Summary for Vector Calculus and Complex Calculus
(Math 321)

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1 Vector Calculus

1.1 Parametrization

Curves, surfaces, or volumes can be parametrized. Below, I'll talk about $3D$ case.
Suppose we use $\vec{e}_x, \vec{e}_y, \vec{e}_z$ as the basis and then the position vector for any point on the curve (surface, volume) can be written as:

$$\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$$

To parametrize, we need to write x, y, z as functions of our parameters. You need to choose the parameters wisely so that our problem would be easy to solve.

Something you must know:

1. For curve, the degree of freedom is one and thus you should only have ONE parameter in your parametrization
2. You should have 2 independent parameters for surface
3. 3 independent parameters for volume

Example:

- For the circle $(x-a)^2 + (y-b)^2 = r^2$, we can use the angle and $x = a + r\cos\theta, y = b + r\sin\theta$ and then your parametrization would be $\vec{r}(\theta) = (a + r\cos\theta)\vec{e}_x + (b + r\sin\theta)\vec{e}_y$
- For the function graph $y = f(x)$, x itself is a convenient parameter.
- for the region $0 \le x, y \le a, z = b$, x, y would be good parameters.

Two important parametrization would be to use the spherical coordinates and cylindrical coordinates. Hope you know the meanings the parameters. To get the Cartesian equation from the parametrization, you just need to eliminate the parameters.

Example: $x = a\tan\theta, y = b\sec\theta$. What is $\vec{r}$? What is the Cartesian equation?

1.2 Curves and line integrals

Suppose you get the parametrization already (given or obtained by yourself) $\vec{r}(t)$:

$$d\vec{r} = \frac{d\vec{r}}{dt}dt = \vec{r}'(t)dt$$

Sometimes, we are given by functions like $\vec{r}(t) = \vec{r}(\theta(t))$ or $\vec{r}(t) = \vec{r}(u(t), v(t))$. It's easy to get the following by chain rule:

$$\begin{aligned} \vec{r}'(t) &= \frac{d\vec{r}}{d\theta}\theta'(t) \\ \vec{r}'(t) &= \frac{\partial\vec{r}}{\partial u}u'(t) + \frac{\partial\vec{r}}{\partial v}v'(t) \end{aligned}$$

Having obtained $d\vec{r}$, everything would be just easy:

- Arclentgh:

  $$L = \int_{t_1}^{t_2} |\vec{r}'(t)|dt$$

  You should know how to parametrize the curve using arclength parameter.

- Area enclosed by the initial vector, end vector and the curve:

  $$A = \int |\vec{r} \times d\vec{r}|$$

- Work

  $$W = \int \vec{F} \cdot d\vec{r}$$

Example: Along the curve $y = \sin x, 0 \le x \le \pi$, if the force is $\vec{F} = F_0\vec{r}'$. Calculate the work done by this force if the particle moves from the origin to $(\pi, 0)$.
Example: #6 in this section

1.3 surface and surface integrals

We have the parametrization $\vec{r}(u, v) = x(u, v)\vec{e}_x + y(u, v)\vec{e}_y + z(u, v)\vec{e}_z$.
Example: For the sphere: $\vec{r}(\theta, \phi) = r\sin\theta\cos\phi\vec{e}_x + r\sin\theta\sin\phi\vec{e}_y + r\cos\theta\vec{e}_z$
For the torus $\vec{r} = \ldots$

1.3.1 Coordinate curves, tangent vectors(coordinate vectors)

You should know the definition of them and use them to determine if the given coordinates are orthogonal coordinates.

Example: If we parametrize $z = h(x, y)$ using x, y when would they be orthogonal parameters?

1.3.2 Surface element

$$d\vec{S} = \vec{N} du dv, \quad \vec{N} = \vec{r}_u \times \vec{r}_v$$

$\vec{N}$ is the normal. Do not just memorize them. You should understand why surface element looks like this.

Sometimes we are interested in $dS = |d\vec{S}|$.

Example: Calculate the area of the torus with outer radius 4 and inner radius 1.

As long as you get the surface element, everything would be nice.

- Flux
$$\int_S \vec{v} \cdot d\vec{S} = \int_A \vec{v} \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

- Pressure force:
$$\int_S p(\vec{r}) d\vec{S}$$

Example: All the problems in section 1.4.

1.4 Volumes and volume integrals

Parametrization $\vec{r}(u, v, w) = x(u, v, w)\vec{e}_x + y(u, v, w)\vec{e}_y + z(u, v, w)\vec{e}_z$.

Example: Parametrize the ball $x^2 + y^2 + z^2 \leq 4$, the cylinder $x^2 + y^2 \leq 1, 0 \leq z \leq 3$ and the solid inside the torus with outer radius 4, inner radius 2. Parametrize the cube $0 \leq x, y, z \leq 1$ and the solid bounded by xy, yz xz $x = 2$, $y = 3$ planes and the surface $z = 10x^2 + 30(y + 2)$

1.4.1 Coordinate curves, coordinate surfaces and coordinate vectors

Suppose we have $\vec{r}(u,v,w) = x(u,v,w)\vec{e}_x + y(u,v,w)\vec{e}_y + z(u,v,w)\vec{e}_z$. What are the coordinate curves and coordinate surfaces? Calculate the coordinate vectors.

1.4.2 Line elements, Surface elements

Given $\vec{r}(u,v,w)$. What are the line elements for the coordinate curves? What is the general line element? What are the surface elements for the coordinate surfaces?

1.4.3 Volume element

$$dV = (\vec{r}_u \times \vec{r}_v) \cdot \vec{r}_w dudvdw$$

In the three cases below find suitable $\vec{r}$ and then calculate the total mass inside the volume:
The density of a kind of material is $2(H-h)^2$. h is the height from the lowest point of that volume and H is the largest height.

- The ball $x^2 + y^2 + z^2 \leq 1$
- The cylinder $x^2 + y^2 = 1, 0 \leq z \leq 1$
- The cube $0 \leq x, y, z \leq 1$
- The solid in the first octant and bounded by $x = 1, y = 1$ and $z = x^2y^2$

1.4.4 Orthogonal coordinates and scale factors

We can always write the coordinate vectors as its magnitude times the direction.

$$h_i = |\partial\vec{r}/\partial q_i|, \quad \hat{q}_i = \frac{1}{h_i}\frac{\partial\vec{r}}{\partial q_i}$$

If the parameters are **orthogonal**, then $\hat{q}_i \cdot \hat{q}_j = \delta_{ij}$. If they satisfy right-handed rule furthermore, then we get a new basis. The surface element and volume element in such cases are quite easy:

$$\begin{aligned} d\vec{S}_3 &= \hat{q}_3 h_1 h_2 dq_1 dq_2 \\ dV &= h_1 h_2 h_3 dq_1 dq_2 dq_3 \end{aligned}$$

Example: Calculate $\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi$ for spherical coordinates. (They are important when we derive the gradient, divergence, etc under spherical coordinates)

1.5 Change of variables for integration

Just remember one thing: *Jacobian*!

Example: Calculate the Jacobian if we change from $2D$ Cartesian coordinates to polar coordinates. Use this to evaluate $\int_{-\infty}^{+\infty} e^{-x^2} dx$

1.6 ∇ operator, gradient, curl, divergence

1.6.1 In Cartesian coordinate

- $f(x, y, z)$ is a given nice scalar function. ∇f is called the gradient of function f, which points to the direction of fastest increasing and its magnitude equals the rate of change with respect to the distance going from one point. According to this meaning:

  $$df = d\vec{r} \cdot \nabla f$$

  Your task: Using the Cartesian form of $d\vec{r} = \ldots$ to get:

  $$\nabla f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z$$

  According to its geometric meaning and this expression, it's quite natural to get the direction derivative:

  $$\frac{\partial f}{\partial n} = \hat{n} \cdot \nabla f$$

- It's convenient to define a vector operator now:

  $$\nabla = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}$$

- Given a vector field $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3$ (here, they just mean the $\vec{e}_x$, $\vec{e}_y$ etc), define the divergence of $\vec{v}$ to be $\nabla \cdot \vec{v}$. One can easily get:

$$\nabla \cdot \vec{v} = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3$$

- Define the curl to be $\nabla \times \vec{v}$ and one can easily get:

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- The Laplacian of f is defined to be $\Delta f = \nabla \cdot \nabla f$. In Cartesian form:

$$\Delta f = \partial_{xx} f + \partial_{yy} f + \partial_{zz} f$$

  If $\Delta f = 0$, we call f a **harmonic function**.

1.6.2 identities

You should know how to prove them

- $\nabla \cdot (\nabla \times \vec{v}) = 0$
- $\nabla \times (\nabla f) = 0$
- $\nabla \cdot (f\vec{v}) = \nabla f \cdot \vec{v} + f\nabla \cdot \vec{v}$
- $\nabla \times (f\vec{v}) = \nabla f \times \vec{v} + f\nabla \times \vec{v}$
- $\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \Delta\vec{v}$

1.6.3 Spherical coordinates

In spherical coordinates, it's useful to use the basis $\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi$. Recall the definition of $\vec{e}_\theta$ for example. $\vec{r} = r\sin\theta\cos\varphi\vec{e}_x + r\sin\theta\sin\varphi\vec{e}_y + r\cos\theta\vec{e}_z$. $\vec{e}_\theta = \frac{1}{|\vec{r}_\theta|}\vec{r}_\theta$ etc. You should know how to derive ∇ operator under spherical coordinates. Given $f(r, \theta, \varphi)$

$$\begin{aligned} df &= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \varphi}d\varphi = d\vec{r} \cdot \nabla f \\ d\vec{r} &= \vec{r}_r dr + \vec{r}_\theta d\theta + \vec{r}_\varphi d\varphi \end{aligned}$$

You just assume that $\nabla f = u\vec{e}_r + v\vec{e}_\theta + w\vec{e}_\varphi$ because it's a vector and then you'll get:

$$\frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \varphi}d\varphi = |\vec{r}_r|udr + |\vec{r}_\theta|vd\theta + |\vec{r}_\varphi|wd\varphi$$

Then you'll have

$$\begin{aligned} u &= \frac{\partial f/\partial r}{h_r} = \frac{\partial f}{\partial r} \\ v &= \frac{\partial f/\partial \theta}{h_\theta} = \frac{1}{r}\frac{\partial f}{\partial \theta} \\ w &= \frac{\partial f/\partial \varphi}{h_\varphi} = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi} \end{aligned}$$

which gives the following **important formula**:

$$\nabla = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

Using the definition of divergence, gradient, curl, Laplacian etc, one can get the expressions for them under spherical coordinates. I won't list them here (You can find them online.). However, you should understand how to derive them even though you don't have to memorize them.

Example: In $\nabla \cdot \vec{v}$, we would have the term $(\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}) \cdot (u\vec{e}_r)$ which is nonzero. Please calculate this term out.

Other coordinates

Please derive ∇ operator using cylindrical coordinates and compare your results with those on Wikipedia.

1.7 Fundamental theorems for vector calculus

- Green's Theorem

$$\oint_C (Fdx + Gdy) = \int_A (\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y})dxdy$$

LHS can be also written as $\oint_C \vec{v} \cdot d\vec{r}$ if we define $\vec{v} = F\vec{e}_x + G\vec{e}_y$

One can check that the above formula is nothing but a special case of Stokes' theorem if the surface is restricted in the x-y plane. The **curl form** of Green's Theorem:

$$\oint_C \vec{v} \cdot d\vec{r} = \int_A \nabla \times \vec{v} \cdot \vec{e}_z dA$$

is just exactly the same formula as above.

Furthermore, if we define $\vec{n}$ as the unit normal vector with the suitable direction, we can write the left hand side as $\oint_C (G\vec{e}_x - F\vec{e}_y) \cdot \vec{n} dr$, and thus we have $\oint_C (G\vec{e}_x - F\vec{e}_y) \cdot \vec{n} dr = \int_A (\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}) dxdy$. Change the notations and we can have

$$\oint_C (F\vec{e}_x + G\vec{e}_y) \cdot \vec{n} dr = \int_A (\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}) dxdy$$

$$\oint_C \vec{v} \cdot \vec{n} dr = \int_A \nabla \cdot \vec{v} dA$$

This is an analogy of Gauss' Theorem in $2D$ and it's called the **divergence form** of Green's Theorem

- Stokes' Theorem

$$\int_S \nabla \times \vec{v} \cdot d\vec{S} = \oint_C \vec{v} \cdot d\vec{r}$$

Here S is any surface in space with boundary C and C is a closed curve in space.

Example: If S is a closed surface, C would be empty. What would the right hand side be? Use Gauss' Theorem to justify this.

- Gauss' Theorem

$$\oint_S \vec{v} \cdot d\vec{S} = \oint_S \vec{v} \cdot \vec{n} dS = \int_V \nabla \cdot \vec{v} dV$$

Here, S is a closed surface (no boundary) and V is the volume enclosed by it (this means the boundary of V is C but C has no boundary).

Example: Show that $\oint_C \nabla \times f \cdot d\vec{r} = 0$ if C is closed in space and f is a good function.

2 Complex Calculus

2.1 Some basic knowledge

- Complex conjugate etc. $(e^z)^* = e^{(z^*)}$. Generally, if f is analytical and takes real values on x-axis then, $(f(z))^* = f(z^*)$.
- $|z_1 z_2| = |z_1||z_2|$, $||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$ etc. Using the first identity, we actually can get some useful expressions: $(ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2)$ etc. For the second one, we can see that $R^2 - |b| \leq |R^2 e^{2i\theta} - b| \leq R^2 + |b|$ when $R \to \infty$
- Geometic series
$$\sum_{n=0}^{N} a_1 q^n = \frac{a_1(1 - q^{N+1})}{1 - q}$$
and if $|q| < 1$, we have
$$\sum_{n=0}^{\infty} a_1 q^n = \frac{a_1}{1 - q}$$
Example: Calculate $\sum\limits_{n=0}^{N} \cos(n\theta)$ and $\sum\limits_{n=0}^{N} \sin(n\theta)$ at the same time.
- Polar form of a complex number and calculate its **complex roots**. $z = re^{i\theta}$. You can get r by taking its magnitude and θ by plotting it in the complex plane. To calculate the complex roots, you can just use this polar form. There are exercises in the last homework.
- Euler's identity $e^{iz} = \cos(z) + i\sin(z)$
- How to calculate $\ln(z), \sin(z), \cos(z)$ etc

2.2 Complex power series

If a series $\sum\limits_{n=0}^{\infty} c_n(z-a)^n$ has a convergence bigger than zero then $f(z) = \sum\limits_{n=0}^{\infty} c_n(z-a)^n$ is analytical and its Taylor series is exactly this series. You should know how to apply Ratio Test etc to calculate the radius of convergence.
You should be able to write out the Taylor series of $e^z, \sin z, \cos z, \frac{1}{1-z}$

2.3 Cauchy-Riemann equations

We can regard any function of x, y as a function of z, $f(z)$. We say it's analytical if $f'(z)$ exists in a domain we are interested in. We know it's analytical if it is the sum of a power series. However, we have a more fundamental criterion–Cauchy-Riemann equations.
A function $f(z) = u(x, y) + iv(x, y)$ is analytical if and only if

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

One amazing fact is that both u and v **should be harmonic**! There are problems in the last homework.

2.4 Complex Integrals

Understand the definition. You should know how to use parametrization to calculate some simple complex integrals. Example: How to prove $\int_{|z|=1} \frac{1}{z} dz = 2\pi i$? How to calculate $\int_{|z|=2} z^* dz$?

2.4.1 Cauchy's Theorem

If a function $f(z)$ is analytical inside a closed curve C and on C(Actually, we only need it to be continuous on C), we then have:

$$\oint_C f(z) dz = 0$$

One important corollary of this theorem is

$$\oint_C (z-a)^n dz = \begin{cases} 2\pi i & n = -1 \textit{ and } C \textit{ encloses } a \\ 0 & \textit{otherwise} \end{cases}$$

2.4.2 Cauchy's formula

If $f(z)$ is analytical inside and on C, then:

$$\oint_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i f^{(n)}(a)}{n!}$$

if a is inside C and 0 otherwise.

This means that we can determine everything about $f(z)$ inside C only using values on C if $f(z)$ is analytical. This is not quite amazing since the real and imaginary parts are harmonic.

2.4.3 Applications

Use Cauchy's formula and Cauchy's theorem to calculate some real integrals.

Example: $\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$ $\int_0^{\infty} \cos(x^2) dx$ etc.