrant $Y$, the lifting property against (TC3) is tautological since (TC3) equals (TA3). Next, the lifting property against (TA2) says that the maps $X(\Omega[T]) \to f^*Y(\Omega[T])$ are Kan fibrations, and the DK condition says that these are Kan equivalences, so that we conclude that such maps have the right lifting property against (TC2). Lastly, given any lifting problem against a map in (TC1), essential surjectivity and Remark 3.43 produce a lifting problem against a map in (TA1) which has a solution, providing a solution to the original problem. This finishes the proof.

For later use, we record the following.

Lemma 3.44. For all $T \in \Omega_G$, the objects $Sc[T], \Omega[T]$ are tame cofibrant in $PreOp^G$.

Proof. The case of $Sc[T]$ follows from the pushout below, where $\bigsqcup_{E(T)} \Omega[\eta]$ is tame cofibrant

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by (TC1) and the left vertical map is a tame cofibration by (TC2) with \( n = 0 \).

\[
\bigcup_{G \in \mathcal{V}_G(T)} \partial \Omega[T_{G^n}] \longrightarrow \bigcup_{E(T)} \Omega[n] \\
\downarrow \quad \downarrow \\
\bigcup_{G \in \mathcal{V}_G(T)} \Omega[T_{G^n}] \longrightarrow S_{\mathcal{C}}[T]
\]

The case of \( \Omega[T] \) follows since (TC3) with \( n = 0 \) says that \( S\mathcal{C}[T] \to \Omega[T] \) is a tame cofibration. \( \square \)

## 4 The Quillen equivalences

In this section we prove our main result, Theorem I, modulo the key Lemma 4.28, which will be proved in §5.

We do so in two steps. After recalling the Dwyer-Kan model structure on equivariant simplicial operads in §4.1, we prove in Theorem 4.39 that the nerve functor is a right Quillen equivalence between \( s\mathcal{O}p^G \) and the tame model structure on \( \text{PreOp}^G \) from §3.4. This yields two zigzags of Quillen equivalences between \( s\mathcal{O}p^G \) and the joint Reedy model structure on \( sd\text{Set}^G \). To conclude our main result, we show that the two associated derived composites agree up to joint equivalence.

### 4.1 Equivariant simplicial operads

We will write \( s\mathcal{O}p^G = \mathcal{O}p^G(\text{sSet}) \) for the category of \( G \)-equivariant colored simplicial operads. There are several possible descriptions of this category: as the algebras for a composition product \( \circ \); as the algebras for a free operad monad \( \mathbb{F} \); as the subcategory of preoperads that satisfy a strict Segal condition.

Our primary goal in this section is to recall (and slightly repackaged) the model structure on \( s\mathcal{O}p^G \) built in [BPc]. The work in loc. cit. is based on the free operad monad perspective on \( s\mathcal{O}p^G \), which is technically involved, but in this paper we will only need a brief overview of that perspective. Instead, and in preparation for the proof of the Quillen equivalence \( \text{PreOp}^G \rightleftarrows s\mathcal{O}p^G \) in §4.2, we will find it useful in this section to also make use of the strict Segal condition perspective, which also plays a key role on both Appendices A,B.

We start by recalling the category \( \Sigma_\mathcal{C} \) of \( \mathcal{C} \)-corollas (cf. Definition 2.16). A typical object in \( \Sigma_\mathcal{C} \) is given by a \( \mathcal{C} \)-colored corolla as on the left below. Then letting \( n \) be the number of leaves of \( \vec{\mathcal{C}} \), which we call the arity of \( \vec{\mathcal{C}} \), and \( \sigma \in \Sigma_n \), the picture below depicts a generic map in \( \Sigma_\mathcal{C} \).

\[
\begin{array}{c}
\vec{\mathcal{C}} \\
\sigma \\
\end{array}
\]

Alternatively, \( \mathcal{C} \)-corollas can be represented simply as strings in \( \mathcal{C} \), which we call \( \mathcal{C} \)-profiles (cf. Remark 3.14). In profile notation, (4.1) then becomes

\[
\vec{\mathcal{C}} = (c_1, \ldots, c_n; c_0) \to (c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(n)}; c_0) = \vec{\mathcal{C}}\sigma^{-1}.
\]

For this reason, we also refer to \( \Sigma_\mathcal{C} \) as the \( \mathcal{C} \)-symmetric category.

Lastly, we note that the notation \( \vec{\mathcal{C}}\sigma^{-1} \) used above comes from the natural right action of \( \Sigma_n \) on \( \mathcal{C} \)-profiles of arity \( n \) via \( (c_1, \ldots, c_n; c_0)\sigma = (c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c_0) \).
Definition 4.2. The category $\mathbf{sSym}$ of simplicial symmetric sequences has:

- objects given by pairs $(C, X)$ with $C \in \mathbf{Set}$ a set of colors and $\Sigma^\text{op}_C \to \mathbf{sSet}$ a functor;
- a map $(C, X) \to (D, Y)$ given by a map of colors $\varphi : C \to D$ and $\Phi$ as below.

More explicitly, we note specifying (4.1) to a map $\overset{\Rightarrow}{\sigma}$ in $\Sigma^\text{op}_C$, one has that a symmetric sequence $X : \Sigma^\text{op}_C \to \mathbf{sSet}$ has structure maps

$$X(\overset{\Rightarrow}{C}) = X(c_1, \ldots, c_n; c_0) \overset{\sim}{\to} X(gc_{\sigma(1)}, \ldots, gc_{\sigma(n)}; gc_0) = X(g\overset{\Rightarrow}{C})$$

for $\overset{\Rightarrow}{C}$ of arity $n$ and $\sigma \in \Sigma_n$.

The free operad monad $F$ is then a monad on $\mathbf{sSym}$ which, when evaluated on $X : \Sigma^\text{op}_C \to \mathbf{sSet}$, is given by the left Kan extension (the real challenge, of course, is that of defining the multiplication $F \circ F \Rightarrow F$)

$$\Omega^{\text{op}}_C \xrightarrow{T \Rightarrow \Pi \Rightarrow \nu(T)} \mathbf{sSet} \xrightarrow{\text{Lan}=FX} \mathbf{sSym}$$

(4.3)

The category $\mathbf{sOp}$ of colored simplicial operads can then be described as the category of $F$-algebras on $\mathbf{sSym}$. For $G$ a finite group, we then write $\mathbf{sOp}^G$ (resp. $\mathbf{sSym}^G$) for the category of $G$-objects on $\mathbf{sOp}$ ($\mathbf{sSym}$) which we call the category of $G$-equivariant colored simplicial operads (symmetric sequences). Note that, by abstract nonsense, $F$ induces a monad on $\mathbf{sSym}^G$ whose category of algebras is $\mathbf{sOp}^G$.

Mirroring (3.8) we now have color set functors

$$\mathbf{sSym} \xrightarrow{\varepsilon} \mathbf{Set} \quad \mathbf{sOp} \xrightarrow{\varepsilon} \mathbf{Set} \quad \mathbf{sSym}^G \xrightarrow{\varepsilon} \mathbf{Set}^G \quad \mathbf{sOp}^G \xrightarrow{\varepsilon} \mathbf{Set}^G.$$

Remark 4.4. Replacing $\mathbf{sSet}$ with $\mathbf{Set}$ in Definition 4.2 and (4.3) one recovers the analogue non-simplicial categories $\mathbf{Sym}$, $\mathbf{Op}$, $\mathbf{Sym}^G$, $\mathbf{Op}^G$. As is well known, there is then a fully faithful inclusion $\mathbf{sOp} \subset \mathbf{Op}^\Sigma$, as those simplicial objects with a constant set of colors.

The color set functors above are all Grothendieck fibrations and, moreover, the monad $F$ is suitably compatible with these fibrations. In [BPc] the fiber perspective is used to describe the fibers $\mathbf{sSym}^G$, $\mathbf{sOp}^G$ of those objects with a fixed $G$-color set $\mathcal{C}$ and maps which are the identity on colors. However, here we will be able to make do with a more elementary approach.

If $\mathcal{C}$ is a $G$-set of colors, one has a left $G$-action of $\mathcal{C}$-profiles. Then, if $X \in \mathbf{sSym}^G$ has colors $\mathcal{C}_X = \mathcal{C}$, on has, generalizing (4.5), that $X$ has structure maps

$$X(\overset{\Rightarrow}{C}) = X(c_1, \ldots, c_n; c_0) \overset{\sim}{\to} X(gc_{\sigma(1)}, \ldots, gc_{\sigma(n)}; gc_0) = X(g\overset{\Rightarrow}{C})$$

(4.5)

for $\overset{\Rightarrow}{C}$ a $\mathcal{C}$-profile of arity $n$ and $(g, \sigma) \in G \times \Sigma_n^\text{op}$.
Note that, implicit in the \( g^C \sigma \) notation in (4.5) is the fact that \( G \times \Sigma_n^{op} \) has a left action on \( C \)-profiles of arity \( n \). As such, given a subgroup \( \Gamma \leq G \times \Sigma_n^{op} \) and \( \bar{C} \) of arity \( n \), we say that \( \Gamma \) stabilizes \( \bar{C} \) if \( g^C \sigma = \bar{C} \) for all \((g, \sigma) \in \Gamma\). In particular, (4.5) then implies that whenever \( \Gamma \) stabilizes \( \bar{C} \) and for \( X \in sSym^G \) (resp. \( O \in sOp^G \)) with \( G \)-color set \( C \), the level \( X(\bar{C}) \) (resp. \( O(\bar{C}) \)) has an action by \( \Gamma \).

**Notation 4.6.** For \( x \in X(\bar{C}) \) with \( \bar{C} \) of arity \( n \) and \((g, \sigma) \in G \times \Sigma_n^{op} \) we write \( g x \sigma \in X(g^C \sigma) \) for the image of \( x \) under (4.5). Note that this defines an action of \( G \times \Sigma_n^{op} \) on \( X(\bar{C}) \) of arity \( n \).

Moreover, if \( x \sigma = x \) only when \( \sigma = id \) then we say \( x \) is \( \Sigma \)-free.

**Remark 4.7.** Let \( G_C \) denote the groupoid with objects the \( C \)-signatures and arrows \( \bar{C} \to g^C \sigma \) for \( \bar{C} \) of arity \( n \) and \((g, \sigma) \in G \times \Sigma_n^{op} \). In other words, \( G_C \) is the coproduct over \( n \geq 0 \) of the action groupoids for the actions of \( G \times \Sigma_n^{op} \) on \( n \)-ary signatures (alternatively, in \([BPc]\) we use an alternative description \( G_C = G \times \Sigma_n \); see \([BPc, Prop. 2.52]\)). Equation (4.5) then identifies \( sSym^G_C = sSet^{G_C} \).

Before describing the model structure on \( sOp^G \) we need to recall two more ingredients.

First, a subgroup \( \Gamma \leq G \times \Sigma_n^{op} \) is called a \( G \)-graph subgroup if \( \Gamma \cap \Sigma_n^{op} = \{e\} \). Equivalently, it is straightforward to show that \( \Gamma \) must be of the form \( \Gamma = \{(h, \phi(h)^{-1}) \mid h \in H\} \) for some subgroup \( H \leq G \) and homomorphism \( \phi: H \to \Sigma_n \).

Second, one has functors

\[
sOp^G \xrightarrow{\pi_0} Op^G \xrightarrow{j^*} \text{Cat}
\]

where \( \pi_0 \) is computed levelwise, i.e. \((\pi_0)(\bar{C}) = \pi_0(O(\bar{C}))\) and \( j^* \) forgets non-unary operations.

Generalizing \([Berge07, CM13b]\), we show in \([BPc]\) that \( sOp^G \) has a Dwyer-Kan style model structure. More precisely, we have the following result, which is \([BPc, Thm. III]\), with the alternative characterization of fibrations provided by \([BPc, Prop. 4.78]\).

**Theorem 4.8.** The category \( sOp^G \) has a cofibrantly generated model structure with weak equivalences (resp. fibrations) those maps \( F:O \to P \) such that:

- \( F \) is fully faithful (resp. a local fibration), i.e. the induced maps
  \[
  O(\bar{C})^\Gamma \to P(F(\bar{C}))^\Gamma
  \]
  are Kan equivalences (resp. Kan fibrations) in \( sSet \) for all \( \mathfrak{C}_O \)-signatures \( \bar{C} = (\epsilon_1, \ldots, \epsilon_n; \epsilon_0) \) and all graph subgroups \( \Gamma \leq G \times \Sigma_n^{op} \) which stabilize \( \bar{C} \);

- \( F \) is essentially surjective (resp. an isofibration), i.e. the induced maps of usual categories
  \[
  j^* \pi_0 O^H \to j^* \pi_0 P^H
  \]
  are essentially surjective (resp. isofibrations) for all \( H \leq G \).

For a fixed \( G \)-set of colors \( \mathfrak{C} \), the fixed color symmetric sequences category \( sSym^G_{\mathfrak{C}} \) admits an auxiliary model structure where (adapting (4.9)) \( X \to Y \) is a (trivial) fibration iff \( X(\bar{C})^\Gamma \to Y(\bar{C})^\Gamma \) is a (trivial) Kan fibration. For instance, using the identification \( sSym^G_{\mathfrak{C}} = sSet^{G_{\mathfrak{C}}} \) from Remark 4.7, this is the model structure from \([BPc, Prop. 3.17]\) with respect to the family of \( F^\Gamma \) such that \( \bar{C} \) consists of the graph subgroups stabilizing \( \bar{C} \) (here we use the fact that the automorphism group of an \( n \)-ary signature \( \bar{C} \) in \( G_{\mathfrak{C}} \) can be naturally viewed as a subgroup of \( G \times \Sigma_n^{op} \)).
**Proposition 4.11.** Let $f: A \to B$ be a map in $s\text{Sym}_C^\ast \approx s\text{Set}^G$. The following are equivalent:

(i) $f$ is a cofibration;

(ii) $f$ is a monomorphism and the stabilizer of every $x \in B \setminus f(A)$ is a graph subgroup;

(iii) $f$ is a monomorphism and every $x \in B \setminus f(A)$ is $\Sigma$-free (cf. Notation 4.6).

**Proof.** By [BPc, Rem. 3.14] the generating cofibrations in $s\text{Sym}_C^\ast$ then have the form $G_\ast(G, -)/\Gamma \cdot (\partial \Delta[k] \to \Delta[k])$ with $\Gamma \in \mathcal{F}_C^\ast$, $k \geq 0$, so that (i) $\iff$ (ii) follows by adapting [Ste16, Prop. 2.16] or [Per18, Prop. 6.5]. (ii) $\iff$ (iii) is straightforward. \qed

In light of (iii) in the previous result, a color fixed map of operads $O \to P$ such that the underlying map of symmetric sequences is a cofibration is called a $\Sigma$-cofibration. Combining Proposition 4.11 with [BPc, Prop. 3.63] (also, see [BPc, Prop. 4.11(ii)]) yields the following,

**Proposition 4.12.** If $f: O \to P$ is a color fixed map in $s\text{Op}^G$ and $O$ is $\Sigma$-cofibrant then $f$ is a $\Sigma$-cofibration. In particular, cofibrant operads are $\Sigma$-cofibrant.

We next recall the operadification-nerve functor adjunctions

$$\tau: d\text{Set} \rightleftarrows \text{Op}: N \quad \tau: \text{PreOp} \rightleftarrows s\text{Op}: N.$$  \hspace{1cm} (4.13)

where we note that the rightmost adjunction is induced by applying the leftmost adjunction to each simplicial level (indeed, by Definition 3.6 Remark 4.4 both $\text{PreOp}$ and $s\text{Op}$ are characterized by demanding that the color sets on all simplicial levels are are the same).

To describe the nerve $N$, we need to recall the operad $\Omega(F) \in \text{Op}$ freely determined by a forest $F \in \Phi$. Explicitly, $\Omega(F)$ is the $E(F)$-colored operad which, when evaluated on a $E(F)$-colored corolla $\overline{C} = (C, c E(C) \to E(F))$, is given by

$$\Omega(F)(\overline{C}) = \begin{cases} \ast & \text{if } c E(C) \to E(F) \text{ defines a map } C \to F \text{ in } \Phi \\ \emptyset & \text{otherwise.} \end{cases}$$

We note that, since all levels of $\Omega(F)$ are either $\ast$ or $\emptyset$, there is always at most one possible way to compose operations, and hence at most one possible operad structure on $\Omega(F)$. That the operad structure indeed exists (i.e. that composition is always defined) is a consequence of the observation that, for any tree $U \in \Omega$, a coloring $c E(U) \to E(F)$ defines a map $U \to F$ in $\Phi$ iff the restrictions $c_v: E(U_v) \to E(F)$ define maps $U_v \to F$ in $\Phi$ for all vertices $v \in V(U)$.

We will also make use of an alternative description of $\Omega(F)$, as follows.

First, if $\overline{C} \in \Sigma_C$ is a $\mathcal{C}$-corolla, we denote its representable functor as $\Sigma_C[\overline{C}] = \Sigma_C^{op}(\overline{C}, -)$ in $\text{Sym}_C = \text{Set}^{\mathcal{C}^{op}}$. Second, for $\overline{F} \in \Phi_C$ a $\mathcal{C}$-forest, we extend the $\Sigma_C[-]$ notation via

$$\Sigma_C[\overline{F}] = \bigsqcup_{v \in V(F)} \Sigma_C[\overline{F}_v],$$

where $u^\mathcal{C}$ denotes the coproduct in $\text{Sym}_C$ (rather than in the larger category $\text{Sym}$). Third, for $F \in \Phi$ an (uncolored) forest we write $F^\tau$ for $F$ together with its tautological $E(F)$-coloring, i.e. $F^\tau = (F, t E(F) \xrightarrow{\sim} E(F))$, and abbreviate $\Sigma_\tau[F] = \Sigma_E(F^\tau)$. All together, one then has an identification

$$\Omega(F) = \mathbb{F} \Sigma_\tau[F]$$  \hspace{1cm} (4.14)

which, informally, says that “$\Omega(F)$ is freely generated by the vertices of $F$”.

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For \( \mathcal{O} \in \mathcal{Op} \) the nerve \( N\mathcal{O} \in \mathcal{dSet} \) from (4.13) is then described by

\[
(N\mathcal{O})(U) = \mathcal{Op}(\Omega(U), \mathcal{O}), \quad U \in \Omega
\]

while \( \tau: \mathcal{dSet} \to \mathcal{Op} \) is the unique colimit preserving functor such that \( \tau(\Omega[U]) \to \Omega(U) \).

We now recall [MW09, Prop. 5.3 and Thm. 6.1] that the nerve \( N:\mathcal{O} \to \mathcal{dSet} \) is then a fully faithful inclusion whose (essential) image can be characterized as those dendroidal sets \( X \in \mathcal{dSet} \) with the strict right lifting property against inner horn inclusions \( \Lambda^i[U] \to \Omega[T] \) for \( U \in \Omega, e \in E(U) \). Next, following either [CM13a, Prop. 2.5 and Cor. 2.6] or [BPa, Props. 3.22 and 3.31], this is in turn equivalent to the strict right lifting property of \( X \) against Segal core inclusions \( S\{U\} \to \Omega[T] \) for \( U \in \Omega \), which is in turn equivalent to the strict Segal condition (cf. Definition 3.19) below, demanding that the maps

\[
X(U) \xrightarrow{\sim} X(S\{U\}), \quad U \in \Omega
\]

\[
X_e(U) \xrightarrow{\sim} \prod_{v \in V(U)} X_{e_v}(U_v), \quad U \in \Omega, c E(U) \to X(\eta)
\]

are all isomorphisms. Moreover, one then has the following alternate formula for the nerve \( N\mathcal{O} \) evaluated at \( U \in \Omega, c: E(U) \to \mathcal{E}_\mathcal{O} \).

\[
(N\mathcal{O})_c(U) = \prod_{v \in E(U)} \mathcal{O}(U_v, e_v) = \prod_{v \in E(U)} \mathcal{O}(\hat{U}_v).
\]

To describe the generating (trivial) cofibrations of the model structure in Theorem 4.8 we will make use of a fibered simplicial tensoring on \( s\mathcal{Op}^G \) which is closely related to the analogue tensoring on \( \mathcal{PreOp}^G \) from §3.3.

First, for \( \mathcal{O} \in s\mathcal{Op}^G \) and \( K \in \mathcal{sSet} \) we define the fiber cotensor \( \{K, \mathcal{O}\}_e \in s\mathcal{Op}^G \) via the pointwise simplicial cotensor, i.e.

\[
\{K, \mathcal{O}\}_e(\vec{c}) = \mathcal{O}(\vec{c})^K.
\]

**Remark 4.17.** The fact that \( \{K, \mathcal{O}\}_e \) as described above has an operad structure can be seen by considering nerves. Indeed, one readily checks that the strict Segal condition (4.15) for \( N\mathcal{O} \) implies the same condition for the preoperad \( \{K, N\mathcal{O}\}_e \) (defined as in (3.27)). Thus, (4.16) describes the levels of the unique (up to isomorphism) operad \( \{K, \mathcal{O}\}_e \) such that \( N\{K, \mathcal{O}\}_e = \{K, N\mathcal{O}\}_e \).

Next, we turn to the fiber tensor \( (\cdot) \otimes_{e_\cdot} K \) adjoint to (4.16). First, note that (4.16) still makes sense at the level of symmetric sequences, i.e. with \( \mathcal{O} \in s\mathcal{Op}^G \) replaced with \( X \in \mathcal{sSym}^G \). Then, at the level of symmetric sequences, the left adjoint construction \( X \times K \in \mathcal{sSym}^G \) is simply given pointwise by \( (X \times K)(\vec{c}) = X(\vec{c}) \times K \). It is now formal that, on a free operad \( \mathbb{F}X \), the tensor \( \mathbb{F}X \otimes_{e_\cdot} K \) adjoint to (4.16) is given by \( (\mathbb{F}X) \otimes_{e_\cdot} K = \mathbb{F}(X \times K) \) so that, for a general \( \mathcal{O} \in s\mathcal{Op}^G \) (which has a standard description \( \mathcal{O} \simeq coeq(\mathbb{F}\mathcal{O} \Rightarrow \mathbb{F}O) \) as a coequalizer of free algebras) it is given by

\[
\mathcal{O} \otimes_{e_\cdot} K \simeq coeq(\mathbb{F}(\mathcal{O} \times K) \Rightarrow \mathbb{F}(\mathcal{O} \times K)).
\]

**Remark 4.18.** In [CM13b, §7.1], the objects \( \Omega(T) \otimes_{e_\cdot} K \) were denoted \( T[K] \) and built by hand.

**Remark 4.19.** The analogues of Remarks 3.32, 3.33 apply mutatis mutandis to the operadic fiber tensor. In particular, one has that the canonical map

\[
u_i \mathcal{O} \otimes_{e_\cdot} K_i \to \mathcal{O} \otimes_{e_\cdot} (\nu_i K_i)
\]

is a cocartesian arrow over the fold map \( \nu_i \mathcal{E} \to \mathcal{E} \).
Proposition 4.21. For all $X \in \text{PreOp}^G$ and $K \in \text{sSet}$, one has a natural identification
\[
\tau(X \otimes_{e_*} K) = \tau(X) \otimes_{e_*} K
\]
with the first (resp. second) $\otimes_{e_*}$ is the fiber simplicial tensoring of $\text{PreOp}^G$ (resp. $\text{sOp}^G$).

Proof. This is equivalent to the already established (cf. Remark 4.17) adjoint identification $N(K, O) e_* = (K, NO) e_*$ for $O \in \text{sOp}^G$.

Remark 4.22. Proposition 4.21 is a slight generalization of [CM13b, Prop. 7.2], which establishes the case $X = \Omega[U], U \in \Omega$ by direct inspection (cf. Remarks 3.30,4.18).

Remark 4.23. Let $\Gamma \leq G \times \Sigma_n^{op}$ be the graph subgroup given by $\Gamma = \{(h, \phi(h)^{-1})| h \in H\}$ for $H \leq G, \phi \in H \to \Sigma_n$. Writing $C_n$ for the $n$-corolla, $\phi$ defines a left $H$-action on $C_n$, so that one obtains an associated $G$-corolla $C = G \cdot_H C_n$. It is then straightforward to check that there are natural identifications (here we view the natural left $G^{op} \times \Sigma_n$-action on $G \cdot C_n$ has a right $G \times \Sigma_n^{op}$-action)
\[
(G \cdot C_n)/\Gamma = G \cdot_H C_n = C \quad \Sigma_\tau[G \cdot C_n]/\Gamma = \Sigma_\tau[G \cdot_H C_n] = \Sigma_\tau[C] \quad (4.24)
\]
in $\Phi^G_*$ and $\text{Sym}^G$, respectively.

We need one final ingredient to describe the generating sets of maps for the model structure on $\text{sOp}^G$ (cf. [BPc, Def. 4.4]).

Definition 4.25. Let $[1]$ denote the free isomorphism category, i.e. the contractible groupoid with two objects 0, 1. An interval is a cofibrant simplicial category $\mathbb{I} \in \text{sCat}_{(0,1)}$ equivalent to $[1]$.

Remark 4.26. Specifying [BPc, Def. 4.19] for $\text{sSet}$, we can rewrite the notation therein via
\[
\mathbb{F}(\Sigma_\tau[G \cdot C_n]/\Gamma \cdot f) = \mathbb{F}(\Sigma_\tau[C] \cdot f) = \mathbb{F}(\Sigma_\tau[C]) \otimes_{e_*} f = \Omega(C) \otimes_{e_*} f
\]
where the first identification is (4.24), the second follows by definition of $\otimes_{e_*}$, and the third is (4.14). We thus have that the generating cofibrations in $\text{sOp}^G$ are the maps
\[
\begin{align*}
\text{(C1)} & \quad \emptyset \to G/H \cdot \Omega(\eta) \quad \text{for} \quad H \leq G \\
\text{(C2)} & \quad \Omega(C) \otimes_{e_*} (\partial \Delta[m] \to \Delta[m]) \quad \text{for} \quad C \in \Sigma_G, m \geq 0.
\end{align*}
\]
while the generating trivial cofibrations are (for the countability condition, see [BPc, Rem. 4.17])
\[
\begin{align*}
\text{(A1)} & \quad G/H \cdot (\eta \to \mathbb{G}) \quad \text{for} \quad H \leq G \quad \text{and} \quad \mathbb{G} \quad \text{an interval with countably many simplices}. \\
\text{(A2)} & \quad \Omega(C) \otimes_{e_*} (\Lambda^k[m] \to \Delta[m]) \quad \text{for} \quad C \in \Sigma_G, m \geq 1, 0 \leq k \leq m.
\end{align*}
\]

In (C2),(A2) above the group $G$ acts only on $\Omega(C)$ and not on the featured simplicial sets. The following lemma will allow us to consider the case where the simplicial sets also have a $G$-action.

Lemma 4.27. For $C \in \Sigma_G$ and $A \to B$ a genuine (trivial) cofibration in $\text{sSet}^G, \Omega(C) \otimes_{e_*}(A \to B)$ is a (trivial) cofibration in $\text{sOp}^G$. 

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Proof. Since $\Omega(C) \otimes (-) : s\text{Set}^G \to s\text{Op}_{\text{E}^G}(C)$ preserves colimits, it suffices to consider the case $(A \to B) = G/H \cdot (K \to L)$ for $K \to L$ a (trivial) cofibration in $s\text{Set}$. Now consider the following diagram.

$$
\begin{align*}
(G/H \cdot \Omega(C)) \otimes \varepsilon_* & \to \Omega(C) \otimes \varepsilon_*(G/H \cdot K) \\
(G/H \cdot \Omega(C)) \otimes \varepsilon_* & \to \Omega(C) \otimes \varepsilon_*(G/H \cdot L)
\end{align*}
$$

By (4.20) the horizontal arrows are cocartesian, while the vertical arrows fix colors, so this is a pushout square. The result now follows since $G/H \cdot \Omega(C)$ decomposes as a coproduct $\bigoplus_i \Omega(C_i)$ with $C_i \in \Sigma_G$.

4.2 Equivalence between preoperads and operads

Our goal in this subsection is to prove Theorem 4.39, showing the Quillen equivalence between preoperads $\text{PreOp}^G_G$ and operads $s\text{Op}^G_G$. The key to proving this result is given by Lemma 4.33 and the subsequent Corollary 4.36, which allow us to understand the counit of the adjunction. These latter results in turn depend on the following key result, whose proof is deferred to §5.

Lemma 4.28. Suppose that $O \in \text{Op}_G^G$ is $\Sigma$-cofibrant. Further, let $C \in \Sigma_G$ be any $G$-corolla, $r \geq 1$ a positive integer and consider a pushout in $\text{Op}_G^G$ of the form

$$
\partial \Omega(C)^{ir} \to O \\
\Omega(C)^{ir} \to \varnothing.
$$

(4.29)

Then the induced map

$$
\Omega[C]^{ir} \cup_{\partial \Omega(C)^{ir}} NO \to NP
$$

(4.30)

is $G$-inner anodyne.

Remark 4.31. Both (4.29) and (4.30) are unchanged if the copowers $(-)^{ir}$ in $\text{Op}_G^G$, $d\text{Set}_G^G$ are replaced with fibered copowers $(-)^{ir \cdot r} \otimes_\varepsilon \{1, \ldots, r\}$ in $\text{Op}_{\text{E}(C)}^G$, $d\text{Set}_{\text{E}(C)}^G$.

Moreover, since $\partial \Omega(C) = \Omega(C) \otimes \varepsilon_\varnothing \otimes_\varepsilon \varnothing$, one is moreover free to replace the left vertical map in (4.29) with $\Omega(C) \otimes \varepsilon_* K \to \Omega(C) \otimes \varepsilon_* L$ for $K \to L$ any inclusion of sets.

Remark 4.32. The integer $r \geq 1$ in Lemma 4.28 is included to match our required application in Lemma 4.33. However, it readily follows by induction on $r$ that one needs only prove the $r = 1$ case. Indeed, writing $O_r$ for the pushout in (4.29) for each $r$, one has a diagram below

$$
\begin{align*}
\Omega[C]^{ir} \cup_{\partial \Omega[C]^{ir}} NO & \to NO_r \\
\Omega[C]^{ir+1} \cup_{\partial \Omega[C]^{ir+1}} NO & \to \Omega[C] \cup_{\partial \Omega[C]} NO_r \to NO_{r+1}
\end{align*}
$$

where the square is a pushout so that induction on $r$ and the $r = 1$ case yield that all horizontal maps are $G$-inner anodyne. The proof of the interesting $r = 1$ case will occupy the entirety of §5.
Lemma 4.33. Let \( A \to B \) be a tame cofibration in \( \text{PreOp}^G \), \( \mathcal{O} \in \mathfrak{sOp}^G \) a \( \Sigma \)-cofibrant \( G \)-operad, and consider a pushout diagram in \( \mathfrak{sOp}^G \) of the form

\[
\begin{array}{ccc}
\tau A & \longrightarrow & \mathcal{O} \\
\downarrow & & \downarrow \\
\tau B & \longrightarrow & \mathcal{P}
\end{array}
\] (4.34)

Then \( \mathcal{O} \to \mathcal{P} \) is a \( \Sigma \)-cofibration and

\[
B \cup_A NO \to NP
\] (4.35)

is a weak equivalence.

Setting \( A = \emptyset, \mathcal{O} = \emptyset \) in the previous result yields the following.

Corollary 4.36. If \( B \in \text{PreOp}^G \) is tame cofibrant, then \( B \to N\tau B \) is a weak equivalence.

Proof of Lemma 4.33. We first consider the case where \( A \to B \) is in one of \( (TC1),(TC2),(TC3) \).

The \( (TC1) \) case is immediate, since \( \mathcal{O} \to \mathcal{O} \cup G[H \cdot \Omega(n)] \) is a \( \Sigma \)-cofibration and \( (4.35) \) is the isomorphism \( NO \cup G[H \cdot \Omega] \cong N(\mathcal{O} \cup G[H \cdot \Omega]) \).

The \( (TC3) \) case is also straightforward: since \( \tau A \to \tau B \) is an isomorphism, one can take \( \mathcal{O} = \mathcal{P} \), so that \( (4.35) \) becomes a section of the map \( NO \to B \cup_A NO \), which is a trivial cofibration (as it is a pushout of \( A \to B \)), and 2-out-of-3 thus implies that \( (4.35) \) is a weak equivalence.

The most interesting case is then \( (TC2) \). Firstly, by Proposition 4.21 the functor \( \tau \) sends maps in \( (TC2) \) to maps in \( (C2) \), so \( \mathcal{O} \to \mathcal{P} \) is indeed a \( \Sigma \)-cofibration by [BP, Prop. 3.63] Next, fixing a simplicial level \( m \geq 0 \), \( A_m \to B_m \) then has the form \( \Omega(C) \otimes \Delta_m(\partial \Delta[n]_m \to \Delta[n]_m) \) so that \( \tau A_m \to \tau B_m \) has the form \( \Omega(C) \otimes \Delta_m(\partial \Delta[n]_m \to \Delta[n]_m) \). But then (following the discussion in Remark 4.31) Lemma 4.28 yields that all levels \( (B \cup_A NO)_m \to (NP)_m \) for \( m \geq 0 \) are equivalences in \( \text{dSet}^G \), showing that \( B \cup_A NO \to NP \) is indeed a complete equivalence in \( \text{PreOp}^G \).

We now turn to the case of \( A \to B \) a general tame cofibration. As usual, \( A \to B \) is a retract of a transfinite composition of pushouts of generating cofibrations. Since the conclusions of the result are invariant under retracts, we are free to assume that \( A \to B \) is a transfinite composite

\[
A = A_0 \to A_1 \to A_2 \to \cdots \to A_\beta \to \text{colim}_{\beta \leq \kappa} A_\beta = B.
\]

where each map \( A_\beta \to A_{\beta + 1} \) is a pushout of a map in one of \( (TC1),(TC2),(TC3) \).

Defining \( \mathcal{O}_\beta \) by replacing \( A \to B \) with \( A \to A_\beta \) in the pushout \( (4.34) \), \( \mathcal{O} \to \mathcal{P} \) becomes the transfinite composite of the maps \( \mathcal{O}_\beta \to \mathcal{O}_{\beta + 1} \) and \( (4.35) \) becomes \( \text{colim}_{\beta \leq \kappa} (NO \cup_{N\tau A_\beta} N\tau B_\beta \to NO_B) \).

It thus suffices to show, by induction on \( \beta < \kappa \), that the maps \( \mathcal{O}_\beta \to \mathcal{O}_{\beta + 1} \) are \( \Sigma \)-cofibrations and that the maps \( NO \cup_{N\tau A_\beta} N\tau B_\beta \to NO_B \) are weak equivalences (sufficiency of the latter condition uses the fact that filtered colimits of weak equivalences in \( \text{PreOp}^G \) are weak equivalences, cf. Theorem 3.10). Consider now the following diagrams.

\[
\begin{array}{ccc}
\tau A & \longrightarrow & \mathcal{O} \\
\downarrow & & \downarrow \\
\tau A_\beta & \longrightarrow & \mathcal{O}_\beta \\
\downarrow & & \downarrow \\
\tau A_{\beta + 1} & \longrightarrow & \mathcal{O}_{\beta + 1}
\end{array} \quad \begin{array}{ccc}
A_\beta \cup_A NO & \longrightarrow & A_{\beta + 1} \cup_A NO \\
\downarrow & & \downarrow \\
NO_B & \longrightarrow & A_{\beta + 1} \cup_{A_\beta} NO_B \\
\downarrow & & \downarrow \\
NO_{\beta + 1} & \longrightarrow & NO_{\beta + 1}
\end{array}
\]

The induction hypothesis states that \( \mathcal{O} \to \mathcal{O}_\beta \) is a \( \Sigma \)-cofibration and that the map \( A_\beta \cup_A NO \to \mathcal{O}_\beta \) is a weak equivalence. Therefore, \( \mathcal{O}_\beta \) is \( \Sigma \)-cofibrant and both vertical maps marked \( \sim \) in the
rightmost diagram above are weak equivalences (this uses the fact that \( \text{PreOp}^G \) is left proper), and thus the induction step will follow provided that the result holds for the map \( A_\beta \to A_{\beta+1} \) and \( O_\beta \). But \( A_\beta \to A_{\beta+1} \) is assumed to be a pushout of a map in \((\text{TC1}),(\text{TC2}),(\text{TC3})\), in which case the result is already known, and thus noting that the result is invariant under pushouts finishes the proof.

Before proving Theorem 4.39, we recall the following, which are adapted from [JT07] (see Proposition 7.15 therein).

**Proposition 4.37.** A cofibration \( A \to B \) is a weak equivalence iff it has the left lifting property against all fibrations between fibrant objects.

**Corollary 4.38.** An adjunction

\[ F:C \rightleftarrows D:G \]

between model categories is a Quillen adjunction provided that \( F \) preserves cofibrations and \( G \) preserves fibrations between fibrant objects.

**Theorem 4.39.** The adjunction

\[ \tau: \text{PreOp}^G \rightleftarrows \text{sOp}^G:N \tag{4.40} \]

is a Quillen equivalence.

**Proof.** We first show that \( N \) preserves and detects weak equivalences. To see this, note first that all objects in the image of \( N \) are Segal operads, so that by Theorem 3.25 a map in the image of \( N \) is a weak equivalence iff it is a Dwyer-Kan equivalence. But it is clear that \( N \) preserves and reflects fully-faithful maps, and since \( N(j^*O^H) \cong j^*((NO)^H) \) for \( H \leq G \) one likewise has that \( N \) preserves and reflects essentially surjective maps.

Next, we use Corollary 4.38 to show that (4.40) is a Quillen adjunction. First, \( \tau \) preserves cofibrations since, by Proposition 4.21, \( \tau \) sends maps in \((\text{TC1}),(\text{TC2})\) to maps in \((\text{OC1}),(\text{OC2})\) and maps in \((\text{TC3})\) to isomorphisms. Second, to show that \( N \) preserves fibrations between fibrant objects, by using the characterization in Theorem 3.39 it suffices, thanks to an adjunction argument, to show that \( \tau \) sends the maps in \((\text{TA1}),(\text{TA2}),(\text{TA3})\) to trivial cofibrations. Moreover, as we already know that \( \tau \) preserves cofibrations, we need only show that \( \tau \) sends \((\text{TA1}),(\text{TA2}),(\text{TA3})\) to weak equivalences. The cases \((\text{TA2}),(\text{TA3})\) are again immediate from Proposition 4.21, but \((\text{TA1})\) requires a different argument (which could also be used for \((\text{TC2}),(\text{TC3})\)). Writing \( A \to B \) for a map in \((\text{TA1})\), one necessarily has that \( A,B \) are tame cofibrant, so that Corollary 4.36 and 2-out-of-3 imply that \( N\tau A \to N\tau B \) is a weak equivalence and thus, since \( N \) reflects weak equivalences, \( \tau A \to \tau B \) itself is a weak equivalence, as desired.

For the Quillen equivalence claim, let \( B \in \text{PreOp}^G \) be tame cofibrant and \( O \in \text{sOp}^G \) be fibrant. We must show the leftmost map below is a weak equivalence iff the rightmost composite is.

\[ \tau B \to O, \quad B \to N\tau B \to NO \]

This now follows from Corollary 4.36 and the fact that \( N \) preserves and detects weak equivalences.

\[ \square \]

### 4.3 The homotopy coherent nerve and the proof of the main equivalence

In this section we prove our main result, Theorem I. We first recall how the \( W: \text{dSet}^G \rightleftarrows \text{sOp}^G:hcN \) adjunction (1.2) is defined.
In the categorical setting the left adjoint $W_t$ admits an explicit description, due to Dugger and Spivak [DS11], in terms of so called necklaces, which we extend to the operadic setting in Appendix A. We now summarize the results in that appendix we will need.

For a tree $U \in \Omega$ there is a simplicial operad $W(U) \in sOp$ with set of colors $E(U)$ and whose $n$-simplices evaluated at a $E(U)$-corolla $\mathcal{C} = (C, c)$

$$W(U)_n(\mathcal{C}) = \begin{cases} \text{factorizations } C \xrightarrow{i} F_0 \xrightarrow{i \cdot p} \cdots \xrightarrow{i \cdot p} F_n \xrightarrow{f \cdot p} U & \text{if } E(C) \xrightarrow{\ell} E(U) \text{ defines a map in } \Omega \\ \emptyset & \text{otherwise.} \end{cases}$$

where we label maps in $\Omega$ as $t/i/f/p$ to indicate they are tall/inner faces/faces/planar (cf. §2.1).

**Remark 4.42.** The factorization description in (4.41) reflects our approach in Appendix A, which makes heavy use of the factorizations in Proposition 2.10. However, there is a simpler and more familiar description of $W(U)(\mathcal{C})$. If one lets $C \xrightarrow{i} U \overset{\alpha \cdot p}{\rightarrow} U$ denote the unique “tall followed by planar outer face” factorization, repeated use of Proposition 2.10 shows that the $F_0 \rightarrow \cdots \rightarrow F_n$ strings in (4.41) are precisely the strings of planar inner faces of $U_C$. And since the latter are in bijection with strings of subsets of inner edges $E'(U_C)$, we have

$$W(U)(\mathcal{C}) = \begin{cases} \Delta[1] \ast E'(U_C) & \text{if } E(C) \xrightarrow{\ell} E(U) \text{ defines a map in } \Omega \\ \emptyset & \text{otherwise}, \end{cases}$$

which recovers the description in [CM13b, §4].

**Remark 4.43.** One neat feature of the description in (4.41) is that the nerve $NW(U)$ can be defined identically (cf. Definition A.17), allowing us to deduce many properties of $W_t$ via systematic use of Proposition 2.10.

The adjunction

$$W_t : dSet \rightleftharpoons sOp : hcN$$

is then defined by

$$W_t X = \operatorname{colim}_{\Omega(U)} X W(U) \quad \text{hcN}(U) = sOp^G(W(U), \mathcal{O})$$

with the equivariant analogue adjunction (1.2) obtained by taking $G$-objects.

For a $G$-tree $T = \sqcup_i T_i = G \cdot H T_*$ in $\Omega_G$ we abbreviate $W(T) = W(\Omega(T))$. Note that, since the $T_i$ have disjoint edge sets, we thus have

$$W(T) = \sqcup_i W(T_i) = G \cdot H W(T_*).$$

The following formalizes some key observations in the proof of [CM13b, Prop. 4.5].

**Lemma 4.45.** For $\eta \neq T \in \Omega^G$ a tree with a $G$-action and $G$-subset $\emptyset \neq E \subseteq E'(T)$, one has pushout diagrams in $sOp^G$

$$\begin{align*}
\Omega(C) \otimes_{\epsilon_*} \partial(\Delta[1] \ast E'(T)) & \rightarrow W_t(\partial \Omega[T]) \\
\Omega(C) \otimes_{\epsilon_*} \Delta[1] \ast E'(T) & \rightarrow W(T) \\
\Omega(C) \otimes_{\epsilon_*} \lambda^E(\Delta[1] \ast E'(T)) & \rightarrow W_t(\Delta^E[T]) \\
\Omega(C) \otimes_{\epsilon_*} \Delta[1] \ast E'(T) & \rightarrow W(T)
\end{align*}$$

(4.46)
where \( C = \text{lr}(T) \), and \( \partial (\Delta[1] \times E^r(T)) \rightarrow \Delta[1] \times E^r(T) \) and \( \lambda^E (\Delta[1] \times E^r(T)) \rightarrow \Delta[1] \times E^r(T) \) are the iterated pushout products

\[
(\partial \Delta[1] \rightarrow \Delta[1])^{E^r(T)}, \quad (\partial \Delta[1] \rightarrow \Delta[1])^{E^r(T)} \rtimes (\{1\} \rightarrow \Delta[1])^{E^r}
\]

with \( G \)-action induced by the action on \( E^r(T) \).

**Proof.** Note first that

\[
\Omega[C] \otimes_{e_r} K \simeq (\mathbb{F} \Sigma_r[C]) \otimes_{e_r} K = \mathbb{F} (\Sigma_r[C] \times K).
\]

Further noting that

\[
(\mathbb{F} (\Sigma_r[C] \times K)) (\mathbb{L} ; r) = (\Sigma_r[C] \times K)(\mathbb{L} ; r) = K,
\]

the top horizontal maps in (4.46) are the unique maps given by the identity at the \( \mathbb{L} ; r \) level, as per the calculations of \( W_\mathbb{L} (\partial \Omega[T]) (\mathbb{L} ; r) \), \( W_\mathbb{L} (\Lambda^E[T]) (\mathbb{L} ; r) \) in Examples A.45, A.47.

Lastly, to see that the squares in (4.46) are pushout squares note that, after taking nerves, it is clear that the left vertical inclusions attach precisely those dendrices missing from the right vertical inclusions. In other words, (4.46) induces pushouts in \( \dSet^\mathbb{L} \) upon applying the nerve functor. The result now follows since the nerve reflects colimits. \( \square \)

**Proposition 4.47** (cf. [CM13b, Prop. 4.9]). \( W_\mathbb{L} \colon \dSet^G \rightleftarrows \sOp^G \colon \text{hcN} \) is a Quillen adjunction.

**Proof.** Note first that, combining the pushouts in Lemmas 4.45 and 4.27 one has that \( W_\mathbb{L} \) preserves cofibrations and sends \( G \)-inner anodyne extensions to trivial cofibrations. By adjunction, the latter claim implies if \( f : \mathcal{O} \rightarrow \mathcal{P} \) is a fibration between fibrant objects then \( \text{hcN}(f) : \text{hcN}\mathcal{O} \rightarrow \text{hcN}\mathcal{P} \) is a \( G \)-inner fibration between \( G \)-\( \infty \)-operads. Hence, to check the needed claim that \( \text{hcN} \) preserves fibrations between fibrant objects (cf. Corollary 4.38), it now suffices to check that the maps \( \tau^* (\text{hcN}_d(f)^H) \simeq \tau^* (\text{hcN}_d(f)^H) \) for \( H \leq G \) are isofibrations of (usual) categories (cf. Theorem 2.34).

But since by definition of fibration in \( \sOp^G \) the maps \( \tau^* \pi_0 f^H \) in (4.10) are isofibrations, the result follows by the identification \( \pi_0 \mathcal{Q} \simeq \tau (\text{hcN} (\mathcal{Q})) \) for fibrant operads \( \mathcal{Q} \in \sOp \), cf. [CM13b, Prop. 4.8]. \( \square \)

**Remark 4.48.** The identification \( \pi_0 \mathcal{Q} \simeq \tau (\text{hcN} (\mathcal{Q})) \) [CM13b, Prop. 4.8] used in the previous proof identifies two procedures of discretizing a simplicial operad \( \mathcal{Q} \in \sOp \) to obtain its homotopy operad \( \pi_0 \mathcal{Q} \in \Op \).

Notably, however, neither Proposition 4.47 nor the original [CM13b, Prop. 4.9] require the full strength of [CM13b, Prop. 4.8], as essential surjectivity depends only on the the category part within operads. Nonetheless, and in light of the fully faithful inclusions in (2.36) it is natural to ask whether [CM13b, Prop. 4.8] generalizes to the context of genuine equivariant operads. The answer to this question is affirmative, and is provided by Proposition B.12 in Appendix B.

We now turn to the proof of our main result, Theorem I.

Recall that, given an object \( X \) in a model category \( \mathcal{M} \), a simplicial frame for \( X \) is a fibrant replacement \( c_i(X) \rightarrow X(\bullet) \) of the constant simplicial object \( c_i(X) \) in the Reedy model structure on \( \mathcal{M}^{\Delta^r} \). Moreover, if \( X \) was already fibrant one is free to assume that \( X(0) = X \).

**Remark 4.49.** The proof of [BPa, Prop. 4.5(ii)] (or, alternatively, adapting [BPa, Prop. 4.24(ii)]) shows that a Reedy fibrant \( X(\bullet) \in \dSet^{\Delta^r} \) is joint fibrant (i.e. its transpose swapping the two simplicial directions is also Reedy) iff the vertex maps \( X(m) \rightarrow X(0) \) are Kan equivalences.
Lemma 4.50. If $X \in (\text{sdSet}^G)^{\Delta^{op}}$ is Reedy fibrant over the dendroidal Reedy model structure on $\text{sdSet}^G$ and the vertex maps $X(m) \to X(0)$ are simplicial equivalences in $\text{sdSet}^G$ then the two maps

$$X(0) \to \delta^* X \leftarrow X_0$$

are also simplicial equivalences in $\text{sdSet}^G$.

Proof. By definition, we need to show that for each $T \in \Omega_G$ the maps

$$X(0)(\Omega[T]) \to \delta^* X(\Omega[T]) \leftarrow X_0(\Omega[T])$$

are Kan equivalences in $\text{sSet}$. Both of these equivalences will follow from [BPn, Prop. 4.5(iv)] provided we show that $X(\Omega[T])$ is a joint fibrant object in $\text{sSet}$. And since the vertex maps $X(\Omega[T])(m) \to X(\Omega[T])(0)$ are Kan equivalences by assumption on $X$, by Remark 4.49 it remains only to check that $X(\Omega[T])$ is Reedy fibrant in $\text{sdSet}^{\Delta^{op}}$. For this last claim, note first that the Reedy fibrancy assumption on $X$ is that the matching maps $X(m) \to M_m X(\bullet)$ are dendroidal fibrations in $\text{sdSet}^G$. Unpacking definitions, this means that for every normal monomorphism $A \to B$ in $\text{dSet}^G$ the maps

$$X(m)(B) \to X(m)(A) \times_{M_m X(\bullet)(A)} M_m X(\bullet)(B)$$

are Kan fibrations in $\text{sSet}$. But now setting $A = B$ to be the map $\emptyset \to \Omega[T]$ we obtain that the maps $X(\Omega[T])(m) \to M_m X(\Omega[T])(\bullet)$ are Kan fibrations, i.e. that $X(\Omega[T])$ is indeed Reedy fibrant in $\text{sdSet}^{\Delta^{op}}$. \qed

Proof of Theorem I. Consider the square of adjunctions on the left below (where we depict only the right adjoints). We already know that all four adjunctions therein are Quillen, and that those adjunctions other than the $(W_1, hcN)$ adjunction are Quillen equivalences. Next, we consider the induced diagram of homotopy categories and derived functors on the right. Crucially, note that while the right Quillen functors $N$ and $hcN$ must be right derived, the left Quillen functors $\gamma^*$ and $c_\circ$ do not, since they preserve all weak equivalences.

$$\begin{array}{ccc}
\text{PreOp}^G & \xleftarrow{N} & \text{sOp}^G \\
\gamma^* \downarrow & & \downarrow hcN \\
\text{sdSet}^G & \xleftarrow{\epsilon_\circ} & \text{dSet}^G \\
\end{array} \quad \begin{array}{ccc}
\text{HoPreOp}^G & \xleftarrow{RN} & \text{Ho} \text{sOp}^G \\
\gamma^* \downarrow & & \downarrow hcN \\
\text{Ho} \text{sdSet}^G & \xleftarrow{\epsilon_\circ} & \text{Ho} \text{dSet}^G \\
\end{array} \quad (4.51)

Recalling that a Quillen adjunction is a Quillen equivalence iff the induced adjunction of homotopy categories is an equivalence adjunction, the desired claim that $(W_1, hcN)$ is a Quillen equivalence will thus follow provided we show that the right square in (4.51) commutes up to natural isomorphism. In other words, we will show that for each fibrant operad $O \in \text{sOp}^G$ there is a natural zigzag of weak equivalences between $\gamma^* NO$ and $c hc NO$.

We now discuss this zigzag. Assume $O \in \text{sOp}^G$ is fibrant. First, choose a (functorial) fibrant simplicial frame $\overline{O}(\bullet) \in (\text{sOp}^G)^{\Delta^{op}}$, where we assume $\overline{O}(0) = O$. Next, let $\gamma^* NO \overline{O}(\bullet) \to \overline{Q}(\bullet)$ be a Reedy fibrant replacement in $(\text{sdSet}^{G})^{\Delta^{op}}$. We note that both $\overline{O}$ and $\overline{Q}$ have two simplicial directions: the frame direction, whose levels are written as $\overline{O}(n), \overline{Q}(n)$ and an internal direction (determined by the simplicial levels in $\text{sOp}^G, \text{sdSet}^G$), whose levels are written $\overline{O}_n, \overline{Q}_n$. Our desired zigzag of weak equivalences in $\text{sdSet}^G$ will have the form below.

$$\gamma^* NO = \gamma^* N \overline{O}(0) \xrightarrow{\sim (a)} \overline{Q}(0) \xrightarrow{\sim (b)} \delta^* \overline{Q} \xleftarrow{\sim (c)} \overline{Q}_0 \xrightarrow{\sim (d)} \big( \gamma^* N \overline{O} \big)_0 \xrightarrow{\sim (e)} hcN \overline{O} \xrightarrow{\sim (f)} c hc NO \quad (4.52)$$
Firstly, the map (a) is a weak equivalence by definition of \(\overline{Q}\).

Next, the maps (b),(c) are weak equivalences (in fact, simplicial equivalences) by Lemma 4.50. Here, we note that while the vertex maps \(\overline{Q}(m) \to \overline{Q}(0)\), which are a priori only joint equivalences, must in fact be simplicial equivalences since the levels \(\overline{Q}(m)\) are joint fibrant.

To see that the map (d) is a weak equivalence, note that one has identifications

\[
\overline{Q}_0(\Omega[T]) \xrightarrow{\sim} \text{sdSet}^G(c_0\Omega[T], \overline{Q}) \xrightarrow{\sim} \text{PreOp}^G(c_0\Omega[T], \gamma_\ast \overline{Q})
\]

(4.53)

in sSet for each \(T \in \Omega_G\). Next, since the counit maps \(\gamma_\ast \gamma_\ast \overline{Q} \to \overline{Q}\) are joint equivalences in \(\text{sdSet}^G\) one has that the maps \(\overline{N}\overline{Q}(m) \to \gamma_\ast \overline{Q}(m)\) are weak equivalences in \(\text{PreOp}^G\). Therefore, and since \(\gamma_\ast\) is right Quillen, both \(\overline{N}\) and \(\gamma_\ast\) are simplicial framings for \(\overline{N}\) in the tame model structure \(\text{PreOp}^G_{\text{tame}}\). Thus, the fact that (d) is a weak equivalence follows since both halves of (4.53) compute the mapping space from \(\Omega[T]\) to \(\overline{N}\) in the tame model structure (this uses the observation that \(\Omega[T]\) is tame cofibrant, cf. Lemma 3.44).

For (e), we consider the identifications

\[
(\gamma_\ast N\overline{Q})_0(\Omega[T]) \to \text{PreOp}^G(c_0\Omega[T], N\overline{Q}) \to \text{sOp}^G(c_0\Omega[T], \overline{Q})
\]

(4.54)

in sSet for each \(T \in \Omega_G\). Thus, since \(W(T) \to \Omega(T)\) is a weak equivalence of cofibrant operads in \(\text{sOp}^G\) and \(\overline{Q}\) is a simplicial frame for \(\Omega\), it follows that (4.54) likewise computes mapping spaces, and thus (e) is indeed a weak equivalence.

Lastly, the claim that (f) is a weak equivalence follows since \(c_0hcN\overline{Q} = hcN\overline{Q}\), the map \(c_0\overline{Q} \to \overline{Q}\) is a levelwise equivalence of levelwise fibrant operads and \(hcN: \text{sOp}^G \to \text{sdSet}^G\) is right Quillen. \(\square\)

**Remark 4.55.** The previous proof is a close variation of the proof of [CM13b, Thm. 8.14], although the equivariant context forces us to use a more formal argument.

More precisely, the given proof of [CM13b, Thm. 8.14] relies on [CM13b, Thm 5.9(v)], which states that a pre-operad \(X \in \text{PreOp}\) is equivalent in \(\text{sdSet}\) to the presheaf \(T \mapsto \text{Map}(\Omega[T], X)\) for \(T \in \Omega\) (where Map(\(-,-\)) denotes the homotopy space of maps). However, in the equivariant context the assignment \(T \mapsto \text{Map}(\Omega[T], X)\) for \(T \in \Omega_G\) does not produce a presheaf in \(\text{sdSet}^G\) (since the levels of such presheaves are indexed by \(U \in \Omega\)) but rather a presheaf in the category \(\text{sdSet}_G\), which is not featured in (4.51).

As such, rather than attempt to formulate and use an analogue of [CM13b, Thm 5.9(v)], our proof replaces the role of that result with an explicit analysis of the simplicial framings needed to define the homotopy mapping spaces featured in [CM13b, Thm 5.9(v)].

**Remark 4.56.** There is a natural way to attempt to simplify the zigzag (4.52) in the a previous proof. Namely, one may attempt to replace the first four maps therein with the simpler two map zigzag

\[
\gamma_\ast N\overline{Q}(0) \to \delta_\ast \gamma_\ast N\overline{Q} \leftarrow (\gamma_\ast N\overline{Q})_0.
\]

(4.57)

As it turns out, it can be shown that (4.57) consists of weak equivalences, but our argument for this is substantially more involved that the argument for (4.52).

Briefly, if \(\overline{X}\) in \((\text{PreOp}^G_{\text{tame}})^{\Delta^m}\) is a simplicial frame for some \(X\) in \(\text{PreOp}^G\), one can find a levelwise simplicial equivalence \(\overline{X} \to \overline{Y}\) with \(\overline{Y}\) a simplicial frame in \((\text{PreOp}^G_{\text{normal}})^{\Delta^m}\). One then
has that $\tilde{X}(0) \to \delta^* \tilde{X} \leftarrow \tilde{X}_0$ consists of weak equivalences iff $\tilde{Y}(0) \to \delta^* \tilde{Y} \leftarrow \tilde{Y}_0$ does, and the latter can be shown by following the (rather involved) proof of [BPa, Prop. 5.41] with $\tilde{Y}$ taking the role of $X'$ therein.

As a side note, [BPa, Prop. 5.41] is one of the keys to our proof of [BPa, Thm. 5.48], which establishes the DK description of weak equivalences between fibrant objects in $\text{PreOp}^G, \text{sdSet}^G$, so our proof of Theorem 1 does still indirectly rely on [BPa, Prop. 5.41].

5 Nerves of free extensions are homotopy pushouts

This section will be dedicated to proving the following key lemma, which is the equivariant analogue of [CM13b, Prop. 3.2].

**Lemma 5.1.** Suppose that $\mathcal{O} \in \text{Op}^G$ is $\Sigma$-cofibrant. Further, let $C \in \Sigma_G$ be any $G$-corolla and consider a pushout in $\text{Op}^G$ of the form

$$
\begin{array}{ccc}
\partial \Omega(C) & \longrightarrow & \mathcal{O} \\
\downarrow & & \downarrow \\
\Omega(C) & \longrightarrow & \mathcal{P}.
\end{array}
$$

Then the induced map

$$\Omega(C) \cup_{\partial \Omega(C)} N\mathcal{O} \to N\mathcal{P}$$

is $G$-inner anodyne.

5.1 The characteristic edge lemma

**Notation 5.3.** Let $Y \in \text{dSet}^G$ be a $G$-equivariant dendroidal set and $y: \Omega[U^y] \to Y$ a dendrex, $U^y \in \Omega$.

We write $(y) = y(\Omega[U^y])$ and refer to $(y) \subseteq Y$ as the principal subpresheaf generated by $y$.

Moreover, if some (and thus any) non-degenerate representative $y$ is free with respect to the $\text{Aut}(U^y)$-action (via precomposition), we say $y$ and $(y)$ are $\Sigma$-free. If all dendrices $y$ are $\Sigma$-free, we say $Y$ itself is $\Sigma$-free.

Given a map of trees $V \to U^y$ we write $\partial_V y$ for the composite to $\Omega[V] \to \Omega[U^y] \xrightarrow{y} Y$.

**Remark 5.4.** Note that $(y) = (\bar{y})$ iff $y, \bar{y}$ are both degeneracies of a common non-degenerate dendrex. In particular, if the chosen representatives $y, \bar{y}$ are both nondegenerate, there must exist an isomorphism $\varphi: U^y \xrightarrow{\cong} U^\bar{y}$ (which is unique if $(y)$ is $\Sigma$-free) such that $y = \bar{y} \circ \varphi$.

**Notation 5.5.** Given a $\Sigma$-free $(y)$, a coherent inner edge set $E^{(y)}$ for $(y)$ is a collection of subsets $E^y \subseteq E(U^y)$ for each non-degenerate representative $y$ of $(y)$, and such that $E^y = \varphi(E^\bar{y})$ for the unique $\varphi$ with $y = \bar{y} \circ \varphi$. Note that $E^{(y)} = \{E^y\}$ is entirely determined by any of the $E^y$.

**Remark 5.6.** Recalling that $G$ acts on dendrices by postcomposition, i.e. $gy$ is the composite $\Omega[U^y] \xrightarrow{y} Y \xrightarrow{gy} Y$ we see that $U^{gy} = U^y$. Moreover, the action extends to principal subpresheaves and $g(y) = (gy)$.

As such, if $(y)$ is $\Sigma$-free, a coherent inner edge set $E^{(y)} = \{E^y \subseteq E(U^y)\}$ for $(y)$ gives rise to a coherent inner edge set $gE^{(y)} = \{E^y \subseteq E(U^{gy})\}$ for $gy$ with the same edge sets $E^y$.  

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The following essentially replicates [BPa, Def. 3.1] as generalized in [BPa, Rem. 3.7], except with dendrices $y : \Omega[[U^y]] \to Y$ mostly replaced with the principal presheaves $\langle y \rangle \subseteq Y$. The reformulation of (Ch0.2) and the descending chain condition are discussed in Remarks 5.8, 5.9.

**Definition 5.7.** Let $f : X \to Y$ be a monomorphism in $dSet^G$ and $\{ \langle y \rangle \}$ a set of $\Sigma$-free principal subpresheaves of $Y$. Suppose further that $\{ \langle y \rangle \}$ is equipped with a poset structure compatible with the natural $G$-action and which satisfies the descending chain condition. For each $\langle y \rangle$ denote

$$X_{\langle y \rangle} = X \cup \bigcup_{\langle y \rangle \subset \langle (y) \rangle} \langle y \rangle$$

Given a coherent inner set $\Xi^{(y)} = \{ \Xi^y \subseteq E(U^y) \}$, non-degenerate representative $y : \Omega[[U^y]] \to Y$, and a subface $V \to U^y$, we write $\Xi^y_V = \Xi^y \cap E(V)$.

We say $\{ \Xi^{(y)} \}$ is a characteristic inner edge collection of $\{ \langle y \rangle \}$ with respect to $X$ if for some (and thus any) choice of non-degenerate representatives $y : \Omega[[U^y]] \to Y$ one has that:

(Ch0.1) $y : \Omega[[U^y]] \to Y$ is injective away from $y^{-1}(X_{\langle (y) \rangle})$;

(Ch0.2) $\{ \langle y \rangle \}$ and $\{ \Xi^{(y)} \}$ are $G$-equivariant, in the sense that $g(\langle y \rangle) \in \{ \langle y \rangle \}$ and $g\Xi^{(y)} = \Xi^{g(y)}$, i.e. $\Xi^y = \Xi^{g(y)}$;

(Ch1) if $V \to U^y$ is an outer face and $\Xi^y_V = \emptyset$, then $\langle \partial_V y \rangle \subseteq X_{\langle (y) \rangle}$;

(Ch2) if $V \to U^y$ is any face and $(\partial_{V \times \Xi^y} y) \subseteq X$, then $\langle \partial_V y \rangle \subseteq X_{\langle (y) \rangle}$;

(Ch3) if $\langle \hat{y} \rangle \not\subseteq \langle y \rangle$, $V \to U^y$, and $(\partial_{V \times \Xi^y} y) \subseteq \langle \hat{y} \rangle$, then $\langle \partial_V y \rangle \subseteq X_{\langle (y) \rangle}$.

**Remark 5.8.** In [BPa, Rem. 3.7] the role of each presheaf $\langle y \rangle$ is played by a special chosen representative, which we here denote by $y^{pl} \in \langle y \rangle$. The motivation for this is that in some key examples, such as in [BPa, Ex. 3.9], one can choose preferred “planar representatives”, allowing for a pictorial depiction of the dendrices and poset as in [BPa, Fig. 3.1].

There is then a bijection $\{ \langle y \rangle \} = \{ y^{pl} \}$ between principal presheaves and the set of representatives, but while the former has a $G$-action the latter a priori does not (as $gy^{pl}$ may not be planar). Translating the $G$-action along this bijection one has that the action of $g$ on $y^{pl}$ is $(gy^{pl})^{pl}$ and (ii),(iii) in (Ch0.2) of [BPa, Rem. 3.7] precisely encode this action on planar representatives.

**Remark 5.9.** Recall that a poset satisfies the descending chain condition if there are no infinite descending chains or, equivalently, if any non-empty subset has a minimal element. As such, while the proof of [BPa, Lemma 3.4] assumed the poset $\{ \langle y \rangle \}$ was finite, since that proof follows by iteratively adding elements to $G$-equivariant convex subsets of the poset (cf. the last paragraph of the proof in loc. cit.), the argument generalizes to any poset satisfying the descending chain condition.

**Lemma 5.10** (cf. [BPa, Lemma 3.4]). If $\{ \Xi^{(y)} \}$ is a characteristic inner edge collection of $\{ \langle y \rangle \}$ with respect to $X$ then

$$X \to X \cup \bigcup_{\langle y \rangle} \langle y \rangle$$

is $G$-inner anodyne.
5.2 Proof of the key lemma

This section is dedicated to proving Lemma 5.1 as an application of the characteristic edge lemma, Lemma 5.10. This will require a fair amount of preparation, starting with a description of the pushout operad \( \mathcal{P} \) in (5.2).

First, we let \( \mathcal{C} = \mathcal{C}_G \) and write \( \mathcal{C} : E(C) \to \mathcal{C} \) for the induced map of colors. Then, denoting \( \mathcal{C} = (C, \mathcal{C}) \) for the associated colored \( G \)-forest, one has identifications

\[
\iota(C) = \iota(F\Sigma_r[C]) \sim F(\iota\Sigma_r[C]) \sim F(\Sigma_{\mathcal{C}})
\]

which allow us to rewrite the pushout (5.2) as the alternative pushout in \( \mathcal{O}_\mathcal{C} \)

\[
\begin{array}{c}
\mathcal{F}(\mathcal{C}) \\
\downarrow \\
\mathcal{F}(\Sigma_{\mathcal{C}})
\end{array}\]

By [BPC, Lemma 3.44] with \( w : X \to Y \) the map \( \mathcal{C} \to \Sigma_{\mathcal{C}} \) (and as further detailed in [BPC, Remark A.51]) one then has, for each \( \mathcal{C} \)-corolla \( \mathcal{D} \in \Sigma_{\mathcal{C}} \), the formula

\[
\mathcal{P}(\mathcal{D}) \simeq \bigoplus_{[\mathcal{T}]} \prod_{\mathcal{T} \in \text{iso}(\mathcal{D} \Omega_{\mathcal{C}})} \left( \prod_{v \in \mathcal{V}_{\mathcal{C}}(\mathcal{T})} \mathcal{O}(\mathcal{T}_v) \times \prod_{v \in \mathcal{V}^{\text{in}}(\mathcal{T})} \Sigma_{\mathcal{C}}(\mathcal{T}_v) \right) \cdot \text{Aut}_{\mathcal{V}}(\mathcal{T}) \cdot \text{Aut}_{\Sigma_{\mathcal{C}}}(\mathcal{D})
\]  \hspace{1cm} (5.12)

(where we note that since \( X = \mathcal{C} \) it is likewise always \( Q^m[u] \) in [BPC, Lemma 3.44]).

The decomposition (5.12) will be the key to verifying the characteristic edge conditions in Definition 5.7. To do so, we will first find it useful to discuss a number of special types of dendrices and principal subpresheaves of \( N\mathcal{P} \), suggested by (5.12). Recall that, by the strict Segal condition characterization of nerves [CM13a, Cor. 2.7], a dendrex \( p : \Omega(U) \to N\mathcal{P} \) is uniquely specified by the tree \( U \in \Omega \) together with a choice of operations \( \{ p_v \in \mathcal{P}(U_v) \}_{v \in \mathcal{V}(U)} \). Moreover, we will throughout use make use of the decomposition

\[
(\Omega[C] \cup_{\partial \Omega[C]} N\mathcal{O})_\eta(U) \sim \Sigma_{\mathcal{C}}(\mathcal{D}) \cup \mathcal{O}(\mathcal{D}).
\]

Definition 5.13. A dendrex \( p : \Omega(U) \to N\mathcal{P} \) is called:

- **elementary** if for each vertex \( U_v \to U \) it is \( \partial U_v \cdot p \in \mathcal{O} \cup \Sigma_{\mathcal{C}} \); 
- **alternating** if \( U \in \Omega^a \) is an alternating tree and for each active (resp. inert) vertex \( U_v \to U \) it is \( \partial U_v \cdot p \in \mathcal{O} \) (resp. \( \partial U_v \cdot p \in \Sigma_{\mathcal{C}} \));
- **canonical** if it is non-degenerate and has a degeneracy which is alternating.

Definition 5.14. Let \( \langle p \rangle \subseteq N\mathcal{P} \) be a principal subpresheaf. We say \( \langle p \rangle \) is:

- **unital** if there is a representative \( p : \Omega(U) \to N\mathcal{P} \) with \( U = \eta \) the stick tree;
- **reduced** if there is a representative \( p : \Omega(U) \to N\mathcal{P} \) with \( U \in \Sigma \), i.e. with \( U \) a corolla;
- **elementary** if there is an elementary representative \( p : \Omega(U) \to N\mathcal{P} \);
• canonical if there is a canonical (equivalently, alternating) representative \( p: \Omega[U] \to \mathcal{NP} \).

**Remark 5.15.** A dendrex is elementary iff its degeneracies are elementary, so the definition of elementary subpresheaf does not depend on the choice of representative.

**Notation 5.16.** Recalling that any tree \( U \) has an associated corolla \( \text{lr}(U) \), we abbreviate \( \partial_v = \partial_{\text{lr}(U)v} \) and call \( \langle \partial_v, p \rangle \) the reduction of \( \langle p \rangle \).

**Remark 5.17.** Equation (5.12) implies that for each reduced principal subpresheaf \( \langle r \rangle \subseteq \mathcal{NP} \) there exists an alternating dendrex \( a \) of \( \mathcal{NP} \), unique up to isomorphism, such that \( \langle \partial_a, a \rangle = \langle r \rangle \). Moreover, \( \langle a \rangle \) is thus the only canonical subpresheaf whose reduction is \( \langle r \rangle \), and we write \( \langle r \rangle_X = \langle a \rangle \) to denote this.

Lastly, note that one thus has that \( \langle p \rangle \) is canonical iff \( \langle p \rangle = \langle \partial, p \rangle_X \).

**Remark 5.18.** A reduced subpresheaf \( \langle r \rangle \) is unital iff \( \langle r \rangle_X \) is unital, in which case \( \langle r \rangle = \langle r \rangle_X \).

**Remark 5.19.** If \( \varepsilon: \Omega[U^c] \to \mathcal{NP} \) is an elementary dendrex, the tree \( U^c \) can be naturally regarded as an \( (\mathcal{O}, \mathcal{C}) \)-labeled tree by labeling each vertex \( U^c_e \to U^c \) according to whether it is \( \partial_{U^c.e} \in \mathcal{O} \) or \( \partial_{\mathcal{C}}.e \in \mathcal{C} \).

By the alternating tree analogue of [BPb, Prop. 5.49], there is hence an unique alternating tree \( U^a \) together with a tall planar \( \mathcal{C} \)-inert label map \( U^a \to U^c \), and then it follows that \( \partial_{U^a.e} \) is an alternating dendrex so that \( \langle \partial_{U^a.e} \rangle = \langle \partial_e \rangle_X \). In particular, this shows that \( \langle \partial_e \rangle_X \subseteq \langle e \rangle \).

**Definition 5.20.** Let \( \varepsilon: \Omega[U^c] \to \mathcal{NP} \) be a non-degenerate elementary dendrex. We write \( \Xi^c \subseteq E'(U) \) for the subset of inner edges of \( U^c \) which are adjacent to at least one \( \mathcal{C} \)-labeled vertex.

**Remark 5.21.** Let \( \varepsilon: \Omega[U^c] \to \mathcal{NP} \) be a non-degenerate elementary dendrex, \( U^a \to U^c \) be as in Remark 5.19, and write \( a = \partial_{U^c.e} \). Since \( U^a \) is alternating, all of its inner edges are adjacent to a \( \mathcal{C} \)-labeled vertex. Therefore, the fact that \( U^a \to U^c \) is a tall \( \mathcal{C} \)-inert label map implies that \( \Xi^c \) consists of those inner edges which are in the image of \( U^a \).

**Proposition 5.22.** Let \( \varepsilon: \Omega[U^c] \to \mathcal{NP} \) be a non-degenerate elementary dendrex. Then \( \langle e \rangle \) is canonical iff \( \Xi^c = E'(U) \).

**Proof.** We use the notation in Remark 5.21. Since \( \langle a \rangle = \langle \partial_e \rangle_X \), \( \langle e \rangle \) is canonical iff \( \langle a \rangle = \langle e \rangle \), i.e. iff \( U^a \to U^c \) is a degeneracy. But since a map of trees is a degeneracy if it is tall and surjective, it follows that \( U^a \to U^c \) is a degeneracy iff its image includes all inner edges, i.e. iff \( \Xi^c = E'(U) \).

**Remark 5.23.** If \( U \models \eta \) is not the stick tree, then \( \text{lr}(U) \models U - E'(U) \) is the inner face removing all inner edges.

**Lemma 5.24.** For any principal subpresheaf \( \langle p \rangle \subseteq \mathcal{NP} \), there exists an elementary subpresheaf \( \langle e \rangle \subseteq \mathcal{NP} \), non-degenerate representative \( e: \Omega[U^c] \to \mathcal{NP} \), and a subset \( E \subseteq \Xi^c \) such that \( \partial_{U^c.e} \) is non-degenerate and \( \langle p \rangle = \langle \partial_{U^c.e} \rangle_X \). In particular, \( \langle p \rangle \subseteq \langle e \rangle \).

**Proof.** Let \( p: \Omega[U] \to \mathcal{NP} \) be a non-degenerate subpresheaf. We first build \( e \).

For each vertex \( U_v \to U \), write \( p_v = \partial_{U_v} p \) and, noting that \( \langle p_v \rangle \) is reduced, we further write \( e_v: \Omega[U^c_v] \to \mathcal{NP} \) for some canonical representative of \( \langle p_v \rangle_X \) (cf. Remark 5.17).

The identity \( \langle p_v \rangle = \langle \partial_{U_v} e_v \rangle \) implies that \( U_v \models \text{lr}(U^c_v) \), so that by choosing tall maps \( U_v \to U^c_v \) one obtains a \( U \)-substitution datum [BPb, Def. 3.38] which by [BPb, Prop. 3.41] can be assembled into a tree \( U^c \) together with a tall map \( U \to U^c \) such that for every vertex \( U_v \) the “tall map followed by outer face” factorization of the composite \( U_v \to U \to U^c \) is given by \( U_v \to U^c_v \to U^c \).
Since each vertex of $U^c$ is in exactly one of the outer trees $U^c_v$, we define $e: \Omega[U^c] \to N\mathcal{P}$ as the unique dendrex such that $\partial_{U^c} e = e_v$. Note that since the $e_v$ are non-degenerate then so is $e$.

Since $\partial_{U^c} e = p$ was chosen to be non-degenerate, it remains to show that $U \to U^c$ identifies $U = U^c - E$ for some $E \subseteq \Xi^c$. Since $\langle p_v \rangle$, $\langle p_v \rangle$, are non-unital (by assumption on $p$ and Remark 5.18), the $U^c_v$ are not stick trees, and Remark 5.23 implies $U = U^c - E$ for $E = u_c v(U) E(U^c)$. That $E \subseteq \Xi^c$, i.e. that any edge in $E$ is adjacent to a $\mathcal{C}$-labeled vertex, follows from Proposition 5.22.

**Lemma 5.25.** Suppose $c: \Omega[U^c] \to N\mathcal{P}$ is elementary and $(\partial_r c)$ is not unital. Then there exists an inner face map $U^c \to U^c$ such that $\partial_{U^c} c$ is canonical.

**Proof.** Let $U^a \to U^c$ be as in Remark 5.19.

Writing $a = \partial_{U^a} c$, $r = \partial_r c$, and letting $c: \Omega[U^c] \to N\mathcal{P}$ be a canonical representative of $\langle a \rangle = \langle r \rangle$, one has that $a$ is a degeneracy of $c$, i.e. there is a degeneracy map $U^a \to U^c$ such that $\partial_{U^a} c = a$. Since $\langle r \rangle = \langle \partial_r c \rangle$ is not unital, neither is $\langle r \rangle$ (cf. Remark 5.18), so that $U^c$ can not be the stick tree $\eta$, and thus $U^a \to U^c$ has a section which is an inner face (this follows from [Per18, Cor. 5.38] since no edge of $U^c$ is both a root and a leaf). But then the composite $U^c \to U^a \to U$ must be a face (or else $c = \partial_{U^c} c$ would be degenerate) and is tall, and is hence an inner face.

**Lemma 5.26.** Let $c: \Omega[U^c] \to N\mathcal{P}$ be a non-unital canonical dendrex, $e: \Omega[U^c] \to N\mathcal{P}$ an elementary dendrex, and $r(U^c) \to U^c$ a tall map.

Then, if the solid diagram below commutes, there exists a tall dashed map making the diagram commute.

\[
\begin{array}{ccc}
\Omega[\text{lr}(U^c)] & \rightarrow & \Omega[U^c] \\
\downarrow & & \downarrow e \\
\Omega[U^c] & \rightarrow & N\mathcal{P}
\end{array}
\] (5.27)

**Remark 5.28.** The requirement that $\langle e \rangle$ is non-unital is essential, as there may exist non-unital $\langle e \rangle \in \mathcal{N}^c$ such that $\langle \partial_r e \rangle = \langle \partial_r c \rangle$ is unital, in which case no dashed arrow as in (5.27) can exist.

**Proof.** Since commutativity of (5.27) implies $\langle \partial_r c \rangle = \langle \partial_r c \rangle$, which is not unital by assumption, by Lemma 5.25 there is an inner face $U^c \to U^c$ such that $e = \partial_{U^c} c$ is canonical.

By definition of canonical dendrex, there are degeneracies $U^a \to U^c$, $U^a \to U^c$ with $U^a, U^a$ alternating trees and such that the composites $\Omega[U^a] \to \Omega[U^c] \to N\mathcal{P}$, $\Omega[U^a] \to \Omega[U^c] \to N\mathcal{P}$ are alternating dendrices. And since $\text{lr}$ sends tall maps to isomorphisms, we can form the diagram

\[
\begin{array}{ccc}
\Omega[\text{lr}(U^c)] & \xrightarrow{\gamma} & \Omega[\text{lr}(U^a)] \\
\downarrow & & \downarrow z \\
\Omega[U^c] & \xrightarrow{\zeta} & \Omega[U^a] \\
\downarrow & & \downarrow z \\
\Omega[U^c] & \xrightarrow{e} & N\mathcal{P}
\end{array}
\]

We will argue that all dashed vertical isomorphisms exist. That the first vertical isomorphism exists is trivial. The existence of the second vertical isomorphism follows from (5.12) which implies that, for $a, \bar{a}$ alternating dendrices, all isomorphisms $\partial_r a \simeq \partial_r \bar{a}$ are induced from an isomorphism $a \simeq \bar{a}$. Lastly, the existence of the third isomorphism follows from the fact that the factorization of degenerate dendrices through non-degenerate dendrices is unique up to (unique) isomorphism [Per18, Prop. 5.62].

**Lemma 5.29.** Let $e: \Omega[U^c] \to N\mathcal{P}$ be a non-degenerate elementary dendrex, $e: \Omega[U^c] \to N\mathcal{P}$ an elementary dendrex, and $U^c - E \to U^c$ a map where $E \subseteq \Xi^c$. 39
Then, if the solid diagram below commutes, there exists a dashed map making the diagram commute.

\[
\begin{array}{ccc}
\Omega[U^c - E] & \rightarrow & \Omega[U^c] \\
\downarrow & & \downarrow \\
\Omega[U^c] & \rightarrow & N\mathcal{P}
\end{array}
\] (5.30)

**Proof.** We abbreviate \( U' = U^c - E \). Note first that, by applying the “tall map followed by outer face” factorization to \( U' \rightarrow U^c \) to obtain \( U' \rightarrow \tilde{U} \rightarrow U^c \), the dendrex \( \partial \tilde{U} \tilde{e} \) is still elementary (being an outer face of an elementary dendrex), so we reduce to the case where \( U' \rightarrow U^c \) is a tall map.

For each vertex \( U'_v \rightarrow U' \) we apply the “inner face followed by outer face factorization” to the composites \( U'_v \rightarrow U' \rightarrow U^c \), \( U'_v \rightarrow U^c \rightarrow U^c \) to get \( U'_v \rightarrow U^c \rightarrow U^c \), \( U'_v \rightarrow U^c \rightarrow U^c \) and, further writing \( e_v = \partial U'^c_v, \tilde{e}_v = \partial \tilde{U} \tilde{e}_v \), we obtain solid diagrams

\[
\begin{array}{ccc}
\Omega[U'_v] & \rightarrow & \Omega[U^c] \\
\downarrow & & \downarrow \\
\Omega[U^c] & \rightarrow & N\mathcal{P}
\end{array}
\] (5.31)

We now claim that \( e_v \) is canonical. Indeed, \( e_v \) is non-degenerate elementary since it is an outer face of \( e \), which is also non-degenerate elementary. And since all inner edges of \( U'^c_v \) are in \( E \subset \Xi^c \), they are all adjacent to \( \tilde{C} \)-labeled vertices, so \( e_v \) is indeed canonical by Proposition 5.22.

Since \( \{e_v\} \) is non-unital (or \( U'_v \rightarrow U^c \) would be a degeneracy), by Lemma 5.26 there is a tall dashed arrow in (5.31) for each \( v \in V(U') \), i.e. a \( U^c \)-substitution datum. Thus by [BPb, Prop. 3.41] we obtain the desired dashed arrow in (5.30).

**Corollary 5.32.** If \( e: \Omega[U^c] \rightarrow N\mathcal{P} \) is a non-degenerate elementary dendrex and \( E \subset \Xi^c \), then \( \{e\} \) is the smallest elementary subpresheaf containing \( \langle \partial U^c - E \rangle \), i.e. if \( \langle \partial U^c - E \rangle \subset \{\tilde{e}\} \) is elementary then \( \{e\} \subset \{\tilde{e}\} \).

**Proof.** \( \langle \partial U^c - E \rangle \subset \{\tilde{e}\} \) yields the diagram (5.30) and the dashed arrow therein shows \( \{e\} \subset \{\tilde{e}\} \).

**Corollary 5.33.** An elementary subpresheaf \( \{e\} \) is canonical iff it is the smallest elementary subpresheaf containing \( \langle \partial e \rangle \).

**Proof.** As noted in Remark 5.17, \( \{e\} \) is canonical iff \( \{e\} = \langle \partial e \rangle \). If \( \langle \partial e \rangle \) is unital, then \( \langle \partial e \rangle = \langle \partial e \rangle \chi \), which is elementary (cf. Remark 5.18), so the claim is clear. Otherwise, letting \( e: \Omega[U^c] \rightarrow N\mathcal{P} \) be a canonical representative of \( \langle \partial e \rangle \chi \), one has \( \Xi^c = E'(U^c) \) and \( \langle \partial e \rangle = \langle \partial U^c - E \rangle \) so, by Corollary 5.32, \( \langle \partial e \rangle \chi \) is the smallest elementary containing \( \langle \partial e \rangle \).

**Corollary 5.34.** Suppose \( N\mathcal{P} \) is \( \Sigma \)-free and let \( e: \Omega[U^c] \rightarrow N\mathcal{P} \) be an elementary non-degenerate dendrex and \( E, E' \subset \Xi^c \). Then if \( \langle \partial U^c - E \rangle = \langle \partial U^c - E' \rangle \) it must be \( E = E' \).

**Proof.** If \( \langle \partial U^c - E \rangle = \langle \partial U^c - E' \rangle \) then one can find a solid diagram as below

\[
\begin{array}{ccc}
\Omega[U^c - E] & \rightarrow & \Omega[U^c] \\
\downarrow & & \downarrow \\
\Omega[U^c - E'] & \rightarrow & \Omega[U^c]
\end{array}
\]

and thus by Lemma 5.29 one can also find the vertical dashed arrow. But since \( N\mathcal{P} \) is \( \Sigma \)-free by assumption the dashed arrow must be the identity, so that \( U^c - E = U^c - E' \) and \( E = E' \).
We follow Notation 2.5, and label a map in $\Omega$ by either of the letters d/i/o/t/f/p to indicate that we write $E_{\phi}$ given as in Definition 5.20. Note that by Lemma 5.24 every dendrex of $NP$ is in some $(e)$, so that the map (5.11) is indeed $NO_{|_{\partial\Omega[C]}}\Omega[C] \to NP$.

Let $e: \Omega[U^e] \to NP$ be a non-degenerate elementary dendrex. We note the following:

(a) if $\Xi^e = \emptyset$ then either all vertices of $U^e$ are $\mathcal{O}$-labeled, i.e. $e \in NO$, or $U^e$ is a $\mathcal{C}$-labeled corolla, i.e. $e \in \Sigma_e[\mathcal{C}]$. In other words, $\Xi^e = \emptyset$ iff $(e) \subseteq X = NO_{|_{\partial\Omega[C]}}\Omega[C]$.

(b) any outer face of $e$ is again elementary, as is any inner face $\partial_{UV-E}f$ such that $f \not\in \Xi^e$ (since then both vertices adjacent to $f$ are $\mathcal{O}$-labeled). Therefore, by Corollary 5.32, we see that a face of $e$ is not in some elementary $(e) \not\subseteq (e)$ iff it is of the form $\partial_{UV-E}f$ for some $E \subseteq \Xi^e$.

We now check the characteristic edge conditions. (Ch0.2) is clear.

For (Ch1), by (b) any proper outer face of $e$ is in $X_{e(e)}$, so we need only consider the case of $V = U^e$ with $\Xi^e = \emptyset$, in which case $(e) \subseteq X \subseteq X_{e(e)}$ by (a).

For (Ch2),(Ch3), by (a) and the first half of (b) one needs only consider the case of $V \subseteq U^e - E$ where $E \subseteq \Xi^e$ and $\Xi^e \neq \emptyset$. But then $V - \Xi^e \subseteq U^e - \Xi^e$ and by the second half of (b) one has $(\partial_{UV-E}f) \subseteq X_{e(e)}$ iff $(e) \subseteq X_{e(e)}$ iff $(\partial_{UV-E}f) \subseteq X_{e(e)}$, so (Ch2),(Ch3) follow.

Lastly, we address (Ch0.1). By (a) we need only consider the case of $\Xi^e \neq \emptyset$, so that by (b) the complement of the preimage $e^{-1}(X_{e(e)})$ consists of the faces isomorphic to $U^e - E$ for $E \subseteq \Xi^e$.

Injectivity of $e$ within each isomorphism class of the faces away from $e^{-1}(X_{e(e)})$ follows from $NP$ being $\Sigma$-free, while injectivity across distinct isomorphism classes of faces is Corollary 5.34. \qed

**Remark 5.35.** Condition (Ch0.1) is, by some margin, the subtlest condition in the previous proof, and the main reason for the chosen formulations of Lemmas 5.26, 5.29. In particular, we note that injectivity of $e_{\Omega[U^e]} \to NP$ will in general fail away from $e^{-1}(X_{e(e)})$. For example, two edges/vertices of $U^e$ may be assigned the same color/operation, and similarly for larger outer faces. In fact, injectivity may even fail on inner faces $T^e - E$ where $E \not\subseteq \Xi^e$.

### A The dendroidal $W$-construction

In this appendix we discuss the $W$-construction in the operadic context by extending the work of Dugger and Spivak in [DS11] (which deals with the categorical context).

Our discussion will make systematic use of the factorizations in $\Omega$ given by Proposition 2.10. We follow Notation 2.5, and label a map in $\Omega$ by either of the letters d/i/o/t/f/p to indicate that the map is a degeneracy/inner face/outer face/tall face/planar. Moreover, given a map $\phi: S \to T$, we write

$$S \xrightarrow{d} \phi \xrightarrow{\phi} \bar{S} \xrightarrow{\phi^p} T$$

for the (strictly) unique factorization of $\phi$ with the indicated properties (cf. Remark 2.12).

We now define the notion of dendroidal necklace, generalizing the key notion in [DS11].

**Definition A.1** (cf. [DS11, §3]). A necklace is a planar inner face map $w: J \to T$ in $\Omega$. Moreover:

(i) $J$ is called the inner face of joints of the necklace;

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(ii) for each vertex \( v \in V(J) \), the outer face \( n_J v = T_v \rightharpoonup T \) is called a bead of the necklace.

**Example A.2.** Consider the trees \( J \) and \( T \) below. The labeling of the edges indicates a map \( E(J) \to E(U) \) which encodes an inner face map.

Here, \( J = T - \{b, f\} \), and the two beads are

In the following, we write \( \text{Face}_{sc}(J) \) for the Segal core poset of \( J \), consisting of those planar outer faces with no inner edges (which consist of either a single edge or a single vertex of \( J \)).

**Definition A.3.** Given a necklace \( n: J \to T \) we define its representable presheaf \( \Omega[n] \in dSet \) by

\[
\Omega[n] = \text{colim}_{U \in \text{Face}_{sc}(J)} \Omega[nU] = \bigcup_{U \in \text{Face}_{sc}(J)} \Omega[nU]
\]

where the union formula is taken inside \( \Omega[T] \).

The category \( \text{Nec} \) of necklaces is then the full subcategory of \( dSet \) spanned by the \( \Omega[n] \).

**Remark A.4.** The \( \Omega[n] \) presheaves interpolate between the usual Segal core and representable presheaves. More explicitly, each tree \( T \in \Omega \) gives rise to necklaces \( n\leftarrow T \) and \( lr(T) \to T \) for which

\[
\Omega(T \leftarrow T) = \text{Sc}[T], \quad \Omega[lr(T)] = \Omega[T].
\]

In particular, one obtains a natural inclusion \( \Omega \subset \text{Nec} \) given by \( T \mapsto (lr(T) \to T) \). However, we caution that the assignment \( T \mapsto \text{Sc}[T] \) is not functorial on \( T \) (more precisely, it is functorial only with respect to convex maps of trees, in the sense of Remark 2.9).

**Remark A.5.** In the context of linear trees, a necklace is an injective map \( n: [n] \to [m] \), with beads \([m_1], 1 \leq n \) such that \( m_1 + \cdots + m_k = m \). One then has an identification \( \Omega[n] = n(\Delta[m_1] \lor \cdots \lor \Delta[m_k]) \) where each \( \lor \) symbol indicates that the last edge of \( \Delta[m_i] \) is identified with the first edge of \( \Delta[m_{i+1}] \), thus recovering the original definition of necklace due to Dugger-Spivak [DS11, §1].

**Lemma A.6.** Let \( n: J \to T \) be a necklace. Then

(i) a face \( U \rightharpoonup T \) is in \( \Omega[n] \) iff its outer closure \( \bar{U} \) is;

(ii) an outer face \( U = \bar{U} \rightharpoonup T \) is in \( \Omega[n] \) iff \( E'(J) \cap E'(U) = \emptyset \);

(iii) there is a decomposition \( E(T) = E(J) \cup \bigsqcup_{v \in V(J)} E(T_v) \).
Proof. (i) follows since $\Omega[n]$ is an union of outer faces.

The arguments for (ii),(iii) are by induction on the number of inner edges $E_i(J)$, with the base case of $E_i(J) = \emptyset$ being obvious. Otherwise, letting $e \in E_i(J)$, since $e$ is an inner edge of both $J$ and $T$ one has grafting decompositions $J = J' \cup_e J''$, $T = T' \cup_e T''$ together with inner face maps $n': J' \to T'$, $n'': J'' \to T''$. One then has that $U$ is in $\Omega[n]$ iff it is in either $\Omega[n']$ or in $\Omega[n'']$, yielding the induction step for (ii). The induction step for (iii) likewise follows. \hfill \Box

Remark A.7. If $S \xrightarrow{d} S'$ is a degeneracy, the vertices of $S'$ are naturally identified with the vertices of $S$ that are not collapsed to edges. Thus, by factoring a tall map $\varphi: S \xrightarrow{t} T$ as $S \xrightarrow{d} \varphi S \xrightarrow{t} T$ the decomposition (iii) in Lemma A.6 generalizes to

$$E(T) = E(\varphi S) \cup \bigcup_{v \in V(S)} E(\varphi S_v). \quad \text{(A.8)}$$

Notation A.9. Given a necklace $n: J \to T$ and outer face $F \to T$ we write $n_F: J_F \to F$ for the necklace characterized by

$$E_i(J_F) = E_i(J) \cap E_i(\overline{U}).$$

Example A.10. Let $n: J \to U$ be the necklace in Example A.2, and consider the outer faces $F$ and $F'$ of $T$ depicted below. Then $J_F$ is as depicted and $J_{F'} = F'$.

\begin{center}
\begin{tikzpicture}
\node at (0,0) [circle,draw] (a) {c};
\node at (1.2,0) [circle,draw] (b) {d};
\node at (2.4,0) [circle,draw] (c) {e};
\node at (2.4,-2) [circle,draw] (d) {f};
\node at (0,-2) [circle,draw] (e) {r};
\node at (2.4,0) [circle,draw] (f) {J_F};
\node at (5,0) [circle,draw] (g) {a};
\node at (6.2,0) [circle,draw] (h) {b};
\node at (7.4,0) [circle,draw] (i) {c};
\node at (8.6,0) [circle,draw] (j) {d};
\node at (8.6,-2) [circle,draw] (k) {f};
\node at (7.4,-2) [circle,draw] (l) {J_{F'} = F'};
\draw (a) -- (b) -- (c) -- (d) -- cycle;
\draw (e) -- (b);
\draw (e) -- (f);
\draw (g) -- (h) -- (i) -- (j) -- cycle;
\draw (g) -- (j);
\end{tikzpicture}
\end{center}

In particular, note that one need not have a map $J_F \to J$ since $E(J)$ may not contain $E(J_F)$.

Corollary A.11. Let $n: J \to T$ be a necklace and $F \to T$ be an outer face. Then

$$\Omega[n_F] = \Omega[n] \cap \Omega[F]$$

where the intersection is taken inside $\Omega[T]$.

Proof. Combining (i),(ii) in Lemma A.6 we see that a face $U \to F$ is in $\Omega[n]$ iff $E(J) \cap E(U) = \emptyset$, where (since $F$ is outer) the outer closure $U$ can be taken in either $T$ or $F$. But this is equivalent to $E(J_F) \cap E(U) = \emptyset$, i.e. to $U$ being in $\Omega[n_F]$. \hfill \Box

Before proceeding, we will need to better understand the maps in Nec.

Proposition A.12. Let $n: J \to T$ and $n': J' \to T'$ be necklaces. Then:

(i) A map $n \to n'$ in Nec is uniquely determined by some map $T \to T'$ in $\Omega$. More precisely, there exists an unique dashed arrow making the following commute.

$$\begin{array}{ccc}
\Omega[n] & \longrightarrow & \Omega[T] \\
\downarrow & & \downarrow \exists! \\
\Omega[n'] & \longrightarrow & \Omega[T']
\end{array} \quad \text{(A.13)}$$

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(ii) A map of trees \( \varphi : T \to T' \) in \( \Omega \) induces a map \( n \to n' \) in \( \text{Nec} \) iff \( \varphi J \supseteq J'_{\varphi T'} \).

Proof. We first address (i). The composite \( \Omega[n] \to \Omega[n'] \to \Omega[T'] \) in (A.13) induces compatible maps \( \Omega[\overline{nU}] \to \Omega[T'] \) in \( \text{dSet} \), and hence compatible maps \( nU \to T' \) in \( \Omega \). Hence, the result follows from the identification \( T \equiv \text{colim} \; nU \), where the colimit is now in \( \Omega \), cf. [BPb, Cor. 3.70].

We now turn to (ii). By (i),(ii) in Lemma A.6 the map \( \varphi \) defines a map of necklaces precisely if, for each \( v \in V(J) \), one has

\[
\emptyset = E'(J') \cap E'(\varphi T_v) = E'(J') \cap E'(\varphi J_v).
\]  

(A.14)

Writing \( \tilde{\varphi} \) for the composite \( J \to T \xrightarrow{\varphi} \tilde{T} \) and noting that \( \tilde{\varphi} \) is tall, (A.8) becomes

\[
E(\tilde{\varphi}T) = E(\varphi J) \cup \bigcup_{v \in V(J)} E'(\varphi J_v).
\]

Thus (A.14) amounts to \( E'(J') \cap E(\varphi T) \subseteq E(\varphi J) \), which is equivalent to the desired \( \varphi J \supseteq J'_{\varphi T} \) (as these trees have the same outer edges).

**Remark A.15.** Let \( n, n', T, T' \) be as in Proposition A.12 and suppose \( \varphi : T \to T' \) defines a map \( n \to n' \). Then for every outer face \( F \to T \) it follows from Corollary A.11 that the restriction \( F \to \varphi F \) likewise induces a restriction \( n_F \to n'_{\varphi F} \) from which it follows that \( \varphi J_F \supseteq (J'_F)_{\varphi F} \).

**Remark A.16.** Let \( n = (J \to T) \), \( n^* = (J^* \to T^*) \) be necklaces, and \( \varphi : T \to T' \) be a face map which induces a map of necklaces \( n \to n^* \).

Then for each bead \( T_v \to T \) of \( n \) there is a unique bead \( T^*_v \to T^* \) of \( n^* \) such that \( T_v \to T^* \) factors as \( T_v \to T^*_v \to T^* \). In particular, this defines a map of sets of beads \( \varphi^* : B(n) \to B(n^*) \).

**Definition A.17.** Let \( T \in \Omega \) be a tree. We define \( W(T) \in \text{sOp} \) to be the operad whose nerve is the preoperad \( NW(T) \) with \( n \)-simplices given by

\[
(W(T)_n)(S) = \left\{ \text{factorizations } S \xrightarrow{1} J_0 \xrightarrow{i_0} J_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} J_n \xrightarrow{f_n} T \text{ in } \Omega \right\}
\]

if \( \text{s} : E(S) \to E(T) \) defines a map \( \phi : S \to T \) in \( \Omega \) and \( ((W(T))_n)(S) = \emptyset \) otherwise.

Alternatively, it suffices to require that all maps \( S \to F_i \) are tall and all maps \( J_i \to T \) are planar face maps.

The functoriality of \( NW(T) \) with respect to a map \( (S', \text{s}') \to (S, \text{s}) \) is described by the diagram

\[
\begin{array}{ccc}
S' & \xrightarrow{1} & J'_0 \xrightarrow{i_0} J'_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} J'_n \xrightarrow{f_n} T \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S & \xrightarrow{1} & J_0 \xrightarrow{i_0} J_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} J_n \xrightarrow{f_n} T
\end{array}
\]

where \( S' \to J'_0 \to J_0 \) (resp. \( J'_{k+1} \to J_{k+1} \)) is defined as the “tall followed by outer face” factorization of composite \( S' \to S \to J_0 \) (resp. \( J'_k \to J_k \to J_{k+1} \)).

More generally, given a necklace \( n : J \to T \), we define \( NW(n) \subseteq NW(T) \) as the subpresheaf of those factorizations with the property that \( J_0 \supseteq J_{\text{face}} \).

Note that the fact that this is a presheaf follows since

\[
E'(J'_0) = E'(J_0) \cap E'(\text{face}) \supseteq E'(J_{\text{face}}) \cap E'(\text{face}) = E'(J_{\text{face}}).
\]

**Remark A.18.** Note that all \( NW(n) \) defined above are indeed nerves of operads, i.e., \( NW(n)(\eta) \) is discrete and \( NW(n) \) satisfies the strict Segal condition (cf. [BPb, Cor 3.69]).
Next, we discuss the functoriality of $NW(T)$ with respect to $T \in \Omega$.
For each map $T \to T^*$ in $\Omega$ we define $(NW(T))_{n,s}(S) \to (NW(T^*))_{n,s}(S)$ via the diagram
\[
\begin{array}{ccc}
S \xrightarrow{t} J_0 \xrightarrow{i,p} J_1 \xrightarrow{i,p} \cdots \xrightarrow{i,p} J_n \xrightarrow{f,p} T \\
S \xrightarrow{t} J_0^* \xrightarrow{i,p} J_1^* \xrightarrow{i,p} \cdots \xrightarrow{i,p} J_n^* \xrightarrow{f,p} T^*
\end{array}
\]
where $J_n \to J_k^* \to T^*$ (resp. $J_{k-1} \to J_k^* \to J_k^*$) is the "degeneracy followed by face" factorization of the composite $J_n \to T \to T^*$ (resp. $J_{k-1} \to J_k \to J_k^*$).

**Proposition A.20.** For any map $T \to T^*$ in $\Omega$, the induced map $(NW(T))_{s}(S) \to (NW(T^*))_{s}(S)$ in (A.19) is functorial on $(S,s)$.

**Proof.** First, note that the composite $(NW(T))_{s}(S) \to (NW(T))_{s'}(S') \to (NW(T^*))_{s'}(S')$ is computed by the left diagram below, where $S' \to J_i' \to J_i$ and $J_i' \to (J_i')^* \to T^*$ are the unique factorizations with the indicated properties. On the other hand, the composite $(NW(T))_{s}(S) \to (NW(T^*))_{s}(S) \to (NW(T^*))_{s'}(S')$ is computed as on the right with $J_i \to J_i^* \to T'$ and $S' \to (J_i')^* \to J_i^*$ the unique indicated factorizations.

\[
\begin{array}{ccc}
S \xrightarrow{t} J_i \xrightarrow{f,p} T \\
S' \xrightarrow{t} J_i' \xrightarrow{f,p} T^* \\
S' \xrightarrow{t} (J_i')^* \xrightarrow{f,p} T^*
\end{array}
\]

The key to the proof is to show that the planar faces $(J_i')^*$ and $(J_i^*)'$ of $T^*$ coincide, since it will then be automatic that all maps connecting the $(J_i')^*$ and $(J_i^*)'$ and $S'$, $T^*$ likewise match.

To see this, we consider the following diagram.

\[
\begin{array}{ccc}
S' \xrightarrow{t} J_i' \xrightarrow{f,p} T \\
S \xrightarrow{t} J_i \xrightarrow{f,p} T \\
S \xrightarrow{t} J_i^* \xrightarrow{f,p} T^*
\end{array}
\]

Both faces $(J_i')^*$ and $(J_i^*)'$ can alternatively be built by factoring the composite $J_i' \to J_i \to J_i^*$, with $(J_i')^*$ coming from the degeneracy-face factorization and $(J_i^*)'$ coming from the tall-outer factorization. But since $J_i' \to J_i \to J_i^*$ is a composite of convex maps (see Remark 2.9) it is again convex, so the two factorizations coincide, finishing the proof.

**Corollary A.21.** Let $n: J \to T$ and $n^*: J^* \to T^*$ be necklaces and suppose $\psi:T \to T^*$ induces a map $n \to n^*$. Then the induced map $NW(T) \to NW(T^*)$ restricts to a map $NW(n) \to NW(n^*)$.

**Proof.** We need to show that the map $NW(T) \to NW(T^*)$ sends simplices such that $J_0 \supset J_{\psi(S)}$ to simplices such that $J_0^* \supset J_{\psi(S)}$. This follows since $J_0' = \psi(J_0) \supset \psi(J_{\psi(S)}) \supset J_{\psi(S)} = J_{\psi(S)}$.
where the third step is Remark A.15.

We now introduce a notation that plays an important role in two key technical results, Propositions A.24 and A.29. Recall that, for any tree \( U \in \Omega \), the poset \( \text{Face}_{inn}(U) \) of planar inner faces is in fact a lattice, with the join \( F \lor F' \) the characterized by \( E'(F \lor F') = E'(F) \lor E'(F') \).

**Notation A.22.** Let \( \psi : J \to T \) be a necklace and \( \phi : S \to T \) a map in \( \Omega \).
We write \( S^n = \phi S \lor \overline{\psi S} \), where the join is taken in \( \text{Face}_{inn}(\overline{\psi S}) \).

**Remark A.23.** In the context of Notation A.22 one has natural identifications

\[
NW(n)_{\phi}(S^n) \xrightarrow{\iota} NW(n)(S) \xleftarrow{\iota} NW(S^n \to \overline{\psi T})_{\phi}(S)
\]

induced by the natural maps \( S \to S^n \) between trees and \( (S^n \to \overline{\psi T}) \to n \) between necklaces.

**Proposition A.24.** Let \( \psi : J \to T \) be a necklace. Then one has an identification

\[
W(n) \simeq \colim_{U \in \text{Face}_{\iota}(J)} W(T_U) \tag{A.25}
\]

where the colimit takes place in \( s\text{Op} \).

**Proof.** We will verify (A.25) by working with nerves throughout. More explicitly, we will show, that for any \( X \in \text{PreOp} \) satisfying the strict Segal condition, giving a map \( NW(n) \to X \) is the same as giving compatible maps \( NW(T_U) \to X \).

Moreover, clearly both sides of (A.25) yield \( E(T) \) when evaluated on \( \eta \). As such, we are free to fix throughout a map of colors \( E(T) \to X(\eta) \) and verify the universal property for maps respecting this color assignment. In particular, we are free to evaluate \( W(n), X \) on suitable \( E(T) \)-colored trees, rather than on uncolored trees.

Given maps \( NW(T_U) \to X \) we now define the map \( NW(n) \to X \) via (where \( S^n \) is as in Notation A.22)

\[
NW(n)_{\phi}(S^n) \xrightarrow{\iota} NW(n)(S) \xleftarrow{\iota} NW(S^n \to \overline{\psi T})_{\phi}(S)
\]

where the arrows (II) and (V) are isomorphisms by the Segal condition while (III) is an isomorphism since \( S^n_b \to T \) factors through \( T_{\phi, b} \).

Moreover, the arrow (IV) in (A.26) is induced by the chosen maps \( NW(T_U) \to X \), so clearly (A.26) denotes the only possible compatible map \( NW(T) \to X \).

It only remains to check that (A.26) is indeed a map in \( \text{PreOp} \), i.e. that it is natural on \( \overline{S} = (S, \phi) \). To see this, one first readily checks that a map \( \psi : \overline{S} \to \overline{T} \) induces a compatible inclusion \( \psi : S^n \to R^n \) showing the naturality of arrows (I),(VI) in the zigzag.

Next, by Remark A.16 one has a map of bead sets \( \psi : B(\eta_{\psi}) \to B(\eta_{\phi}) \) for which one has further compatible maps \( S^n_b \to R^n_{\psi, b} \), showing the naturality of the arrows (II),(V). Lastly, for any bead \( b \in B(\eta_{\phi}) \) one has \( T_{\psi, b} \simeq T_{\psi, \psi, b} \) showing naturality of the arrows (III),(IV).
Remark A.27. Let $I \xrightarrow{\Delta} \text{PreOp}$ be a diagram of preoperads and let $A = \text{colim}_{i \in I} A_i$.

For each $A(\eta)$-colored tree $\bar{S} = (S, s)$ let us write $I_{\bar{S}^i}$ for the category whose objects are factorizations $E(S) \to A_i(\eta) \to A(\eta)$ for some $i \in I$, which we represent by $E(S) \to A_i(\eta)$, together with maps $i \to i'$ in $I$ satisfying the obvious compatibility.

Then

$$A_s(S) \simeq \text{colim}_{(E(S) \to A_i(\eta)) \in I_{\bar{S}^i}} A_{i,s}(S)$$

(A.28)

where in the expression $A_{i,s}(S)$ we write $s_i$ for the coloring given by $E(S) \to A_i(\eta)$.

Proposition A.29. Let $X \in \text{dSet}$ and define $NW(X) \in \text{PreOp}$ by

$$NW(X) = \text{colim}_{(n \to X) : X \in \text{Nec}_{/X}} NW(n).$$

where $\text{Nec}_{/X} = \text{Nec} \downarrow X$ is the over category of maps $\Omega[n] \to X$, and the colimit is taken in $\text{PreOp}$.

Then $NW(X)$ satisfies the strict Segal condition. In particular, since $N$ is fully-faithful one has that $NW(X)$ is the nerve of the simplicial operad

$$W(X) = \text{colim}_{\Omega[n] \to X \in \text{Nec}_{/X}} W(n).$$

where the colimit is now taken in simplicial operads $s\text{Op}$.

Proof. We will evaluate $NW(X)$ at each $X(\eta)$-colored tree $\bar{S} = (S; s)$ using Remark A.27. We write $\text{Nec}_{\bar{S}^i/X} = (\text{Nec}_{/X})_{\bar{S}^i}$ for the category whose objects are pairs of arrows $E(S) \to \Omega[n] \to X$ whose composite is the coloring $s$.

(A.28) then says that

$$NW(X)_s(S) \simeq \text{colim}_{(E(S) \to \Omega[n] \to X) : X \in \text{Nec}_{\bar{S}^i/X}} NW(n)_{s_i}(S).$$

(A.30)

To show that $NW(X)$ satisfies the strict Segal condition, we will rewrite (A.30) by identifying appropriate subcategories of $\text{Nec}_{\bar{S}^i/X}$. First, write $\text{Nec}_{\bar{S}^i/X}^\Omega \subset \text{Nec}_{\bar{S}^i/X}$ for the full subcategory of those objects for which the map $E(S) \to E(T)$ defines a map $S \to T$ in $\Omega$.

Additionally, we write $\text{Nec}_{\bar{S}^i/X}^\Omega \subset \text{Nec}_{\bar{S}^i/X}$ for the full subcategory of “normalized factorizations”, which are defined by the property that $S \to T$ is a tall map and $J \supset \varphi(S)$.

Moreover, there is a retraction $\text{Nec}_{\bar{S}^i/X}^\Omega \onto \text{Nec}_{\bar{S}^i/X}^{\Omega,nor}$ which sends $E(S) \to \Omega[J \xrightarrow{n} T] \to X$ to $n(n) = (J \xrightarrow{\varphi} S \xrightarrow{\varphi} \bar{S}) = (S \xrightarrow{\varphi} \bar{S})$ (cf. Notation A.22). Recall (cf. Remark A.23) that the natural map $n(n) \to n$ in $\text{Nec}$ induces isomorphisms $NW(n(n))_{s_i}(S) \to NW(n)_{s_i}(S)$.

Since $\text{Nec}_{\bar{S}^i/X}^\Omega$ is a cosieve of $\text{Nec}_{\bar{S}^i/X}$ and $NW(n)_{s_i}(S) = \emptyset$ whenever $n$ is not in $\text{Nec}_{\bar{S}^i/X}^\Omega$, one can replace $\text{Nec}_{\bar{S}^i/X}$ with $\text{Nec}_{\bar{S}^i/X}^\Omega$ in (A.30). Moreover, by the discussion in the previous paragraph one can likewise further replace it with $\text{Nec}_{\bar{S}^i/X}^{\Omega,nor}$.

Next, note that the normalization conditions imply that $\text{Nec}_{\bar{S}^i/X}^{\Omega,nor} \simeq \prod_{\psi \in \varphi(S)} \text{Nec}_{\bar{S}^i/X}^{\Omega,nor}$. Putting
Consider the following diagram, were the functors labeled $W$ are as defined by Definition A.17 and Proposition A.29.

$$
\begin{array}{ccc}
\Omega & \xrightarrow{W} & \text{Nec} \\
\text{dSet} & \xrightarrow{W} & \text{sOp}
\end{array}
$$

Then both triangles in this diagram are left Kan extensions. In particular, the functor $W: \text{dSet} \to \text{sOp}$ coincides with the functor $W: \text{dSet} \to \text{sOp}$ as defined in (4.44).

**Remark A.33.** Recall that $\tau: \text{dSet} \to \text{Op}$ is the left Kan extension along $\Omega \to \text{dSet}$ of the functor $\Omega \to \text{Op}$ given by $T \mapsto \Omega(T)$.

It is then straightforward to check that that for any necklace $n = (J \to T)$ one has $\tau \Omega[n] \approx \Omega(T)$. Adapting the proofs of Proposition A.29 and Theorem A.32 one then has

$$
(N\tau X)_a(S) \approx \colim_{(E(S) \to \Omega[J \to T]) \in \text{Nec}_{S|\Omega}^{\Omega[n]}} \Omega[T]_a(S) \approx \colim_{(E(S) \to \Omega[J \to T]) \in \text{Nec}_{S|\Omega}^{\Omega[n]}} \Omega[T]_a(S).
$$

Moreover, the normalization conditions guarantee it is always $\Omega[T]_a(S) = *$ in the rightmost formula. Thus, specifying for the case of $S = C$ a corolla one has that operations in $\tau_! X(C)$ are represented by data of the form $\Omega[C] \xrightarrow{t} \Omega[T] \xleftarrow{\Omega[J \to T]} \Omega[J \to T] \to X$ subject to the equivalence relation generated by deeming two such data to be equivalent whenever there exists a map of necklaces $J \to T \to (J^* \to T^*)$ making the diagram below commute.

$$
\begin{array}{c}
\Omega[C] \xrightarrow{t} \Omega[T] \\
\downarrow \quad \downarrow \\
\Omega[C] \xrightarrow{t} \Omega[T^*] \\
\end{array}
\xleftarrow{\Omega[J \to T]} \xrightarrow{\Omega[J^* \to T^*]} X
$$
Remark A.34. All the work in this appendix can be adapted to the categories dSet$_G$ and Op$_G$ of genuine dendroidal sets and genuine operads discussed at the end of §2.3. In particular, the “genuine operadification” functor $\tau_G: dSet_G \to Op_G$ first mentioned in (2.37) can be described via an analogue of Remark A.33.

Briefly, an equivariant necklace is a map $n: J \to T$ of $G$-trees that is a planar orbital inner face. Alternatively, this means that $n$ is an ordered isomorphism on roots/components which is a planar inner face on tree components. Then, just as in Remark A.33 one has that, for each $G$-corolla $C$, the operations in $\tau_GX(C)$ can be represented by data $\Omega[C] \xrightarrow{t,r} \Omega[T] \leftarrow \Omega[J \to T] \to X$ (where the map labeled $t,r$ induces an ordered isomorphism on roots which is tall in each component) subject to the equivalence relation generated by diagrams

$$
\begin{align*}
\Omega[C] \xrightarrow{t,r} \Omega[T] & \leftarrow \Omega[J \to T] \to X \\
\Omega[C] \xrightarrow{t,r} \Omega[T^*] & \leftarrow \Omega[J^* \to T^*] \to X.
\end{align*}
$$

The remainder of this appendix is dedicated to providing a more explicit description of $W(X)$ for $X \in dSet$. Combining the proof of Proposition A.29 with the description of the simplices in Definition A.17 gives the following.

Corollary A.35 (cf. [DS11, Cor. 4.4]). Let $X \in dSet$. Then the simplices in $NW(X)_{n,\phi}(S)$ are equivalence classes of quadruples $(n, S \xrightarrow{\phi} T, \Omega[n] \xrightarrow{\eta} X, J_\bullet)$ where:

(i) $(J \xrightarrow{n} T) \in \text{Nec}$ is a necklace;

(ii) $S \xrightarrow{\phi} T$ is a tall map in $\Omega$ which induces a map $\Sigma[S] \to \Omega[n]$, i.e. $J \geq \phi(S)$;

(iii) $\Omega[n] \to X$ is a map in $dSet$ such that the induced composite $E(S) \to E(T) \to X(\eta)$ is the coloring $s$;

(iv) $J_\bullet$ denotes a tall simplex in $NW(n,\phi)$, i.e. a factorization of $\phi$

$S \xrightarrow{t} J_0 \xrightarrow{i_1} J_1 \xrightarrow{i_2} \cdots \xrightarrow{i_n} J_n \xrightarrow{i_{n+1}} T$ \hspace{1cm} (A.36)

such that $J_0 \geq J$.

The equivalence relation is generated by considering $(n, \phi, x, J_\bullet)$ and $(n^*, \phi^*, x^*, J^*_\bullet)$ to be equivalent if there is a map $\varphi: \Omega[n] \to \Omega[n^*]$ such that $\phi^* = \varphi \phi$, $x = x^* \varphi$ and $J^*_\bullet = \varphi J_\bullet$ (i.e $J^*_\bullet$ is obtained by push forwarding $J_\bullet$ along $\varphi$ in the sense of (A.19)).

Our final goal in this appendix is to show that, among the representatives in Corollary A.35 one can always identify a nice representative that is suitably unique.

Firstly, we discuss uniqueness of the maps $\Omega[n] \xrightarrow{\eta} X$ up to degeneracy.

Definition A.37. A map of necklaces $(J \to T) \to (J^* \to T^*)$ is called a degeneracy if the associated map $\varphi: T \to T^*$ is a degeneracy and $\varphi J = J^*$.

Definition A.38 (cf. [DS11, §4]). Let $J \xrightarrow{n} T$ be a necklace and $X \in dSet$. A map $\Omega[n] \to X$ is called totally non-degenerate if for all beads $T_v$ of $n$ the induced dendrex $\Omega[T_v] \to X$ is non-degenerate.
Lemma A.39 (cf. [DS11, Prop. 4.7]). Any map $\Omega[n] \to X$ has a factorization, unique up to unique isomorphism, as

$$\Omega[J \overset{\alpha}{\to} T] \to \Omega[J' \overset{\alpha'}{\to} T'] \to X$$

where the first map is a degeneracy of necklaces and the second map is totally non-degenerate.

Proof. The proof is by induction on the size of $E'(J)$. The base case is that of $E'(J) = \emptyset$ (note that then it must also be $E'(J') = \emptyset$), in which case the result reduces to [CM11, Prop. 6.9] or [Per18, Prop. 5.62].

Otherwise, let $e \in E'(J)$ and consider the grafting decomposition $T = R \cup_e S$. By the induction hypothesis, one has factorizations, unique up to unique isomorphism, $\Omega[n_R] \to \Omega[n'_R] \to X$, $\Omega[n_S] \to \Omega[n'_S] \to X$. Writing $n'_R = (J'_R \to R')$ and $n'_S = (J'_S \to S')$, we then set $n' = (J'_R \cup_e J'_S \to T'_R \cup_e T'_S)$.

The uniqueness up to unique isomorphism property of $tau'$ is easily seen to be inherited from the analogue property for $n'_R, n'_S$ (note that the “unique isomorphism” clause implies that there is no ambiguity concerning the grafting edge $e$), finishing the proof.

Next, we also need a preferred form for the tall simplex data in (A.36).

Definition A.40 (cf. [DS11, §4]). A tall simplex as in (A.36) is called flanked if $J_0 = J$ and $J_n = T$. Further, a quadruple $(n, \phi, x, J_*)$ is called flanked if $J_*$ is.

Remark A.41. Suppose $(n, \phi, x, J_*)$ is a flanked quadruple and set $n_k = (J_k \to T)$. Then the structure maps in (A.36) induce a diagram of necklace maps

$$\xymatrix{\Sigma e[T] \ar[r] & \Omega[n_k] \ar[r] & \Omega[n_{k-1}] \ar[r] & \cdots \ar[r] & \Omega[n_0] \ar[r] & \Omega[n] \ar[l] & \Sigma e[S]}$$

Remark A.42. If both simplices $J_*, J'_*$ in a pushforward diagram (A.19) are flanked, then the associated map of necklaces $n \to n^*$ is a degeneracy.

In what follows we say a quadruple $(n, \phi, x, J_*)$ as in Corollary A.35 is flanked if $J_*$ is and totally non-degenerate if $x$ is.

Lemma A.43 (cf. [DS11, Lemma 4.5]). (i) Any quadruple $(n, \phi, x, J_*)$ as in Corollary A.35 is equivalent a flanked one;

(ii) if two flanked quadruples are equivalent, then the equivalence can be described via a zigzag involving only flanked quadruples.

Proof. The key to the proof is the fact that the map $J_n \to T$ induces a map of necklaces $(J_0 \to J_n) \to (J \to T)$. This map of necklaces then induces a pushforward of simplices

$$\xymatrix{S \ar[r]^t & J_0 \ar[r]^{i_p} & J_1 \ar[r]^{i_p} & \cdots \ar[r]^{i_p} & J_n \ar[r]^i & J_n \ar[r] & J_n \ar[l]_j \ar[r] & S}$$

where the top simplex (and thus the associated quadruple) is now flanked, so (i) follows.

(ii) then follows by noting that the procedure above is natural. More precisely, an arbitrary pushforward of tall simplices along the necklace map $(J, T) \to (J^* \to T^*)$ as in (A.19) induces a pushforward of flanked simplices

$$\xymatrix{S \ar[r]^t & J_0 \ar[r]^{i_p} & J_1 \ar[r]^{i_p} & \cdots \ar[r]^{i_p} & J_n \ar[r]^i & J_n \ar[r] & J_n \ar[r] & J_n \ar[l]_j \ar[r] & S}$$

$$\xymatrix{S \ar[r]^t & J_0' \ar[r]^{i_p} & J_1' \ar[r]^{i_p} & \cdots \ar[r]^{i_p} & J_n' \ar[r]^i & J_n' \ar[r] & J_n' \ar[r] & J_n' \ar[l]_j \ar[r] & S}$$

$$\xymatrix{\ar[r]^t & J_0' \ar[r]^{i_p} & J_1' \ar[r]^{i_p} & \cdots \ar[r]^{i_p} & J_n' \ar[r]^i & J_n' \ar[r] & J_n' \ar[r] & J_n' \ar[l]_j \ar[r] & S}$$

$$\xymatrix{\ar[r]^t & J_0' \ar[r]^{i_p} & J_1' \ar[r]^{i_p} & \cdots \ar[r]^{i_p} & J_n' \ar[r]^i & J_n' \ar[r] & J_n' \ar[r] & J_n' \ar[l]_j \ar[r] & S}$$
along the necklace map \((J_0 \to J_n) \to (J_0' \to J_n')\). \hfill \Box

**Corollary A.44** (cf. [DS11, Cor. 4.8]). Each quadruple \((n, \phi, x, J_n)\) as in Corollary A.35 has a representative, unique up to unique isomorphism, which is both flanked and totally non-degenerate.

**Proof.** By Lemma A.43(i) any quadruple is equivalent to a flanked quadruple and by Lemma A.39 any flanked quadruple is equivalent to a flanked quadruple that is also totally non-degenerate.

As for the uniqueness condition, by Lemma A.43(ii) we need only consider zigzags of equivalences of flanked quadruples, which are induced by necklace degeneracies, cf. Remark A.42. Thus, arguing by induction on the size of the zigzag, Lemma A.39 implies that all flanked quadruples in the zigzag have the same totally non-degenerate quotient, so the desired uniqueness claim reduces to the uniqueness claim in Lemma A.39. \hfill \Box

**Example A.45.** Let \(U \in \Omega\) be a tree. We will apply Lemma A.43 to describe \(W_i(\partial \Omega[U])\).

Firstly, on any signature \(\widehat{\mathcal{C}}\) that is not isomorphic to the leaf-root signature one readily sees that \(W_i(\partial \Omega[U])((\widehat{\mathcal{C}})) = W(U)(\widehat{\mathcal{C}})\).

For the interesting case of \(W_i(\partial \Omega[U])(\widehat{\mathcal{C}})\), note that a map \(\Omega[J \to V] \to \Omega[U]\) is totally non-degenerate iff the inducing map \(V \to U\) is a face map. Moreover, we are free to choose the unique planar representative. Next, note that \(\Omega[J \to V] \to \Omega[U]\) factors through \(\partial \Omega[U]\) iff either \(V \neq U\) or \(J \notin \text{lr}(U)\). In other words, the \(n\)-simplices of \(W_i(\partial \Omega[U])(\widehat{\mathcal{C}})\) are uniquely represented by strings

\[
J_0 \xrightarrow{1:p} J_1 \xrightarrow{1:p} \ldots \xrightarrow{1:p} J_n = V \xrightarrow{1:p} U \tag{A.46}
\]

such that either \(J_n \neq U\) or \(J_0 \notin \text{lr}(U)\).

Put another way, \(W_i(\partial \Omega[U])(\widehat{\mathcal{C}})\) is the boundary of the nerve of the poset of inner faces \(\text{Face}_{\text{inn}}(U) \simeq (0 \to 1)^{\mathcal{E}(U)}\). In particular, \(W_i(\partial \Omega[U])(\widehat{\mathcal{C}})\) is identified with the *domain* of the iterated pushout product

\[
((0,1) \to \Delta[1])^{\mathcal{E}(U)}.
\]

**Example A.47.** Let \(U \in \Omega\) and \(\emptyset \neq E \subseteq \mathcal{E}(U)\). Just as in Example A.45 one has \(W_i(\Lambda\mathcal{E}[U])(\widehat{\mathcal{C}}) = W(U)(\widehat{\mathcal{C}})\) whenever \(\widehat{\mathcal{C}} \notin (\widehat{\mathcal{C}})\) (though we note that this fails for outer horns).

Then, one has that \(\Omega[J \to V] \to \Omega[U]\) factors through \(\Lambda\mathcal{E}[U]\) iff either \(V \neq U - E\) or \(J \notin \text{lr}(U)\). I.e., the \(n\)-simplices of \(W_i(\Lambda\mathcal{E}[U])(\widehat{\mathcal{C}})\) are uniquely represented by strings (A.46) such that either \(J_n \neq U - E\) or \(J_0 \notin \text{lr}(U)\). Put another way, \(W_i(\Lambda\mathcal{E}[U])(\widehat{\mathcal{C}})\) is identified with the *domain* of the iterated pushout product

\[
((0,1) \to \Delta[1])^{\mathcal{E}(U)}_{-E} \sqcup ((1) \to \Delta[1])^{\mathcal{E}}.
\]

**B. The homotopy genuine equivariant operad**

The goal of this appendix is to establish Proposition B.12, which compares two procedures of discretizing an equivariant operad \(\mathcal{O} \in \mathfrak{sOp}^G\), and is the full equivariant generalization of [CM13b, Prop. 4.8] (cf. Remark A.48).

Most of the work will be spent describing the “genuine operadification” functor \(\tau_G: \mathbf{dSet}_G \to \mathbf{Op}_G\) first mentioned in (2.37). For a general \(Z \in \mathbf{dSet}_G\) one can describe \(\tau_G Z\) as in Remark A.34, but this description is somewhat cumbersome in practice. Instead, here we will focus on the case of \(Z \in \mathbf{dSet}_G\) a genuine \(G\)-∞-operad, in which case \(\tau_G Z\) admits a more explicit description as a *homotopy operad*, which we denote \(\mathfrak{ho}(Z)\) (Definition B.9), mimicking the description of \(\tau X\) when \(X \in \mathbf{dSet}\) is an ∞-operad [MW09, §6].
**Definition B.1.** $Z \in \text{dSet}_G$ is a genuine $G$-$\infty$ operad if it has the right lifting property against all maps $v_{G,*}(\Lambda^E[T] \to \Omega[T])$ for $T \in \Omega_G$ and $G$-subset $E \subseteq E'(T)$.

**Remark B.2.** Since $v_{G,*}$ is fully faithful one has that $X \in \text{dSet}_G$ is a $G$-$\infty$-operad iff $v_{G,*}X \in \text{dSet}_G$ is a genuine $G$-$\infty$-operad.

We now turn to the task of describing $\tau_G Z$ for $Z \in \text{dSet}_G$ a genuine $G$-$\infty$-operad.

We start with some notation. Given a multiset $I \in \Sigma_G$ determined by the number $0 \leq k$ of leaf orbits and isotropy subgroups $H_i \leq H_0 \leq G$ for $0 \leq i \leq k$, where $H_0$ is the isotropy of a (chosen) root edge. Pictorially, such a $G$-corolla has the orbital representation given on the left below, but in this section we will find it more convenient to label edge orbits using coset notation as on the right below, so that $[e_i] = Ge_i$ denotes the $G$-orbit of $e_i$.

\[
\begin{array}{ccc}
G/H_1 & \cdots & G/H_k \\
G/H_0 & & [e_1] \cdots [e_k] \\
C & & C
\end{array}
\]

We will then abbreviate $\sigma' C = \sigma[e_i] C$, and write $e_i, e'_i$ for the two edges of $\sigma' C$ that degenerate the edge $e_i$ of $C$, with $e_i$ denoting the inner edge and $e'_i$ the outer edge.

\[
\begin{array}{ccc}
[[e_1]] & \cdots & [[e_k]] \\
| & & | \\
[e_0] & & [e_0] \\
\sigma' C & \cdots & \sigma' C
\end{array}
\]

The $G$-tree $\sigma' C$ then has an orbital inner face\(^2\) $\sigma' C - [e_i]$ obtained by removing $[e_i]$ as well as an orbital outer face obtained by removing $e'_i$, which we denote $\sigma' C - [e'_i]$. Moreover, note that we have natural identifications $C \simeq \sigma' C - [e_i]$, $C \simeq \sigma' C - [e'_i]$.

In what follows, we will find it convenient to simplify notation by denoting maps $v_{G,*} \Omega[T] \to Z$, where $T \in \Omega_G$ and $Z \in \text{dSet}_G$, simply as $T \to Z$.

**Definition B.3.** Let $Z \in \text{dSet}_G$ be a genuine $G$-$\infty$-operad and $C$ a $G$-corolla with edge orbits $[e_0], \ldots, [e_k]$. Given two parallel operations $f, g \colon C \to Z$, we write $f \sim_i g$ if there exists a map $H : \sigma' C \to Z$ such that

- $f$ equals the restriction $H|_{\sigma' C - [e'_i]}$;

\(^2\)See [BPa, Defn. 2.16] for a discussion of orbital inner faces.
• \( g \) equals the restriction \( H|_{\sigma^j C \rightarrow \{e_i\}} \);
• the restriction \( H|_{\sigma^j(e_i)} \) is the degeneracy \( \sigma^j(e_i) \rightarrow \{e_i\} \rightarrow C \rightarrow Z \).

**Remark B.4.** Note that if \( f \sim_1 g \) then it must be \( f|_{\partial C} = g|_{\partial C} \).

**Example B.5.** Let \( G = \mathbb{Z}/2 = \{\pm 1\} \) and consider the \( G \)-corolla with orbital and expanded representations as given on the left below.

\[
\begin{array}{c}
G \cdot e \\
G/G \cdot r \\
C
\end{array}
\begin{array}{c}
\sigma(e,-e) \\
\rightarrow
\end{array}
\begin{array}{c}
G(e) \\
G/G \cdot r \\
C
\end{array}
\begin{array}{c}
\sigma(e,-e) \\
\rightarrow
\end{array}

C \text{ then has a single leaf } G\text{-edge orbit } [e] = G \cdot e, \text{ so that for } f,g : C \rightarrow Z \text{ it is } f \sim_1 g \text{ if there exists a dendrex } H: \sigma(e,-e) \rightarrow Z \text{ such that}

\[
f = H|_{\sigma(e,-e)C \rightarrow \{e',-e'\}} \quad g = H|_{\sigma(e,-e)C \rightarrow \{e,-e\}}
\]

\[
H_{\sigma(e),H|_{\sigma(e,-e)}} \text{ are degenerate. (B.6)}
\]

It is worthwhile to compare this equivariant relation with the relations obtained if one forgets the \( G \)-actions. Indeed, while (B.6) implicitly assumes that all of \( f, g, H \) are \( G \)-equivariant, by omitting that assumption one can reinterpret (B.6) as defining a relation \( f \sim_{[e]} g \) between not necessarily \( G \)-equivariant maps \( f,g : C \rightarrow Z \).

A priori, the \( \sim_{[e]} \) relation differs from the non-equivariant \( \sim_e \) and \( \sim_{-e} \) relations obtained by regarding \( C \) as a non-equivariant corolla. However, for \( f,g,H \) as in (B.6) one has

\[
f = H|_{\sigma(e,-e)C \rightarrow \{e',-e'\}} \sim_{-e} H|_{\sigma(e,-e)C \rightarrow \{e,-e\}} \sim_{e} H|_{\sigma(e,-e)C \rightarrow \{e,-e\}} = g \quad (B.7)
\]

so that, by Lemma B.8(b) below one has that \( f \sim_{[e]} g \) in fact implies \( f \sim_{e} g \). Moreover, the converse statement follows immediately by using degeneracies.

More generally, similar considerations show that the \( \sim \) relations are compatible with restricting the \( G \)-actions.

**Lemma B.8** (cf. [MW09, Prop. 6.3 and Lemma 6.4]). Let \( Z \in \mathcal{dSet}_G \) be a genuine \( G\)-\( \infty \)-operad and \( C \) a \( G \)-corolla with edge orbits \([e_0],\ldots,[e_k]\). Then:

(a) each of the relations \( \sim_i \) in Definition B.3 is an equivalence relation;

(b) all the equivalence relations \( \sim_i \) coincide.

**Proof.** We first address (a).

For reflexivity \( f \sim_i f \), we take the exhibiting homotopy \( H \) to be the degeneracy \( \sigma^i C \rightarrow C \rightarrow Z \).
Both symmetry and transitivity will use the tree $\sigma^{ji}C$ below, which degenerates $[e_i]$ twice.

For symmetry, suppose $f \sim_i g$ with $H: \sigma^i C \to Z$ the exhibiting homotopy. Define a map $H: \Lambda_0^{[ei]}[\sigma^{ii}C] \to Z$ by

$$H|_{\sigma^{ii}C-[e_i]} = H, \quad H|_{\sigma^{ii}C-[e_j]} = f \circ \sigma^i, \quad H|_{\sigma^{ii}[e_i]} = f|[e_i] \circ \sigma^{ii} = g|[e_i] \circ \sigma^{ii}.$$ 

Since the orbital inner horn inclusion $H: \Lambda_0^{[e_i]}[\sigma^{ii}C] \to \Omega(C)$ is $G$-inner anodyne by [BPa, Prop. 3.13], $H$ admits an extension $\tilde{H}: \sigma^{ii} C \to Z$. The restriction $\tilde{H}|_{\sigma^{ii}C-[e_i]}$ then provides the homotopy exhibiting $g \sim_i f$, and symmetry of $\sim_i$ follows.

Next, suppose $f \sim_i g$ and $g \sim_i h$, and let $H: \sigma^i C \to Z$, $K: \sigma^j C \to Z$ be the exhibiting homotopies. Define a map $\tilde{H}: \Lambda_0^{[e_i]}[\sigma^{ii}C] \to Z$ by

$$\tilde{H}|_{\sigma^{ii}C-[e_i]} = H, \quad \tilde{H}|_{\sigma^{ii}C-[e_j]} = K, \quad \tilde{H}|_{\sigma^{ii}[e_i]} = f|[e_i] \circ \sigma^{ii} = g|[e_i] \circ \sigma^{ii} = h|[e_i] \circ \sigma^{ii}.$$ 

$\tilde{H}$ again admits an extension $\bar{H}: \sigma^{ii} C \to Z$, and the restriction $\bar{H}|_{\sigma^{ii}C-[e_i]}$ provides the homotopy exhibiting $f \sim_i g$, and transitivity of $\sim_i$ follows.

We next turn to (b). Consider the tree $\sigma^{ij}C$ which degenerates $C$ once along each of $[e_i]$ and $[e_j]$.

Suppose $f \sim_i g$ with $H: \sigma^i C \to Z$ the associated homotopy. Define a map $\tilde{H}: \Lambda_0^{[e_i]}[\sigma^{ij}C] \to Z$ by

$$\tilde{H}|_{\sigma^{ij}C-[e_i]} = H, \quad \tilde{H}|_{\sigma^{ij}C-[e_j]} = f \circ \sigma^i, \quad \tilde{H}|_{\sigma^{ij}C-[e_j]} = f \circ \sigma^j.$$ 

Yet again, $\tilde{H}$ admits an extension $\bar{H}: \sigma^{ij} C \to Z$, and the restriction $\bar{H}|_{\sigma^{ij}C-[e_i]}$ provides a homotopy exhibiting $g \sim_j f$. (b) now follows.

In light of Lemma B.8, given parallel operations $f,g: C \Rightarrow Z$ with $C$ a $G$-corolla and $Z$ a genuine $G$-$\infty$-operad, we will henceforth write $f \sim g$ whenever $f \sim_i g$ for some (and thus all) $i$. We now extend the $\sim$ relation.
**Definition B.9.** Let $T \in \Omega_G$ be a $G$-tree and $Z \in dSet_G$ be a genuine $G$-$\infty$-operad.

Given dendrices $x,y: T \to Z$ we write $x \sim y$ if there are equivalences of restrictions $\pi_\varepsilon T_x \sim \pi_\varepsilon T_y$ for all $G$-vertices $\varepsilon \in V_G(T)$.

Further, we define $\text{ho}(Z)(T) = Z(T)/\sim$.

**Proposition B.10.** Let $Z \in dSet_G$ be a genuine $G$-$\infty$-operad. Then the assignment $T \mapsto \text{ho}(Z)(T)$ is a contravariant functor on $T \in \Omega_G$, i.e. $\text{ho}(Z) \in dSet_G$.

**Proof.** It suffices to show that the $\sim$ equivalence relations are compatible with the generating classes of maps in $\Omega_G$, namely degeneracies, inner faces, outer faces, and quotient maps.

The cases of degeneracies and outer faces are obvious. In the case of quotients, since any quotient $\tilde{T} \to T$ of $G$-trees induces quotients on $G$-vertices, it suffices to consider the case of a quotient $\tilde{C} \xrightarrow{\pi} C$ of $G$-corollas. But it is then straightforward to check that a homotopy exhibiting $f \sim g$ also induces a homotopy exhibiting $f \circ \pi \sim g \circ \pi$ (notably, the same needs not be true for the relations $f \sim g$ when $0 < i$, in which case the exhibiting homotopy may instead exhibit a string of relations $f \circ \pi \sim \cdots \sim g \circ \pi$ as in (B.7)).

It remains to address the most interesting case, that of inner faces. Since inner faces can be factored as composites of inner faces that collapse a single inner edge orbit, it suffices to consider the case of faces $D \to T$ where $T$ has a single inner edge orbit; that is, we can assume that there are $G$-corollas $C_1, C_2$ such that $T = C_1 \cup_{[e_i]} C_2$ and $D = T - [e_i]$, as illustrated below.

![Diagram](image)

The claim is now that if $x,y: T \to Z$ are such that $x|C_1 \sim y|C_1$ and $x|C_2 \sim y|C_2$, then it is also $x|D \sim y|D$. This will follow from the following two claims:

(i) if $x,y: T \to Z$ are such that $x|C_1 = y|C_1$ and $x|C_2 = y|C_2$, then $x|D = y|D$;

(ii) given $x:T \to Z$, $f:C_1 \to Z$ and $g:C_2 \to Z$ such that $f \sim x|C_1$, $g \sim x|C_2$, there exists $y:T \to Z$ such that $y|C_1 = f$, $y|C_2 = g$ and $y|D = x|D$.

To show (i) and (ii), consider the degeneracies $\sigma^0 T$ and $\sigma^i T$ pictured below.

![Diagram](image)
Given $x, y$ as in (i), one can now build a map $H: \Lambda^e_{\sigma^0}[\sigma^0 T] \to Z$ by

$$H|_{\sigma^0 T-[e_1]} = x, \quad H|_{\sigma^0 T-[e'_1]} = y, \quad H|_{\sigma^0 C_1} = x|_{C_1} \circ \sigma^0 = y|_{C_1} \circ \sigma^0.$$  

Letting $\overline{H}: \sigma^0 T \to Z$ be an extension of $H$, the restriction $H|_{\sigma^0 T-[e_1]}$ provides the desired homotopy $x|_D \sim y|_D$, showing (i).

Lastly, let $x, f, g$ be as in (ii), and let $K: \sigma^C_1 \to Z$ exhibit the relation $f \sim_i x|_{C_1}$ and $K: \sigma^C_2 \to Z$ exhibit the relation $x|_{C_2} \sim_i g$ (note the reversed order). Now build the map $H: \Lambda^e_{\sigma^C_1}[\sigma^C T] \to Z$ by

$$H|_{\sigma^C T-[C_1]} = x, \quad H|_{\sigma^C C_1} = K, \quad H|_{\sigma^C C_2} = K.$$

Again letting $\overline{H}: \sigma^C T \to Z$ be a lift, the restriction $\overline{H}|_{\sigma^C T-[C_1]}$ provides the desired $y T \to Z$ in (ii), finishing the proof.

**Corollary B.11.** Let $Z \in dSet_G$ be a genuine $G$-∞-operad. Then:

(a) $\text{ho}(Z) \in dSet_G$ is a genuine equivariant operad, i.e. it satisfies the strict right lifting property against the Segal core inclusions $\nu_G, *: (Sc[T] \to \Omega[T])$ for $T \in \Omega_G$;

(b) the quotient map $Z \to \tau_G(Z)$ is the universal map from $Z$ to a genuine equivariant operad.

In particular, (a) and (b) yield a natural identification $\text{ho}(Z) \simeq \tau_G(Z)$.

**Proof.** Note first that by Remark B.4 any map $\text{Sc}[T] \to \tau_G(Z)$ admits a factorization $\text{Sc}[T] \to Z \overset{\nu}{\to} \tau_G(Z)$.

The right lifting property for $\tau_G(Z)$ against the maps $\text{Sc}[T] \to \Omega[T]$ is then automatic from the lifting property for $Z$.

For strictness, note that Definition B.9 can be reinterpreted as saying that a pair of dendrices $\Omega[T] \Rightarrow Z$ give rise to the same point of $\tau_G(Z)$, i.e. the composites $\Omega[T] \Rightarrow Z$ coincide, iff the composites $\text{Sc}[T] \to \Omega[T] \Rightarrow Z$ coincide, showing strictness, and thus (a).

For (b), since $\tau_G(Z)$ is a quotient of $Z$, it suffices to show that any map of the form $F: Z \to Y$ with $Y$ a genuine equivariant operad must also enforce the $\sim$ relation. For a $G$-corolla $C$ and $f, g: C \Rightarrow Z$ such that $H: \sigma^C \to Z$ exhibits $f \sim g$, the strict lifting condition for $Y$ shows that $F \circ H: \sigma^C \to Y$, $F(f) \circ \sigma^*: \sigma^C \to Y$ must coincide, and thus that $F(f) = F(g)$. The further claim that $F$ respects equivalences of general pairs of dendrices $T \Rightarrow Z$ is immediate from Definition B.9.

The following is the analogue of [CM13b, Prop. 4.8].

**Proposition B.12.** Let $O \in sOp^G$ be a fibrant operad. Then there is a natural isomorphism of genuine equivariant operads

$$\tau_G(hcN(O)) \overset{\sim}{\Rightarrow} \pi_0(\nu_G, NO). \quad (B.13)$$

**Proof.** To ease notation, we abbreviate $\nu_G, \ast$ as $\nu, \ast$ throughout the proof.

By [BPa, Prop. 5.9], $\pi_0(\nu, NO)$ is a genuine equivariant operad, and the existence of the map in (B.13) will be an application of Corollary B.11(b).

Firstly, note that we have the following identifications, naturally in $T \in \Omega_G$.

$$\nu, hcN(O)(T) \simeq sOp^G(W; \Omega[T], O) \simeq sSet_G(W; \Omega[T], NO) \simeq sSet_G(\nu, NW; \Omega[T], NO)$$
where the second and third identifications use the fact that \( N \colon Op \to dSet \) and \( \upsilon \colon dSet^G \to dSet^G \) are fully faithful inclusions. One now has a map

\[
\text{sdSet}_G(\upsilon_* NW! \Omega[T], \upsilon_* N\Omega) \to \text{sdSet}_G(\upsilon_* NW! \Omega[T], \pi_0 \upsilon_* N\Omega) \\
\cong \text{dSet}_G(\pi_0 \upsilon_* NW! \Omega[T], \pi_0 \upsilon_* N\Omega) \\
\cong \text{dSet}_G(\upsilon_* \Omega[T], \pi_0 \upsilon_* N\Omega) \\
= (\pi_0 \upsilon_* N\Omega)(T)
\]

so altogether we obtain a map \( \upsilon_* hcN(\mathcal{O}) \to \pi_0 \upsilon_* N\mathcal{O} \) and hence, by Corollary B.11, the desired map

\[
\tau_G(hcN(\mathcal{O})) \to \pi_0 \upsilon_* N\mathcal{O}
\]

Moreover, both of these are quotients of \( \upsilon_* hcN(\mathcal{O}) \), so to prove that this map is an isomorphism one needs only show that any two parallel operations \( C \Rightarrow hcN(\mathcal{O}) \) of \( \upsilon_* hcN(\mathcal{O}) \) that are identified in \( \pi_0 \upsilon_* N\mathcal{O} \) were already identified in \( ho(hcN(\mathcal{O})) \). But this now follows immediately from the pushout below, cf. Lemma 4.45.

\[
\begin{array}{ccc}
\Omega(C) \otimes_{\mathcal{E}} \Delta[1] & \to & W(\partial \Omega[\sigma^0C]) \\
& \downarrow & \\
\Omega(C) \otimes_{\mathcal{E}} \Delta[1] & \to & W(\sigma^0C)
\end{array}
\]

\( \square \)

References


