HW 11: Extra problems

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\( T : V \to W \) is a linear transformation. \( \{v_1, \ldots, v_k\} \) is a subset of \( V \).

- Show that if \( \{T(v_1), \ldots, T(v_k)\} \) is linearly independent in \( W \), then \( \{v_1, \ldots, v_k\} \) is linearly independent in \( V \).

**Proof.** Consider the following linear combination

\[
\sum_{i=1}^{n} c_i v_i = 0
\]

Let’s show \( c_i = 0 \) to show the linear independence. By the property of linear transformation, we have:

\[
0_W = T(0_V) = T(\sum_{i=1}^{n} c_i v_i) = \sum_{i=1}^{n} c_i T(v_i)
\]

However, we know that \( \{T(v_i), 1 \leq i \leq n\} \) is linearly independent, we therefore must have \( c_i = 0 \) which proves the linear independence of \( \{v_i, 1 \leq i \leq n\} \)

Remark: some people showed like this: Since \( \{T(v_i), 1 \leq i \leq n\} \) are linearly independent, \( T(c_1 v_1 + \ldots + c_n v_n) = c_1 T(v_1) + \ldots + c_n T(v_n) = 0 \) must gives \( c_i = 0 \). However \( T(0) = 0 \), and then we must have \( c_1 v_1 + \ldots + c_n v_n = 0 \) with only \( c_i = 0 \).

I can’t give such people full credits here. Why? it seems to me that they are using this claim ‘if \( T(v) = 0 \), then \( v = 0 \)’. \( T \) may not be one-to-one and this may not be true!

- Show that if \( \{T(v_1), \ldots, T(v_n)\} \) are linearly dependent, then \( \{v_1, \ldots, v_n\} \) might not be linearly dependent.

We can just use example to show such claims. Why?

**Proof.** Consider the example given in the problem. Define \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( T(x, y) = (0, y) \).

Then, we can see that \( T(1, 0) = (0, 0) \) and \( T(0, 1) = (0, 1) \). \( (0, 0) \) and \( (0, 1) \) are linearly dependent while \( (1, 0) \) and \( (0, 1) \) are not.

- \( T \) is one-to-one. show that if \( \{T(v_1), \ldots, T(v_n)\} \) is linearly dependent, so is \( \{v_1, \ldots, v_n\} \).
**Proof.** Since \( \{T(v_i), 1 \leq i \leq n\} \) is linearly dependent, then we can find \( c_i, 1 \leq i \leq n \) where some are nonzero so that:

\[
\sum_{i=1}^{n} c_i T(v_i) = 0
\]

However, \( \sum_{i=1}^{n} c_i T(v_i) = T(\sum_{i=1}^{n} c_i v_i) \). \( T \) is one-to-one, that means the kernel of \( T \) only contains 0 vector (or the pre-image of 0 is unique which is 0). Therefore, we must have:

\[
\sum_{i=1}^{n} c_i v_i = 0
\]

for the same \( c_i \). Since these \( c_i \)'s are not all zero and thus \( \{v_1, \ldots, v_n\} \) is linearly dependent as well.

\[
\square
\]

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- Since \( \cos^2 x = (1 + \cos 2x)/2 \) which is the linear combination of the first two, we can cross out \( \cos^2 x \) from our list.

We now show that \( \{1, \cos x, \cos 2x, \cos 3x\} \) is linearly independent. Suppose

\[
c_1 \ast 1 + c_2 \ast \cos x + c_3 \cos 2x + c_4 \cos 3x = 0
\]

Letting \( x = \pi/2 \), we have \( c_1 - c_3 = 0 \). Letting \( x = 0 \), we have \( c_1 + c_2 + c_3 + c_4 = 0 \). Letting \( x = \pi \), we have \( c_1 - c_2 + c_3 - c_4 = 0 \). Therefore, we have \( c_1 + c_3 = 0 \) and \( c_2 + c_4 = 0 \). Using the fact that \( c_1 - c_3 = 0 \), we have \( c_1 = c_3 = 0 \).

Thus, finally, we have \( c_2 \ast (\cos x - \cos 3x) = 0 \). Picking \( x = \pi/4 \) gives \( c_2 = 0 \). Thus all coefficients should be 0 and thus these functions are linearly independent.

\[\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x\] and thus the coordinate vector is \( (1/2, 0, -1/2, 0)^T \).

**Remark:** You can cross out 1. You can also cross out \( \cos 2x \). However, you can only cross out one function no matter which one you choose to cross out. To prove the linear independence, you can also define the transformation \( T(f) = [f(0), f(\pi/4), f(\pi/2), f(\pi)] \). This is another method, but personally I don’t like this.

- We can see \( \ln(x^2) = 2 \ln(x) \) and \( \ln(2x) = \ln 2 + \ln x \). Thus, you can cross out \( \ln(x^2) \) and \( \ln(2x) \). Then, you are left with \( \{1, \ln x\} \). They are linearly independent.

\[
c_1 \ast 1 + c_2 \ast \ln x = 0
\]

Letting \( x = 1 \), we have \( c_1 = 0 \). Therefore, \( c_2 = 0 \) as well.

Since \( \ln(x/3) = \ln x - \ln 3 \). The coordinate vector is \( (-\ln 3, 1) \)