Math 321 Exercise

Due: March 27/29

Your Name: *Your Section:*

Let's look at the applications of surface integral in physics.

1. Consider a ball with radius r deep in the water. Suppose the center of the ball is l below the water level. Archimedes principle says that 'Any object immersed in a fluid is buoyed up by a force equal to the weight of the fluid displaced by the object'. Suppose the density of the water is ρ and we know the volume of the ball is $4\pi r^3/3$(you can verify this using volume integral). Then, the buoyancy should be $\frac{4}{3}\pi r^3 \rho g$ and point up. If we create coordinate system at the water level surface, and make z axis point downward, then we would have $\vec{F_b} = -\frac{4}{3}\pi r^3 \rho g \vec{e}_z$. Our goal here is to get this force by using surface integral directly.
   a). We make the center of the ball on the z axis. Then the center has coordinate $(0, 0, l)$. Parametrize the surface of the ball using the suitable angles θ, φ. (**2 thank-you marks**)
   b). The pressure due to the water equals $p(h) = \rho g h$ where h is the depth of the point in the water. Use this expression and surface integral to get the exact buoyancy above. (**4 thank-you marks**)
   Ans: (a). $\vec{r}(\theta, \varphi) = r\sin\theta\cos\varphi \vec{e}_x + r\sin\theta\sin\varphi\vec{e}_y + (l + r\cos\theta)\vec{e}_z$ (Notice here $r \neq |\vec{r}|$)
   (b). $d\vec{S} = \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \varphi} d\theta d\varphi = r^2 \sin\theta(\sin\theta\cos\varphi\vec{e}_x + \sin\theta\sin\varphi\vec{e}_y + \cos\theta\vec{e}_z)d\theta d\varphi$.
   The force is $\vec{F} = -\int_S \rho g h d\vec{S} =$
   $-\int_0^{2\pi}\int_0^{\pi} \rho g(l + r\cos\theta)r^2 \sin\theta(\sin\theta\cos\varphi\vec{e}_x + \sin\theta\sin\varphi\vec{e}_y + \cos\theta\vec{e}_z)d\theta d\varphi$
   x, y components are 0 because of the φ integrals. The z component is:
   $$-\int_0^{2\pi}\int_0^{\pi} \rho g(l + r\cos\theta)r^2 \sin\theta\cos\theta d\theta d\varphi = -r^3\rho g\int_0^{2\pi}\int_0^{\pi}\cos\theta\sin\theta\cos\theta d\theta d\varphi$$
   The answer is $-\frac{4\pi r^3\rho}{3} g\vec{e}_z$
   We have a negative sign for the force, because it is the water which pushes the ball. Then, at each point, the force points inside, but for $d\vec{S}$ we used the outer normal vector.